E. J. Senkler
Jan. 1859.

AN ESSAY ON PROBABILITIES,

and on their application to

LIFE CONTINGENCIES

AND

INSURANCE OFFICES,

BY

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PREFACE.

In order to explain the particular object of this Treatise, it will be necessary to give a brief account of the science on which it treats.

At the end of the seventeenth century, the theory of probabilities was contained in a few isolated problems, which had been solved by Pascal*, Huyghens, James Bernoulli, and others. They consisted of questions relating to the chances of different kinds of play, beyond which it was then impossible to proceed: for the difficulty of a question of chances depending almost entirely upon the number of combinations which may arise, the actual and exact calculation of a result becomes exceedingly laborious when the possible cases are numerous. A handful of dice, or even a single pack of cards, may have its combinations exhausted by a moderate degree of industry: but when a question involves the chances of a thousand dice, or a thousand throws with one die, though its correct principle of solution would have been as clear to a mathematician of the sixteenth century as if only half a dozen throws had been considered; yet the largeness of the numbers, and the

* Un problème relatif aux jeux de hasard, proposé à un austère janseniste par un homme du monde, a été l'origine du calcul des probabilités.
Poisson.

consequent length and tediousness of the necessary operations, would have formed as effectual a barrier to the attainment of a result, as difficulty of principle, or want of clear perception.

There was also another circumstance which stood in the way of the first investigators, namely, the not having considered, or, at least, not having discovered, the method of reasoning from the happening of an event to the probability of one or another cause. The questions treated in the third chapter of this work could not therefore be attempted by them. Given an hypothesis presenting the necessity of one or another out of a certain, and not very large, number of consequences, they could determine the chance that any given one or other of those consequences should arrive; but given an event as having happened, and which might have been the consequence of either of several different causes, or explicable by either of several different hypotheses, they could not infer the probability with which the happening of the event should cause the different hypotheses to be viewed. But, just as in natural philosophy the selection of an hypothesis by means of observed facts is always preliminary to any attempt at deductive discovery; so in the application of the notion of probability to the actual affairs of life, the process of reasoning from observed events to their most probable antecedents must go before the direct use of any such antecedent, cause, hypothesis, or whatever it may be correctly termed. These two obstacles, therefore, the mathematical difficulty, and the want of an inverse method, prevented the science from extending its views beyond problems of that simple nature which games of chance present. In the mean time, it was judged by its fruits;

and that opinion of its character and tendency which is not yet quite exploded, was fixed in the general mind.

Montmort, James Bernoulli, and perhaps others, had made some slight attempts to overcome the mathematical difficulty; but De Moivre, one of the most profound analysts of his day, was the first who made decided progress in the removal of the necessity for tedious operations. It was then very much the fashion, and particularly in England, to publish results and conceal methods; by which we are left without the knowledge of the steps which led De Moivre to several of his most brilliant results. These however exist, and when we look at the intricate analysis by which Laplace obtained the same, we feel that we have lost some important links * in the chain of the history of discovery. De Moivre, nevertheless, did not discover the inverse method. This was first used by the Rev. T. Bayes, in *Phil. Trans.* liii. 370.; and the author, though now almost forgotten, deserves the most honourable remembrance from all who treat the history of this science.

Laplace, armed with the mathematical aid given by De Moivre, Stirling, Euler, and others, and being in possession of the inverse principle already mentioned, succeeded both in the application of this theory to more useful species of questions, and in so far reducing the difficulties of calculation that very complicated problems may be put, as to method of solution, within the reach of an ordinary arithmetician. His contribution to the science was a general method (the analytical beauty and power of which would alone be sufficient to give him a high rank among mathematicians) for the solution of

* The same may be said of several propositions given by Newton.

all questions in the theory of chances which would otherwise require large numbers of operations. The instrument employed is a table (marked Table I. in the Appendix to this work), upon the construction of which the ultimate solution of every problem may be made to depend.

To understand the demonstration of the method of Laplace would require considerable mathematical knowledge; but the manner of using his results may be described to a person who possesses no more than a common acquaintance with decimal fractions. To reduce this method to rules, by which such an arithmetician may have the use of it, has been one of my primary objects in writing this treatise. I am not aware that such an attempt has yet been made: if, therefore, the fourth, and part of the fifth chapters of this work, should be found *difficult*, let it be remembered that the attainment of such results has hitherto been *impossible*, except to those who have spent a large proportion of their lives in mathematical studies. I shall not, in this place, make any remark upon the utility of such knowledge. Those who already admit that the theory of probabilities is a desirable study, must of course allow that persons who cannot pay much attention to mathematics, are benefited by the possession of rules which will enable them to obtain at least the results of complicated problems; and which will, therefore, permit them to extend their inquiries further than a few simple cases connected with gambling. By those who do not make any such concession, it will readily be seen, that the point in dispute may be argued in a more appropriate place than with reference to the question whether others, who hold a different opinion, should, or should not, be supplied with a certain arithmetical method.

The first six chapters of this work (the fourth, and part of the fifth exclusive) may be considered as a treatise on the principles of the science, illustrated by questions which do not require much numerical computation. To this must be added the first appendix, *on the ultimate results of play*. Omitting the first pages of the latter, the discussion on the noted game of *rouge et noir* will, with the problems in page 108. &c., serve to show the real tendency of such diversion. I am informed that this game is not played in England at any of the clubs which are supposed to allow of gambling: but it was permitted in the Parisian *salons* until the very recent suppression of those establishments; and the account given of it will show what has taken place in our own day. The game of hazard is more used in this country; but I have been prevented from giving it the same consideration by the want of a clear account of the manner in which it is played. Nothing can be more unintelligible than the description given by the celebrated Hoyle.

The fourth chapter has been already alluded to: it contains the method of using the tables at the end of the work in the solution of complicated problems. The seventh chapter, and the fourth appendix, contain the application of the preceding principles to instruments of observation in general.

The remainder of the work is devoted to the most common application of this theory, the consideration of life contingencies and pecuniary interests depending upon them, together with the main principles of the management of an insurance office. As this portion was not written for the sake of the offices. but of those who deal with them, I have confined myself to such points as I considered most requisite to be generally known. Common as life insurance has now become, the present amount of capital so invested is trifling compared with what will be the case when its principles are better understood; provided always that the offices continue to act with prudence until that time arrives. At present, while the public has little except results to judge by, the failure of an office would cause a panic, and perhaps retard for half a century the growth of one of the most useful consequences of human association: but the time will come when knowledge of the subject will be so diffused, that even such an event as that supposed, if it could then happen, would not produce the same result.

There are, however, one or two things to which I should call the attention of those whose profession it is to calculate life contingencies: —

1. The notation for the expression of such contingencies (pp. 197—204.). This notation was suggested by that of Mr. Milne, from which it differs in what I believe to be a closer representation of the analogies which connect different species of contingencies. Thus, an annuity to last a number of years certain does not differ from a life annuity in any circumstance which requires a difference of notation; nor an insurance from an annuity certain of one year deferred till a life drops. Since writing the pages above referred to, I have learned that I was not the first who considered an insurance in that light. Some years ago the government granted annuities for terms certain, to commence at the death of an individual; but refused to insure lives: the consequence was, that, by a very obvious evasion, insur-

ances were effected by buying annuities for one year certain, to commence at the death of a person named. This had the effect of putting an end to such annuities.

2. The form of the rule for computing the value of fines, and its introduction into the method of calculating the present value of a perpetual advowson (pp. 231. 236. and Appendix the Second). It will be found that the rule of every writer on the subject is palpably wrong in principle, with the exception of that of Mr. Milne.

3. The rule for the valuation of uniformly increasing or decreasing annuities, given in the fifth appendix. A simple application of the differential calculus is made a striking instance of the position, that the labour of a person of competent knowledge is seldom lost. The annuities given by Mr. Morgan and Mr. Milne, are for every rate of interest, from three to eight per cent.; and perhaps those gentlemen may have had some doubts as to the necessity of inserting the two last rates. It now appears, however, that, in consequence of the extent to which their tables are carried, the values of increasing or decreasing annuities, can be calculated with great accuracy for three and four per cent., and with sufficient nearness for five per cent.; and with very little trouble, compared with that which it must have cost Mr. Morgan to calculate the table referred to in page xxviii. of the Appendix.

The rules, in page xxix. of the Appendix, contain a point which, as no demonstration is given, may cause some difficulty. In turning an annuity or insurance which cannot be extinguished during the life of the party into one which can, a direction to *add* is given which will at first sight, perhaps, be supposed to be a mistake, and that *subtract* should be written instead. But it must be remembered that an annuity of, say £3 a year, diminishing by £1 every year, is equivalent, by the first part of the rule, to an annuity of which the successive payments are as follows:

£3, £2, £1, £0, £(—1), £(—2), £(—3), &c.

That is, the first part of the rule, when the annuity is extinguished during the tabular life of the party, gives the value of his interest upon the supposition that he is to begin to *pay* as soon as he ceases to *receive*. If then, this is not to be the case, the value of his interest must be *increased* accordingly.

4. The method of the *balance of annuities*, or the determination of complicated annuities by the addition and substraction of simple ones. This has been done before; but it has not, to my knowledge, been carried to the extent of making all the questions which commonly occur deducible from the fundamental tables, without the aid of any new series. It is desirable that the beginner should be accustomed to deduction by reasoning, without having recourse to the mechanism of algebra, which, as a quaint editor of Euclid observed, "is the paradise of the mind, where it may enjoy the fruits of all its former labours, without the fatigue of thinking." Of no part of algebra is this more true, than of the method by which complicated annuities are deduced from simple ones, by the resolution of the series which represent them into the simpler series of which they are composed. The education of an actuary does not necessarily imply the study of geometry; and such processes, for instance, as those by which are found the values of a contingent insurance or a temporary insurance (pp. 222. 226.), will serve, as far as they go, to ac-

custom him to make those efforts of mind, and to bear that tension of thought, the necessity for which is the distinction between a problem of geometry, and one of ordinary algebra.

The considerations contained in this volume have, in my opinion, a species of value which is not directly derived from the use which may be made of them as an aid to the solution of problems, whether pecuniary or not. Those who prize the higher occupations of intellect see with regret the tendency of our present social system, both in England and America, with regard to opinion upon the end and use of knowledge, and the purpose of education. Of the thousands who, in each year, take their station in the different parts of busy life, by far the greater number have never known real mental exertion; and, in spite of the variety of subjects which are crowding upon each other in the daily business of our elementary schools, a low standard of utility is gaining ground with the increase of the quantity of instruction, which deteriorates its quality. All information begins to be tested by its *professional* value; and the knowledge which is to open the mind of fourteen years old is decided upon by its fitness to manure the money-tree.

Such being the case, it is well when any subject can be found which, while it bears at once upon questions of business, admits, at the same time, the application of strict reasoning; and by its close relation to knowledge of a more wide and liberal character, invites the student to pursue from curiosity a path not very remote from that which he entered from duty or necessity. Such a subject is the theory of life annuities, which, while it will attract many from its commercial utility, can hardly fail to be the gate through which some will find their way to the general theory of probabilities, and, perhaps, from thence to the pursuit of other branches of science. There are strong instances in favour of such a supposition. Many persons in this country have begun by the common studies of an accountant, have been led to an elementary knowledge of algebra and to the use of logarithms by seeing the value of such information in their particular pursuit, and have ended by becoming, in many cases well informed, and in some instances eminent, mathematicians.

Nothing is of more importance, as a help in holding out every bait by which students may be drawn to the exact sciences, than the co-operation of the universities; which, though they do not possess much power of introducing subjects into general study, yet have great influence in the settlement of the manner in which those things shall be learned, the advantages of which have been, or may be, felt by the community at large. If ever it should happen that a particular branch of knowledge becomes in request, it would be of much advantage if those institutions would forthwith appropriate and liberalise it; to do which nothing more would be necessary than to promote the study of it among their aspirants to distinction. The consequence would be, that it would find a place in the elementary works which so frequently appear; and not only *a* place, but *its* place; that is, in proper connection with other branches of learning, and treated by methods which would preserve that connection. Those who begin to study it in their younger days for professional purposes would be led to the method which bore the sanction of the universities, and not unfrequently to the pursuit of other subjects immediately connected with it.

The theory of insurance, with its kindred science of annuities, deserves the attention of the academical bodies. Stripped of its technical terms and its commercial associations, it may be presented in a point of view which will give it strong moral claims to notice. Though based upon self-interest, yet it is the most enlightened and benevolent form which the projects of self-interest ever took. It is, in fact, in a limited sense, and a practicable method, the agreement of a community to consider the goods of its individual members as common. It is an agreement that those whose fortune it shall be to have more than average success, shall resign the overplus in favour of those who have less. And though, as yet, it has only been applied to the reparation of the evils arising from storm, fire, premature death, disease, and old age; yet there is no placing a limit to the extensions which its application might receive, if the public were fully aware of its principles, and of the safety with which they may be put in practice.

It is of great importance at the present moment that sound principles on the subject of insurance should be widely and rapidly disseminated. Within the last twenty years, many new institutions have been founded; and during the busy portion of the London year, seldom a month passes without the announcement of a novel plan. Of many of these projects I am unable to speak, from not having paid particular attention to them. But of one thing I am certain, that the magnificent style in which the prospectuses frequently indulge might often remind their readers of the unparalleled benefits which are promised by another description of traders, who vie with each other in describing the rare qualities of their several blackings. If there be in this country a person whose ambition it is to walk in the brightest boots to the cheapest insurance office, he has my pity: for, grant that he is ever able to settle where to send his servant, and it remains as difficult a question to what quarter he shall turn his own steps. The matter would be of no great consequence if persons desiring to insure could be told at once to throw aside every prospectus which contains a puff: unfortunately this cannot be done, as there are offices which may be in many circumstances the most eligible, and which adopt this method of advertising their claims. If these pompous announcements be intended to profess that every subscriber shall receive more than he pays, their falsehood is as obvious as their meaning; if not, their meaning is altogether concealed.

Public ignorance of the principles of insurance is the thing to which these advertisements appeal: when it shall come to be clearly understood that in every office some must pay more than they receive, in order that others may receive more than they pay, such attempts to persuade the public of a certainty of universal profit will entirely cease. To forward this result, I have endeavoured, as much as possible, to free the chapters of this work which relate to insurance offices from mathematical details, and to make them accessible to all educated persons. Whether they act by producing conviction, or opposition, a step is equally gained: nothing but indifference can prevent the public from becoming well acquainted with all that is essential for it to know on a subject, of which, though some of the details may be complicated, the first principles are singularly plain.

August 3. 1838.

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AN ESSAY

ON

PROBABILITIES.

CHAPTER I.

ON THE NOTION OF PROBABILITY AND ITS MEASUREMENT; ON THE PROVINCE OF MATHEMATICS WITH REGARD TO IT, AND REPLY TO OBJECTIONS.

When the speculators of a former day were busily employed in constructing celestial tables for the use of prophets, or investigating the qualities of bodies for the manufacture of gold, no one could guess that they were accelerating the formation of sciences which should themselves be among the most essential foundations of navigation and commerce, and, through them, of civilisation and government, peace and security, arts and literature. That good plants of such a species require the warmth of mysticism and superstition in their early growth is not a rule of absolute generality, for there are cases in which cupidity and vacancy of mind will do as well. Cards and dice were the early aliment of the branch of knowledge before us; but its utility is now generally recognised in all the more delicate branches of experimental science, in which it is consulted as the guide of our erroneous senses, and the corrector of our fallacious impressions. And more than this, it is the source from whence we draw the means of equalising the accidents of life, and contains the principles on which it is found practicable to induce many to join together, and consent that all shall bear the average lot in life of the whole. But the ill educated offspring of a vicious parent is frequently fated to bear the stigma of his descent, long after his own conduct has created the good opinion of those who know him. The science which I endeavour, and I believe almost for the first time, to render practically accessible in its higher and more useful parts to readers whose knowledge of mathematics extends no farther than common arithmetic, is still often considered as foreign to the pursuits, and dangerous in the conduct, of life. It is said to be necessary only to gamblers, and calculated to excite a passion for their worthless and degrading pursuit. This refers to its practical and moral consequences: with regard to its title to confidence, it is often supposed to rest upon pure conventions of an uncertain order, and to depend for the connection of results with principles upon the higher branches of mathematics; things understood by very few, and frequently distrusted, if not by those who have reached them, by those who have passed some way up the avenue which leads to them. All these impressions must necessarily be removed before the theory of probabilities can occupy its proper place; and it is, therefore, my preliminary task to meet the arguments which arise out of them. There is an indefinite dislike in many minds to all knowledge which they cannot reach; it may tend to remove this if I show that results, at least, are very easily attained, and methods practised: but the notion that asserted knowledge is not knowledge must be met by preliminary reasoning, and imperfect as it must necessarily be, considered as a view of the subject, it may yet afford the means of dwelling on the first principles to a greater extent than is usually done in formal treatises on recognised subjects.

Human knowledge is, for the most part, obtained under the condition that results shall be, at least, of that degree of uncertainty which arises from the *possibility* of their being false. However improbable it may be, for instance, that the barbarians did not overturn the Roman empire, we do not recognise the same sort of *sensible certainty* in our *moral certainty* of the fact which we have in our knowledge that fire burns, or that two straight lines do not enclose space. And we perceive a difference in the quality of our knowledge, when any alteration takes place in our circumstances with respect to exterior objects. That fire *does* burn is more certain than the account of the fall of Rome: that fire yet to be lighted *will* burn may or may not be more certain than the historical fact, according to the temperament and knowledge of the individual. And thus we begin to recognise differences even between our (so called) *certainties;* and the comparative phrases of more and less certain are admissible and intelligible. It is usual to begin the subject by saying that our certainties are only very high degrees of probability. This is not practically true at the outset; yet so far as deductions can be made numerically. with respect to our impressions of assent or dissent, it *will be shown* to be correct so to consider the subject. We have a process to go through before we can arrive at such a conclusion, as follows: — When a child is born, there is a certain degree of force which we allow to the assertion that he will die aged 50. To it we answer that it may be, but that that particular age is unlikely compared with all the rest, though, at first sight, as likely as any other. If the assertion be made of two children, that one or other will die aged 50, we readily admit that our "it may be, but it is not likely," is no longer the same assertion as it was before. It is of the same sort, but not of the same strength: the assertion is *more probable,* and wherever we have the notion of *more* and *less,* we feel the possibility of an answer to the question, "how much more or less?" and which we should produce if we knew how. First impressions would induce us to suppose it twice as probable that the assertion may be made of one or other of two children, as of one alone; and so on. Let this false measure (for such it is) remain; we are not here considering what is the proper measure, but whether we can conceive the possibility of a measure or not. Let the preceding method of measurement be admitted; and let us ask how we stand with regard to the same assertion, predicated of one or other of a million of children born together. The answer is, we feel quite certain, that many of them will die at the age of 50. Supposing humanity to endure 50 years, we feel as confident of the truth of the assertion, as we do that Rome was taken by Alaric, or that fire will burn. Without entering into the very different sources through which conviction comes to us, we put four propositions together: —

The Roman empire was overturned by northern barbarians.	Two straight lines cannot enclose a space.	Fire will burn.	Of 1,000,000 of children born, some will die aged fifty, if the race of man last fifty years.

and, we ask, if you were to receive a certain advantage upon naming a truth from among these four assertions, what would guide your choice? There is certainly a little difference in the impressions of assent with which we regard the four; but whether it be of any real strength, we may test in this way: — Supposing the benefit in question to be 1000*l.*, would you not let another person choose for you, almost at his pleasure, and certainly for a shilling?

On this we remark, firstly, that by it we feel sensible of our assent and dissent to propositions derived in very different ways, being a sort of impression which is of the same kind in all. To make this clearer, observe the following: — A merchant has freighted a ship, which he expects (is not certain) will arrive at her port. Now suppose a lottery, in which it is quite certain that every ticket is marked with a letter, and that all the letters enter in equal numbers. If I ask him, which is most probable, that his ship will come into port, or that he will draw no letter if he draw, he will answer, unquestionably, the first, for the second will certainly not happen. If I ask, again, which is most probable, that his ship will arrive, or that he will, if he draw, draw either *a*, or *b*, or *c*, ... or *x*, or *y*, or *z*, he will answer, the second, for it is quite certain. Now suppose I write the following series of assertions: —

He will draw no letter (a drawing supposed).
He will draw *a*.
He will draw either *a* or *b*.
He will draw either *a*, or *b*, or *c*.
....
....
He will draw either *a* or *b* or .... or *y*.
He will draw either *a* or *b* or .... or *y* or *z*.

and making him observe that there are, of their kind, propositions of all degrees of probability, from that which cannot be, to that which must be, I ask him to put the assertion that his ship will arrive, in its proper place among them. This he will perhaps not be able to do, not because he feels that there is no proper place, but because he does not know how to estimate the force of his impressions in ordinary cases. If the voyage were from London Bridge to Gravesend, he would (no steamers being supposed) place it between the last and last but one: if it were a trial of the north-west passage, he would place it much nearer the beginning; but he would find difficulty in assigning, within a place or two, where it should be. All this time he is attempting to compare the magnitude of two very different kinds (as to the sources whence they come) of assent or dissent; and he shows by the attempt that he believes them to be of the same sort. He would never try to place the *weight* of his ship in its proper position in a table of *times* of high water.

We also see, secondly, that the impression called certainty is of the character of a very high degree of probability. Out of 1,000,000 of children born, it is certain some will die aged 50. But by gradual progression, our unassisted judgment makes us believe that we may correctly say that it is 1,000,000 times as probable the assertion will be true of one or other out of 1,000,000 as of one alone. The method of measuring is wrong, but that is here immaterial; suffice it that, come how it may, the multiplication of the degree of assent implied in "there is a remote chance of it" *is found* to give that which is conveyed in "we are quite sure of it." We have thus a sort of freezing and boiling point of our scale of assent and dissent, namely, absolute certainty against on the one hand, absolute certainty for on the other hand, with every description of intermediate state.

Thirdly, we have proposed two ascending scales of assertions, in both of which first impressions would make us suppose the probability of the second is double that of the first, that of the third treble, and so on, as follows: —

A child born will die aged fifty.	*a* must be drawn.
Of two children born, one or other will die aged fifty.	*a* or *b* must be drawn.
Of three children born, one or other will die aged fifty.	*a*, or *b*, or *c* must be drawn.
&c. &c. &c.	&c. &c. &c.

Now it will hereafter be positively proved that our notion is correct in the second case, but incorrect in the first; or at least that it cannot be correct in both. Even then, if we should fail in assigning positive measurements, we may succeed in drawing useful distinctions. When we imagine two things to have a point of resemblance which they have not, it is worth while to investigate methods of correction, even though we cannot assign *how much* the two properties differ which we supposed were alike.

The quantities which we propose to compare are the forces of the different impressions produced by different circumstances. The phraseology of mechanics is here extended: by force, we merely mean cause of action, considered with reference to its magnitude, so that it is more or less according as it produces greater or smaller effect. It is one of the most essential points of the subject to draw the distinction we now explain. Probability is the feeling of the mind, not the inherent property of a set of circumstances. It is frequently referred to external objects, as if it accompanied them independently of ourselves, in the same manner as we imagine colour, form, &c. to abide by them. Thus we hold it just to say, that a white ball may be shut up in a box, and whether we allow light to shine on it or not, it is still a white ball. And if we were to translate the common notion, we should also say that in a lottery of balls shut up in a box, each ball has its *probability of being drawn* inseparably connected with it, just as much as form, size, or colour. But this is evidently not the case: two spectators, who stand by the drawer, may be very differently affected with the notion of likelihood in respect to any ball being drawn. Say that the question is, whether a red or a green ball shall be drawn, and suppose that A feels certain that all the balls are red, B, that all are green, while C knows nothing whatever about the matter. We have here, then, in reference to the drawing of a red ball, absolute certainty for or against, with absolute indifference, in three different persons, coming under different previous impressions. And thus we see that the real probabilities may be different to different persons. The abomination called intolerance, in most cases in which it is accompanied by sincerity, arises from inability to see this distinction. A believes one opinion, B another, C has no opinion at all. One of them, say A, proceeds either to burn B or C, or to hang them, or imprison them, or incapacitate them from public employments, or, at the least, to libel them in the newspapers, according to what the feelings of the age will allow; and the pretext is, that B and C are morally inexcusable for not believing what is true. Now substituting* for what *is* true that which A believes to be true, he either cannot or will not see that it

* The refusal of this substitution is what soldiers call the key of A's position: he himself sees the absurdity of his own arguments the moment it is made; and he is therefore obliged to contend for a sort of absolute truth external to himself, which B or C, he declares, might attain if they pleased.

depends upon the constitution of the minds of B and C what shall be the result of discussion upon them. Let it be granted that the intellectual constitution of A, B, and C is precisely the same at a given moment, and there is ground for declaring that any difference of opinion upon the same arguments must be one of moral character. Granting, then, that it were quite certain A is right, he might be justified in using methods with B and C which are reformative of moral character; that is to say, granting that state punishments are reformative of immoral habits, as well as repressive of immoral acts, he would be justified in direct persecution. But to any one who is able to see with the eyes of his body that the same weight will stretch different strings differently, and with those of his mind that the same arguments will affect different minds differently — by difference not of moral but of intellectual construction — will also see that the only legitimate process of alteration is that of the latter character, not of the former; namely, argument * and discussion. In the mean time, we bring it forward as not the least of the advantages of this study, that it has a tendency constantly to keep before the mind considerations necessarily corrective of one of the most fearful taints of our intellect.

Let us now consider what is the *measure of probability*. Any one thing is said to measure another when the former grows with the growth of the latter, and diminishes with its diminution. For instance, in the tube of a thermometer, the height of the mercury above freezing point (a line) measures the content of a cylinder; not that a line is a solid, but twice as much length belongs to twice as much content, and so on. Again, the content of the cylinder measures the quantity of expansion in a given quantity of mercury (and in this case not only *measures*, but *is*). Thirdly, the

* It is frequently asserted, that opinions dangerous to the existence of public order must not be promulgated. This is a question distinct from the one in the text, so far as it is political. If we grant no morals except expediency, (which, it appears to us, is necessary for the affirmation of the preceding,) the answer is, simply, that persecution is ineffective.

quantity of expansion measures the quantity of heat which produces it.

The exactness of mathematical reasoning depends upon that of our knowledge of the circumstances employed. No theorem about triangles, for instance, is true of any approach to a triangle such as we make on paper; but only more and more nearly true, the more nearly we make our lines *lengths without breadths*, and *straight*. Similarly, we cannot apply any theory of probabilities to the circumstances of life, with any greater degree of exactness than the data will allow. But as in geometry we invent exactness by supposing the utmost limits of our conceptions attainable in practice, so in the present case we begin by reasoning on circumstances defined by ourselves, and require adherence to certain axioms, as they are called, meaning propositions of the highest order of evidence.

Axiom 1. Let it be granted that the impression of probability is one which admits perceptibly of the gradations of more and less, according to the circumstances under which an event is to happen.

Axiom 2. Let it be granted that when one out of a certain number of events must happen, and these events are entirely independent of one another, the probability of one or other of a certain number of events happening must be made up of the probabilities of the several events happening. For instance, in the lottery of letters, in which there are 26 independent possible events, the probability of drawing either *a*, *b*, *c*, or *d* is made up of the probabilities of drawing *a*, of drawing *b*, of drawing *c*, and of drawing *d*, put together.

The latter axiom may excite some discussion; but we must observe that it is the uniform practice of mankind to act upon it, which is a sufficient justification; for what are we doing but endeavouring to represent that which actually exists? With regard to the value of each chance, suppose that one of the letters is a prize of 26*l.*, and that the 26 letters have been bought. If I buy up all the vested interests at less than 1*l.* a piece, I am certain to gain; if at more, I am certain to lose. 1*l.* a piece is what I ought to give for each, *if I buy all:* it is the universal practice to consider that 1*l.* a piece is still the value, if I buy a part. To say this is in fact to say that the force of the impression called certainty should, in this case, be considered as made up of 26 equal parts, each of which is to be considered as the representative of the impression of probability which a right-minded man would derive from the possession of one ticket.

On this I have to remark, 1. That so soon as any notion receives the exactness of mathematical language, though it be thereby not altered, objections are taken to it. The reason is, that we frequently not only use expressions which can be rendered quite exact, but also fairly act upon them as if they were exact, *but not because we consider them exact.* Why does the lottery ticket of the preceding instance bear the character of being exactly worth 1*l.*? Not as any consequence of the accuracy of the preceding process, supposing it accurate, but because we do not know why we should exceed rather than fall short of it. It appears to me that many of our conclusions are derived from this principle, which is called in mathematics *the want of sufficient reason.* A ball is equally struck in two different directions, the table being uniform throughout. In what direction will it move? In the direction which is exactly between those of the blows. Why? No positive reason is assignable (experiment being excluded); but from the complete similarity of all circumstances on one side and the other of the bisecting direction, it is impossible to frame an argument for the ball going more towards the direction of one blow, which cannot immediately be made equally forcible in favour of the other. The conclusion remains, then, balanced between an infinity of possible arguments, of which we can only see that each has its counterpoise. Now whether we adopt the above conclusion as to probability for its exactness or for its want of demonstrable inexactness one way more than the other, it is still a principle of human action, and as such is adopted. Many writers on probability speak of it as being a maxim which, if it were not adopted, ought to be. Certainly, such an assertion has some strong arguments in its favour; but with me they would not outweigh the importance I should attach to exact deduction from the conceptions which actually prevail.

Let the prospect of drawing any given letter be of a degree of force represented by 1, all the several prospects being equal. Then 2 is the chance of drawing one or other out of any given pair; and so on up to 26, which is here the representative of certainty. But if the lottery had 50 letters, the prospect of drawing a given letter would no longer be represented by 1; or if so, the certainty of drawing one out of 50 in the second would be represented by 50, while the certainty of drawing one out of 26 in the first is represented by 26. Now certainty, absolute certainty, should have the same representation whatever contingencies it may be supposed to be compounded of. If a man be sure of 100*l.*, it matters nothing whether his certainty arise from the announcement of a prize in a lottery of 1000 tickets, or of a legacy to which 20 other people were looking forward. To use a common phrase, a man can but be certain; and therefore it would be desirable to use the same symbol for certainty in all cases. Let this symbol be unity or 1; then in the first lottery the chance of any given letter is represented by $\frac{1}{26}$, and in the second by $\frac{1}{50}$. Similarly the chance of 1 out of 10 given letters in the first lottery is $\frac{10}{26}$, and in the second, $\frac{10}{50}$.

Now I pause upon this result, which, in fact, contains *all the theory* I shall be obliged to use; grant this, and you can be constrained, by demonstration, to admit all the rest as simple logical consequences. A writer on this subject, therefore, must take care not to let an opponent of its principles choose his own ground of attack, so as to wait until he can take advantage of the length of a deduction, or of the mathematical character of the steps. Do you admit, 1. That a certainty, if you have it, of drawing a 10*l.* prize in a lottery, is precisely the same thing whether there be 100 or 1000 tickets? and 2. That if there be 3 white balls and 17 black in a lottery, of which either white ball is to be a prize, you are compelled to regard your chances of success and failure with impressions of which it is reasonable to suppose the force to be as 3 to 17; or to say, "the degree in which I fear failure is, to my degree of hope of success, in the proportion of 17 to 3." If you say this, it matters nothing whether you say it because you feel the correctness of the proposition, or because you feel a want of data to deny it in one way more than in the opposite. Provided only that you do not deny it, your occupation of opponent is gone; for all that succeeds is merely a mathematical use of this mathematical definition. In the words of the ritual, Speak now, or ever after hold your tongue.

But it may be asked, with regard to the mathematical part of this subject, What is the province of the science of calculation? Are we, because we reject the higher mathematics, entirely without evidence; or can we obtain any thing like conviction of the truth of our methods? Now it happens unluckily for objectors, that the duty of mathematics in this science is very much more simple in character than the same in astronomy, mechanics, optics, music, or any other part of mathematical physics. For in the whole of these sciences, we have *principles*, as well as *results*, deduced by long trains of mathematical reasoning; whereas, in the science before us, we ask nothing of mathematics but the abbreviation of long numerical operations. For instance: — "If bodies move round another body, circularly, and so that each body, in its own circle, describes equal lengths in equal times, and if, moreover, the squares of the times of revolution are in the same proportion as the cubes of the distances, then it follows that the cause of motion can be *nothing but* an attractive force directed towards the central body, which, for different distances, changes inversely as the square of the distances." This is well known to be a fundamental part of the system of astronomy which has enabled one century to do more towards correct prediction of the state of the heavens than the twenty centuries which preceded it; and yet the apparatus of mathematics which is required to establish this result, which is of the nature of a principle, is enormous. But in the present subject we shall establish all our principles without the aid of any more mathematics than is contained in arithmetic; and when we draw upon the science, it shall be for nothing but abbreviation of long processes. The principle upon which mathematical abbreviation frequently proceeds is this: that where the calculation of a few results materially aids the production of a great many more, it is advisable to calculate a multitude of results, to arrange them in convenient tables of reference, and to publish them; so that by means of one person taking a little more trouble than would otherwise fall to his share, all others may be saved labour altogether. Mathematical tables are frequently nothing but the result of labour performed once for all; but it also sometimes happens, that the principle on which the labour is performed can be exemplified by a familiar case of it. We shall take that of logarithms as an instance.

Every table of logarithms is an extensive table of compound interest. Not to embarrass ourselves with fractions, let us take a table of *cent.* per *cent.* compound interest. We have then the following amounts of 1*l.* in 1, 2, 3, &c. years:—

Yrs.	Am.	Yrs.	Am.	Yrs.	Am.	Yrs.	Am.
0	1	7	128	14	16584	21	2097152
1	2	8	256	15	32768	22	4194304
2	4	9	512	16	65536	23	8388608
3	8	10	1024	17	131072	24	16777216
4	16	11	2048	18	262144	25	33554432
5	32	12	4096	19	524288	26	67108864
6	64	13	8192	20	1048576	27	134217728

The property of this table is, that if we wish to multiply together any two numbers called amounts, we have only to add together the number of years they belong to, and look opposite the sum in the table of years. Thus, 11 and 12, *added* together, give 23; 2048 and 4096, *multiplied* together, give 8,388,608. The reason is as follows: if 1*l.* in 11 years yield 2048*l.*, and if this 2048*l.* be put out for 12 years more, then, since 1*l.* in 12 years yields 4096*l.*, 2048 times as much will yield 2048×4096*l.*; or the amount in $11+12$ years is the product of the amounts in 11 and 12 years. The only reason why the preceding table is not in the common sense of the words a table of logarithms, is, that its construction leaves out most of the numbers. We can deal with 2048 and 4096, but there is nothing between them. The remedy is, to construct a table of compound interest, at such an excessively small interest, that a year shall never add so much as a pound throughout. Certain considerations, by which the table may be shortened, but with which we have here nothing to do, make it convenient to suppose such a rate of interest, that 1*l.* shall increase to 10*l.* in not less than 100,000 years, at compound interest. Or we may suppose interest payable 100,000 times a year, and say, let the whole *yearly* interest be 1000 per cent. per annum. Taking the first supposition, we have a part of a table of logarithms as follows:—

Am.	Yrs.	Am.	Yrs.	Am	Yrs.	Am.	Yrs.
1000	300043	5232	371867	9997	399987	10001	400004
1001	300087	5233	371875	9998	399991	10002	400008
1002	300130	5234	371883	9999	399996	10003	400013
1003	300173	5235	371892	10000	400000	10004	400017
&c.	&c.	&c.	&c.				

This is the light in which a common reader may view a table of logarithms. Let 1 increase to 10 at compound interest in 100,000 equal moments, then 1 will become 5234 in 371,883 such moments; and so on. We can thus manage to put down every number, within certain limits, as an amount; and thus, within those limits, we reduce all questions of multiplication and division to addition and subtraction, by reference to the tables.

We thus perceive a simple principle applied with much labour, but such as is performed once for all. The notion above elucidated was the first on which logarithms were constructed; in time came more easy methods. We now take another abbreviation which is perpetually occurring in our subject. It is the multiplication of all the successive numbers from 1 up to some high number; that is, the continuation of the process following. Let [10], for instance, represent the product of all the numbers, from 1 up to 10, both inclusive, or let

[10] stand for $1 \times 2 \times 3 \times 4 \times 5 \times 6 \times 7 \times 8 \times 9 \times 10 = 3628800$

[1] is 1; [2] is 2; [3] is 6; [4] is 24; [5] is 120; [6] is 720; [7] is 5040; [8] is 40,320; and so on. This labour becomes absolutely unbearable when the numbers become larger; thus, [30] contains 33 places of figures, and [1000] contains 2568 figures. But, nevertheless, we cannot deal with problems in which there are 1000 possible cases without knowing, *either nearly or exactly*, the value of [1000]. It will, however, be sufficient to know this value very nearly; within, say, a thousandth part of the whole; that is, as nearly as when, the answer of a problem being 1000, we find something between 999 and 1001. We now put before the reader who can use logarithms a rule for this approximation, with an example; intending thereby to show the reader who does not comprehend the process how mathematics enter this subject in the abbreviation of tedious computations.

Rule.—To find very nearly the value of [a given number], from the logarithm of that number, subtract ·4342945, and multiply the difference by the given number, for a first step. Again, to the logarithm of the given number add ·7981799, and take half the sum, for a second step. Add together the results of the first and second steps, and the sum is nearly the logarithm of the product of all numbers up to the given number inclusive. For still greater exactness, add to the final result its aliquot part, whose divisor is 12 times the given number.

Example.—What is [30] or $1 \times 2 \times 3 \times \ldots\ldots \times 29 \times 30$?

log. 30	1·4771213		1·4771213	
	·4342945		·7981799	
Subtract	1·0428268		2)2·2753012	Add.
	30			
			1·1376506	Second step.
Multiply	31·284804	First step.		
	1·137651	Result of second step.		
	32·422455	log. of result.		

The result has, therefore, 33 places of figures, of which the first six are (nearly) 264,518; or, if this be increased by its 360th (12 times 30) part, or about 735, the result is 265,253, followed by 27 ciphers; or

the approximate result is —

265253000000000000000000000000000

The true result is —

265252859812191058636308480000000

and the error is not so much of the whole, as one part out of 500,000.

In this way, we are able to do with more than sufficient nearness, and in a few minutes, what it would take days to arrive at by the common method, and with much greater risk of error.

If we wish to find the product of all the numbers, say from 31 to 100, both inclusive, we find [100] and [30] approximately, and divide the first by the second. We shall represent this by [31,100]: thus,

[7, 15] stands for $7 \times 8 \times 9 \times 10 \times 11 \times 12 \times 13 \times 14 \times 15$

But, though we can thus simply put the logarithmic computer in possession of a great acquisition of power we can get through much the greater part of our task without such a process, by means of a table of which it is not possible to explain either the principle, or the reason of its utility, to any but a mathematician. We can only explain its mere construction, as follows: —

Let A B be *one* (inch, for example); and take an indefinitely extended line A X, perpendicular to A B: from A towards X let a curve be conceived to be described, so that every *ordinate* N P shall be connected with its *abscissa* A N, by the following law. Measure A N in inches and parts of inches; and multiply the result by itself; and the product by ·4342945. Find the number to which this product is the common logarithm, and divide 1·1284 by the result. The quotient is the fraction of an inch in N P, and in the table we find, not N P, but the area A N P B expressed as a fraction of a square inch. The curve itself is what is called an *asymptote* to A X, continually approaching, but never reaching, A X: and the whole area, A X being continued for ever, is one square inch. To this table I shall have continual occasion to refer: into it, in fact, is condensed almost the whole use I shall have to make of the higher mathematics.

I have thus drawn the distinction between the principles of the subject, as derived from very obvious results of self-knowledge, and the principles of mathematics, applied merely to the abbreviation of the tedious operations which large numbers require. I now proceed to the several assertions which have been made upon the nature and tendency of the subject.

I. That it is not true. The whole weight of this assertion, and of all arguments in its favour, falls entirely upon the method of measurement in page 11., and ultimately upon the second axiom, in page 9. Again, as we are most unquestionably justified in saying that it is more probable we shall draw one of the two, a or b, than that we shall draw a, the argument must be directed against the method of measurement, not against the possibility of a measure: for wherever *more* or *less* are applicable terms, twice, thrice, &c. must be also conceived to be possible, whether we can ascertain how to find them or not. But no other method of measurement has ever been proposed, nor, in truth, have the assertors been aware that they could be brought to such close quarters, but have generally objected to the theory as a whole, without any particular knowledge of its parts. It will be time enough to refute their notion, when they begin to be so particular that refutation becomes possible.

II. That it is not practical. By this it is either meant, (for practical is one of the words employed in shifting an argument, which are sometimes so convenient), that it has not been reduced to practical form, or else that it is not capable of being so reduced; or perhaps that it is not useful. The working results hitherto obtained may be divided into:—1. The method of obtaining probabilities.—2. The method of estimating the probability of more or less departure from the results indicated by the main branch of the theory as most probable. The first has been frequently made practical; the second not hitherto, except to mathematicians. That the whole can be made practical, I hope to establish by the contents of this work. To the assertion that it is not useful, we oppose:—1. The unanimous opinion of astronomers, (meaning thereby persons capable of applying the subject to astronomy) that the exactness of our present knowledge is very much owing to the application of it, and their uniformly continuing to use it in the deduction of results from the necessary discordances of observations.—2. The extent to which it has been applied in the very choicest view of the word practical, (which frequently means money-making) in concerns which now employ many millions sterling.—3. The light in which it is regarded by a very large majority of those who have studied it, as a corrector of false impressions, and indicator of just and necessary, though not always perceptible, distinctions.—4. The beauty of the study itself, considered merely as a speculation, and as a method of exercising certain powers of mind, which might otherwise lie useless.—5. The necessity of informing the public as to the real nature of the occupation called gambling, and of the class of men who live by it; the latter being persons who are using knowledge of these principles successfully, to the daily loss and ruin of those who are not aware of what constitutes unequal play. If such arguments be not sufficient to counterbalance a simple assertion, to the extent of making it worth while to decide the question by an examination of the subject itself, we may safely dispute the utility of any branch of knowledge.

III. That it has a tendency to promote gambling. Those who make this objection generally use the common signification of the term gambling; and the motives for this pursuit are, in their view, either the pleasure of suspense, acting as a stimulus to a mind weary of its own vacancy, or the desire of gain. On the first notion, the assertion is self-destructive; it amounts to saying that knowledge which diminishes suspense, by giving a better view of the circumstances, has a tendency to promote gambling, by affording the pleasure arising from suspense. So far as the theory of probabilities bears upon gaming in general, its tendency is to convert games of chance into something more resembling games of skill. Now games of skill are seldom made the vehicles of very high play. So far, then, the tendency of our study is to substitute the satisfaction of mental exercise for the pernicious enjoyment of an immoral stimulus. With regard to the desire of gain, we may safely admit that those who are already actuated by this motive in an undue degree, will sometimes be led to gamble by knowing how to do so properly; and just in the same manner some of them will be led to make forgery the means of increasing their store, from knowing how to write. But the fear that those who seek a livelihood by what is commonly called gambling, which always means cards, dice, or horse-racing, &c., would be much increased in number, if at all, by such a pursuit as the mathematical appreciation of probabilities, seems to me grounded upon a want of knowledge of human nature. Putting out of view the tendency of all serious thought to lead the mind to a perception of its own resources, and to furnish methods of employing time; and not even considering that the demand for this baneful excitement is controlled by the opinion of society, and lessened by the amount of education: there still remain the means of showing that the balance is in favour of a study of the theory of probabilities, even as a preventive of this very gambling which it is said to provoke. *Nemo repenté fuit turpissimus:* and, it will be one of our objects to show, that the person who lives by gaming, deserves the strongest form of the adjective. No one ever said to himself, I have not played hitherto, but I will begin henceforward to make it my trade. A young man who is ruined by play in the first instance, or who, at least. has begun by courting as an amusement what he ends by requiring as an occupation, is the subject of which a gambler is made. Now, suppose that all those who have been ruined by play had been trained to understand the true nature of their pursuit. Let it be granted that some of them are so fond of acquisition, that it is only necessary to point out a plausible method to insure their following it: yet we must grant, on the other side, that there will be some who can be persuaded that when they play against a bank or a gamester, they are almost certain of playing on very unequal terms, which is never what they contemplate and intend. The only question is, which of the two numbers will be the greatest; 1. Of those who become gamesters prepense, or, 2. Of those who either take a total or a partial warning; the latter in a degree sufficient to insure a fair chance for themselves. The thoughtlessness of youth will be urged against my opinion, that the latter number would be very much the greatest. I reply, that, comparatively speaking, and with respect to maturer years, young men are thoughtless; but, absolutely speaking, they are not so with respect to dangers of which they know the risks. The ill success of others does not deter them, because they attribute it to fortune; and, because they have superstitions hanging about them with respect to luck which are tolerably prevalent in all classes. They think they are *trying their luck,* as the phrase is; but if they could be convinced that it is not their *luck* which they are trying, but only a fraction of it, their opponent having the rest in his pocket, they would show themselves in this, as in other matters, averse to risks in which it is more than an even chance against them. They come to the consideration of the subject fraught with wrong notions, which have been carefully instilled as preventives. The character of a gambler is represented as dishonest, in the common sense of the word. That is to say, the term gambler is confounded with that of sharper, meaning a person who would mark a card, or load a die. They find the falsehood of this notion in their commerce with the world: gamblers show themselves in the face of day who really appear to be, and are, men of honour in the common sense of the word, and who would scorn any under-handed proceeding, under ordinary temptation at least. What then becomes of the previous warning? It is proved to be false in an essential part; and is therefore lost altogether. Add to this that the principle of the occupation is misrepresented: admonition is given against trying fortune, instead of proof that fortune is not tried. A proposition is advanced which is an absurdity: equal play is supposed, and yet it is maintained that the luck will generally be against the inexperienced. Skill is considered as only adding to the chances against the unskilful, instead of creating a certainty, and arguments drawn from a single game, which are really good, are applied to collections of many games, with regard to which they are not applicable. I will leave it to any one to say, whether the considerations pointed out in the succeeding pages have the tendency to promote the pursuit of fair gaming as a means of profit.

With regard to gambling as a stimulus, it must be observed, that the passion has every where subsided with the increase of education and occupation. If the historians who write for schoolboys could spare a little space among their interminable accounts of kings, treaties, battles, to insert some account of the manners of the several ages of Europe, it would be matter of surprise that the universal rage for games of chance, should have left any time for the (so called) great actions which fill the books. The wars of the middle ages would be looked upon as belonging only to one particular class of the stimuli by which the universal vacancy was sought to be filled up. From the old Germans, who played away, to one another, their wives, their children, and lastly themselves, down to the time * of the French revolution, the continent of Europe (and during a part of the time, Great Britain, though in a less degree,) gives, comparatively to ourselves, the notion of successive races devoted to gambling throughout the upper class, the only one upon whose occupations we get frequent details.

IV. That the basis of it is an irreligious principle. There is a word in our language with which I shall not confuse this subject, both on account of the dishonourable use which is frequently made of it, as an imputation thrown by one sect upon another, and of the variety of significations attached to it. I shall use the word *anti-deism* to signify the opinion that there does *not* exist a Creator, who made and sustains the universe. The charge is, that a theory of probabilities (called chances) is necessarily anti-deistical, because it refers

* Quand, avant la révolution française, les états de certaines provinces étaient assemblés, on y jouait un jeu terrible, et tel que l'endroit où il se tenait, dans la ci-devant province de Brétagne, s'appellait l'*enfer*. — *Dict. des Jeux* (*Encyc. Meth.*) 1792.

events to chance. Various modifications of this assertion present themselves, but they may all be referred either to that just made, or to a *tendency* argument of the same character. All the sciences have had to encounter this aspersion, each in its turn; but it is to be remarked, that philosophy and philosophers have always been charged with the worst thing going. The believers in sorcery never failed to attribute an intimate connection with infernal spirits to all who investigated nature in any form: the believers in anti-deism follow in their steps. There is in the proposition above mentioned, a shifting of the meaning of terms: it has been customary to designate anti-deism as the opinion that the world was made by chance, meaning, without any law or purpose *existing*; but the word chance *, in the acceptation of probability, refers to events of which the law or purpose is not *visible*. Thus a great part of the application of this subject has been destroyed by successive discoveries. When the observatory at Greenwich was founded, the chance errors of observation were large in the fixed stars. Nothing could be said but that there was a deviation which appeared of one sort in one observation, and of another in another, without visible law or order. Bradley's discoveries removed much of this, that is, pointed out law where law was not *seen* to exist before. Improvement of instruments and methods of observation has still more distinguished the error into parts with a visible, and parts with an invisible, cause. As an answer to the species of argument employed, nothing more is necessary: those who can, may consider this science as not bearing on religion, either in one way or the other, so far as anything in the preceding argument is concerned, or in the explanation which is no more than necessary for an answer. But there is a view of the subject, and that one most indispensable, which

* Generally speaking, the abstract singular term *chance* has the anti-deistic meaning, while the plural *chances* is used for the several possibilities of an event happening. Thus Hume says: — "Though there be no such thing as chance in the world ..... there is certainly a probability which arises from a superiority of chances."

better deserves to be made a fundamental principle, than an incidental answer to a futile objection. The past contains our grounds of expectation for the future: Why? Because we cannot help supposing that there were causes which produced the past, and which continue to act. If there be any one to whom this is not a truth, he cannot proceed with us one step. Suppose that 100 drawings out of an urn all give white balls, the presumption is very strong that the 101st will give a white ball also. But if there neither be, nor ever were, some reason why the balls so drawn should be white rather than black, that is, if the event be pure *chance*, then the 100 drawings afford no presumption whatever that the 101st will be either white or black. So far then as we have yet gone we have the following positive and negative conclusion:

The theory of probabilities absolutely requires, in its fundamental principles, the rejection of the notion that pure chance can produce any two events alike; that is, it presumes causation and order of some kind or other, that is, *providence* of some kind or other.

The theory of probabilities, so far as considerations of absolute necessity are concerned, neither denies nor asserts, in whole or in part, any thing whatsoever respecting the moral or intellectual character of the providence which it requires to be granted.

From the preceding we may be certain that no conclusion in any way leading to natural religion, however faint, is tacitly assumed in the premises. If therefore such a conclusion should follow legitimately, it stands upon a basis of absolute security. This is not often the case in arguments drawn from nature in general, on account of the mixture of considerations with which the mind is affected by them. When we speak of the vastness, the regularity, and the permanency of the solar system in general, the very immensity of the argument would prevent the mind from being aware whether there was or was not either an appeal to constitutional feeling, distinct from reason, or even an assumption of the question in the manner of deducing it. The celebrated work of Paley may be considered as a treatment of the following syllogism : — " If there be contrivance, there was design ; but there is contrivance, therefore there was design," the minor of which is proved by appeal to observation. But the author refers the opponent to the beauty and ingenuity of the methods in which the contrivance is brought about : to the general effect on our notions of what *is* done, compared with what *we* could do. It may very often be discovered what the real tenor of an argument is, by observing what would refute it : now imagine an individual possessed with the notion that he could execute * better contrivances, and Paley's argument must (to him) be imagined to be ineffective.† It appears to me that the result of the treatise in question is this: " If there be a contriver, he must be one of infinite power and intellect." But the argument of contrivance against chance cannot, from the complication and non-numerical character of the instances, be illustrated by any reference to what might have been if chance had prevailed. Taking, for example, the chamois, as the result of a contrivance for the support of animal life on frozen mountains, we have no method of comparing the chamois of design with any notion that we can form, and call the chamois of chance. But where a consideration is pure number, we then have other ideas, of the homogeneity of which with that in question, we feel assured : and we can absolutely try the question with chance in precisely the same manner as we try it in the common affairs of life. Let us assume, as we must, that a number produced by chance alone (in the antideistical sense of the word,) might as well have been any other as what it is. And further, let us require before we grant intelligence and contrivance, not merely the presence of an adaptation which would have been unlikely from chance alone, but two such phenomena,

* Such as is said actually to have struck Alfonso of Portugal, when the Ptolemaic theory of the heavens was explained to him.

† The fault of most treatises on Natural Theology is to draw the reader's attention from the mere design, to the complication and ingenuity of the design. The Bridgwater Treatises have a consistent title, and it is worthy of remark, that this was the doing of the testator himself.

perfectly distinct from each other considered as phenomena, each of which might have existed without the other, and both tending to the same object, which would have been defeated by the absence of either. Let it also be granted, to fix our ideas, that we admit as proved, a proposition which has a hundred million to one in its favour.

This being premised, and laying it down as our object to show that the necessary results of the theory of probabilities lead to the conclusion that the existence of contrivance is made at least as certain, *by means of it*, as any other result which can come from it, we proceed to state a consequence: — The action of the planets upon each other, and that of the sun upon all (the most certain law of the universe), would not produce a permanent* system unless certain other conditions were fulfilled which do not necessarily follow from the law of attraction. The latter might have existed without the former, or the former without the latter, for any thing that we know to the contrary.† Two of these conditions are, that the orbital motions must all be in the same direction, and also, that the inclinations of the planes of these orbits must not be considerable. Granting a planetary system which is what ours is, in every respect except either of these two, and it is mathematically shown that such a system must go to ruin: its planets could not preserve their distances from the sun. Neither of these phenomena can be shown to depend necessarily on the other, or on any law which regulates the system in general. For any thing we know to the contrary, then, they are distinct and independent circumstances of the organisation of the whole. Now let us see what are the phenomena in question: —

* Permanent, not liable so to change as to destroy the organisation of the parts. If the earth could ever approach so near to the sun that all the water should be vaporised, the permanency of the system would be destroyed, so far as our planet is concerned

† The only way in which we can guess any two things to be independent. It must be remembered, as a result of the theory, that, of things perfectly unknown, the probability of their coming to act, when known, against an argument, is counterbalanced by the equal probability of the future discovery being on the other side.

I. All the eleven planets yet discovered move in one direction round the sun.

II. Taking one of them (the earth) as a standard, the sum of all the angles made by the planes of the orbits of the remaining ten with the plane of the earth's orbit, is less than a right angle, whereas it might by possibility have been ten right angles.

Now it will hereafter be shewn that causes are likely or unlikely, just in the same proportion that it is likely or unlikely that observed events should follow from them. The most probable cause is that from which the observed event could most easily have arisen. Taking it then as certain that the preceding phenomena would have followed from design, if such had existed, seeing that they are absolutely necessary, *ceteris manentibus*, to the maintenance of a system which that design, if it exist, actually has organised, we proceed to inquire what prospect there would have been of such a concurrence of circumstances, if a state of pure chance had been the only antecedent. With regard to the sameness of the directions of motion, either of which might have been from west to east, or from east to west, the case is precisely similar to the following: — There is a lottery containing black and white balls, from each drawing of which it is as likely a black ball shall arise as a white one; what is the chance of drawing eleven balls, all white. Answer, 2047 to 1 against it. With regard to the other question, our position is this. — There is a lottery containing an infinite number of counters marked with all possible different angles less than a right angle, in such manner that any angle is as likely to be drawn as another; so that in 10 drawings the sum of the angles drawn may be any thing under 10 right angles. What is the chance of 10 drawings giving collectively less than 1 right angle? Answer, 10,000,000 to 1 against it. Now what is the chance of both of these events coming together? Answer, more than 20,000,000,000 to 1 against it. It is consequently of the same degree of probability that there has been something at work which is not chance, in the formation of the solar system. And the preceding does not involve a line of argument addressed to our perceptions of beauty or utility, but one which is applied every day, numerically or not, to the common business of life.

Now whether what precedes amounts to the means of producing rational conviction, it is not necessary for me to stop and inquire. The question is, how do the results of this theory affect those moral and intellectual considerations which it has been stated to have a tendency to overthrow? It matters nothing to my present purpose how much of the preceding a reader will admit; for the point, considered with reference to the objection before us, is this: — Does the preceding deduction *weaken* the probability of the existence of an intelligent creative power? for if not, the objection is overthrown, and whatever strength is conceded to belong to the reply is so much addition to other arguments in favour of the same conclusion. Let us suppose a reader so much biassed against the higher parts of mathematics that he does not feel any confidence in the united work of all ages and countries amounting to more than a millionth of certainty. There still remains 20,000 to 1 in favour of the conclusion above stated, after weakening the preceding by introducing a probability that all the exact sciences may be wrong, such as his state of mind requires. With respect to the bearings of the theory, we may now add the following to the statement in page 24.

Applying the first principles of the theory of probabilities, by means of mathematics, to the phenomena of the universe, it is a necessary conclusion that the existence of something which combines together different and independent arrangements to produce an end which could not, *ceteris manentibus*, be produced without them, must be added to the notion of a Providence, intelligent or not, which is required in the first principles.

With regard to the existence of a revelation from the

Supreme Being, this theory leaves the question exactly where it found it; and the same of all questions of historical evidence. If we were to assume fictitious data, we might, as in all other sciences of inference, produce a consequence which should be as true as the premises, standing or falling with them. The science itself is the deduction of the probability in a complicated case from the probability in a known and simple case. But where is the known and simple case in the historical question? In valuing testimony, no theory of the method in which conflicting evidence should be combined will help us to the original value of the several parts of it, any more than an investigation of the method of solving an equation will help us to a knowledge of the particular equations which apply in any given case.

The two great theoretical questions before us are:— I. What is the measure of probability? II. What is the way of using it? — The necessary preliminary to application is, the result of the measurement in a case to which the method of measuring can be applied, and has been applied. The mistakes which have arisen from confounding these considerations are numerous. For instance, tell me how many times per cent. a given man will be wrong in his judgment, and I can tell you exactly, positively, and mathematically, how much more likely a unanimous jury (not starved) is to have arrived at a true decision, than another in which the voices are 8 to 4. But that does not put me one step nearer to ascertaining what *is* the per centage of erroneous conclusions in the judgments of a single individual. The misconceptions just alluded to are equally prevalent with regard to all the sciences; a person who studies astronomy is frequently asked what the moon is made of.

Much of the objection, religious or not, made against probability in general, is connected with the notion already mentioned, (page 7.) that it is a fundamental quality of events, external to ourselves, which is under consideration: on which the rational feeling must be, that there is no such thing. The term probability is as difficult to explain as gravitation: and the method of proceeding is the same with regard to the properties of both. We cannot tell what they are, in simpler terms, but we know them by, or rather trace and define them by, their manifestations. In both, we first see a compound result, depending upon the patient as well as the agent. In the case of the mental phenomena, we cannot decompose the effect produced, still less ascend a step, and find any of the laws which regulate the human disposition to doubt or expect. I shall conclude by again reminding the reader, that the impression produced by circumstances upon his own mind is the thing in question; and that nothing can be more liable to cause confusion than a lurking notion that the results of theory are anything more, before the event arrives, than a representation of the relative force of his own impressions, as they should be if unassisted reason could follow the legitimate consequences of some simple and universally admitted principles.

I proceed in the next chapter to develope the leading rules of the science.

CHAP. II.

ON DIRECT PROBABILITIES.

We now proceed on the supposition that the probability of an event is measured by the fraction which the number of favourable cases is of all that can happen. Thus, if there be 20 white balls and 27 black ($20+27$ or 47 in all), the probability of drawing a white ball is measured by $\frac{20}{47}$, and that of drawing a black ball by $\frac{27}{47}$. We shall say that these probabilities *are* $\frac{20}{47}$ and $\frac{27}{47}$. Nothing is more common than to substitute a measure for the thing itself: thus we talk of a *temperature* of 60°, when, in fact, each 1° only means a certain *length* on the tube of the thermometer. It will illustrate this to take a case in which we do *not* confound the thing and its measure; in the barometer, for instance, we never say that the air's *weight* is 30 *inches*, but that the height of the barometer is 30 inches.

The technical words, probability and improbability, must now be considered as meaning the same thing in different degrees. If there be only one white ball out of a thousand, we usually say that to draw the white ball is possible, but not probable. We now speak of it as having a small probability, namely $\frac{1}{1000}$; we might say it has a great improbability, namely $\frac{999}{1000}$, but this phrase is not customary. The moment we obtain either numerical measures, or distinctions which are not verbal, the distinctions which *are* verbal frequently become superfluous and inconvenient. Thus in the art of book-keeping, profit and loss never appear as separate words, but only as part of a complex term *profit-and-loss*, meaning, one or the other, according to the side of the account on which the item is found. To a mercantile reader, we should say that probability means probability-and-improbability the first or second, according as its measure is greater or less than $\frac{1}{2}$. When the number of favourable and unfavourable cases is the same, say 50 of each, the probability of the event is $\frac{50}{100}$ or $\frac{1}{2}$; and in this case we say in common life that there is a balance of probabilities, or that the event has an even chance.

By the word *chance* with the article (*a* chance) we mean one single way in which an event may happen, as when we say that every white ball adds a chance to the prospect of drawing a white ball. In the first instance above, the chances of white and black are as 20 to 27. It is also usual to say that the odds are here 27 to 20 in favour of black against white.

Questions on probability are twofold in character: 1. Where we know the previous circumstances and require the probability of an event. 2. Where we know the event which has happened, and require the probability which results therefrom to any particular set of circumstances under which it might have happened. The first I call direct, and the second inverse, questions.

We must begin with direct questions, though the inverse precede the direct in practice. For, as we know the whole range of possible cases in hardly one instance, we cannot proceed with points which have reference to matters of life until we know what presumptions arise with respect to the whole, from observation of a part.

In direct questions of probabilities, the event may either be simple, that is, depending on one indivisible event; or compound, consisting of several events which may happen together. Thus, suppose four events, either of which may happen, and call them A, B, C, D. Knowing the circumstances of each, I may ask the following questions, every one of which states an event simple or compound.

1. What is the chance of A. 2. What is the chance that one shall happen and only one. 3. What is the chance that one or more will happen. 4. What is the chance that one at least will happen, and one at least will not. 5. What is the chance that a given pair, and no others, will arrive. 6. What is the chance that a *given* pair at least will arrive. 7. What is the chance that some pair or other will arrive, but only a pair. 8. What is the chance that a pair at least will happen, &c. &c. All these are most evidently distinct questions when they are clearly proposed; but it is almost as evident that they are very liable to be confounded.

The mathematical definitions and theorems which will be necessary for our purpose, are the following: —

1. A *permutation* means a number of cases selected out of all possible cases, in some particular order; so that different arrangements of the same things make different permutations. Thus if all the possible cases be A, B, C, and D, we have the following

Permutations of one out of four, A, B, C, and D.

Permutations of two out of four, AB, BA, AC, CA, AD, DA, BC, CB, BD, DB, CD, DC.

Permutations of three out of four, ABC, ABD, BAC, BAD, ACB, ACD, CAB, CAD, ADB, ADC, DAB, DAC, BCA, BCD, CBA, CBD, BDA, BDC, DBA, DBC, CDA, CDB, DCA, DCB.

Permutations of four out of four (different arrangements of four), ABCD, ABDC, BACD, BADC, ACBD, ACDB, CABD, CADB, ADBC, ADCB, DABC, DACB, BCAD, BCDA, CBAD, CBDA, BDAC, BDCA, DBAC, DBCA, CDAB, CDBA, DCAB, DCBA.

To find the number of permutations of one number (m) out of another (n), begin with the whole number (n) and write down as many numbers, reckoning downwards as there are units in the number (m) which are to be in each permutation: then multiply all together. For instance: How many permutations are there of 4 out of 12. The answer is.

$$12 \times 11 \times 10 \times 9 \quad \text{or} \quad 11880$$

The following table will furnish examples or serve for reference.

	10	9	8	7	6	5	4	3	2	1
1	10	9	8	7	6	5	4	3	2	1
2	90	72	56	42	30	20	12	6	2	
3	720	504	336	210	120	60	24	6		
4	5040	3024	1680	840	360	120	24			
5	30240	15120	6720	2520	720	120				
6	151200	60480	20160	5040	720					
7	604800	181440	40320	5040						
8	1814400	362880	40320							
9	3628800	362880								
10	3628800									

Thus the number of permutations of 6 out of 9 is opposite to 6 under 9, or 60480.

2. By a *combination* is meant the same thing as a permutation, except only that arrangement is no part of the idea. Thus, of all the permutations of three out of four in A, B, C, D, the following, A B C, B A C, A C B, C A B, B C A, and C B A, are all the same *combination*. Thus we have

a *permutation* : a selection in a certain order.
a *combination* : a selection without reference to order.

To find the number of combinations, divide the number of permutations by the product of all the numbers up to the number in each combination. Thus, how many combinations can be made of 4 out of 12? The answer is found by

$$\left.\begin{array}{ll}\text{dividing} & 12\times 11\times 10\times 9 \\ \text{by} & 1\times 2\times 3\times 4\end{array}\right\}\text{which gives 495.}$$

The process may always be shortened by striking common factors from the dividend and divisor. Thus, if we wish to know the number of different hands which can be held at whist (combinations of 13 out of 52) we must

divide 52. 51. 50. 49. 48. 47. 46. 45. 44. 43. 42. 41. 40.
by 1. 2. 3. 4. 5. 6. 7. 8. 9. 10. 11. 12. 13.

The shortest way is to decompose every number in both into its factors, when it will appear that all the factors of the divisor are found among those of the dividend; as follows:—

13. 2.2. 17.3. 5.5.2.7.7. 2.2.3.2.2. 47. 2.23. 5.3.3. 11.2.2. 43. 2.7.3. 41. 2.2.2.5.
1. 2. 3. 2.2. 5. 3.2. 7. 2.2.2. 3.3. 5.2. 11. 2.2.3. 13.;

and the factors which must be multiplied to give the result are

17. 7. 47. 23. 5. 43. 2.7. 41. 2.2.2.5.
or 17. 47. 23. 43. 41. 35. 560 giving 635013559600.

When the number in each combination is more than half of the whole number, the rule may be shortened.

Thus, if I ask how many combinations of 21 can be taken out of 25, I do in effect ask how many combinations of 4 may be taken. For there are just as many ways of taking 21 as there are of leaving 4.

The following table corresponds to the one preceding: —

	10	9	8	7	6	5	4	3	2	1
1	10	9	8	7	6	5	4	3	2	1
2	45	36	28	21	15	10	6	3	1	
3	120	84	56	35	20	10	4	1		
4	210	126	70	35	15	5	1			
5	252	126	56	21	6	1				
6	210	84	28	7	1					
7	120	36	8	1						
8	45	9	1							
9	10	1								
10	1									

Thus the combinations of 5 out of 9 are 126 in number.

3. When the same event may be repeated in a permutation, the number of permutations is the product of as many numbers, all equal to the whole number of possible events, as there are of events in each permutation. Thus, of four events, A, B, C, D, the number of permutations of two, *with repetition*, is 4 × 4 or 16; without repetition 4 × 3 or 12. The additional 4 in the first case, are A A, B B, C C, and D D. The number of permutations of three, with repetition, is 4 × 4 × 4, or 64; without repetition, 4 × 3 × 2 or 24. Of the additional 40, 10 are where A only is repeated; namely, A A B, A A C, A A D, A B A, A C A, A D A, B A A, C A A,

D A A, that is *nine* where A is repeated twice, and A A A, *one*, where A is repeated three times. Ten more are those in which B is repeated twice, &c.

4. The number of combinations in which there are repetitions must be determined without rule, in every particular case. Suppose I wish to know how many combinations, including repetitions, can be made of four out of seven. Let the seven be A, B, C, D, E, F, G.

I. The number, without repetition, is 35.

II. Those in which A is repeated twice, and no other, are evidently as many as the number of pairs in B, C, D, E, F, and G, that is, 15. Thus we have A A B C, A A C D, &c. There are as many in which B is repeated twice, &c., so that 7×15 or 105, is the number in which one only is repeated twice.

III. The number in which A is repeated three times is evidently 6, and the number in which one or other is repeated three times only, is 6×7 or 42.

IV. The number in which one is repeated four times, is 7.

V. The number in which two are repeated twice, is as many as the number of pairs in 7, or 21.

Consequently, the whole number is made up of 35 105, 42, 7, and 21, or it is 210.

We have not so much to do with combinations allowing repetition, as with permutations of the same kind.

To avoid the perpetual occurrence of long phrases for simple ideas, I shall use the following abbreviations: — By P $\{4,20\}$ is meant the number of permutations of 4 out of 20, without repetition: by P P $\{4,20\}$ the same with repetition. By C $\{4,20\}$ is meant the number of combinations of 4 out of 20, without repetition: by C C $\{4,20\}$ the same with repetition. Again, by [4,20], as in page 15., is meant the product of all numbers from 4 to 20, *both inclusive;* and, by 7^{10}, as in algebra, is meant *ten sevens* multiplied together. Thus, a reference to the preceding rules will show the following: —

$$P\left\{5,12\right\} \text{ is } [12, 8] \quad , \quad PP\left\{5,12\right\} \text{ is } 12^5$$

$$C\left\{4,25\right\} \text{ is } [25, 22] \text{ divided by } [1, 4]$$

5. If there be an event composed of several others in succession, of which the first may happen in, say 10 different ways, the second in 14, and the third (let there be but three) in 6, then the compound event must be one out of 10 × 14 × 6, which is the number of different ways in which it may happen. When all the component events may happen in the same number of ways, this reduces itself, both in principle and rule, to the case of permutation with repetition.

I now proceed to some problems requiring nothing but a knowledge of the measure of probability and the preceding rules. Any event of known circumstances may be familiarly represented by a lottery of balls of different colours. Thus, suppose ten Russian ships, twelve French, and fourteen English, are expected in port, or may have arrived. Let one be as likely to arrive as another, and suppose it known that two have arrived, but not of which nation they are. Let a certain advantage accrue to A, if they should happen to be Russian and French, of which he is desirous of selling his chance immediately. The question is precisely the following : — Let there be a lottery of green, white, and red balls, 10, 12, and 14 in number. Let two have been drawn, and suppose a certain advantage to arise to A, if they be green and white. What should be given to A, certain, in consideration of his chance?

Here we must first consider the number of ways in which two can be drawn. All the balls are 36 in number, and the whole number of pairs is $36 \times 35 \div 2$ or 630. Of these a green ball (1 out of 10) may be paired with a white ball (1 out of 12) in 120 different ways. Consequently, the probability of his gaining the advantage is $\frac{120}{630}$ or $\frac{4}{21}$. The probability against it is therefore $\frac{17}{21}$: or out of every 21 possible events, 4 are in favour of, and 17 against, the advantage. It is there-

17 to 4, or $4\frac{1}{4}$ to 1, against the advantage. If the contingent gain were 21*l.* the value of the chance would be 4*l.*, as will hereafter be more fully shown.

Of all the pairs which can be drawn, there are

10 x 12 = 120 green and white.	10 x 9 ÷ 2 = 45 both green.
12 x 14 = 168 white and red.	12 x 11 ÷ 2 = 66 both white.
14 x 10 = 140 red and green.	14 x 13 ÷ 2 = 91 both red.

The sum of these is 630, as it should be. The reader should explain to himself the reason of the difference of process. The least probable case is both green, the most probable white and red. Nothing is more common than the idea that the event most likely to happen, which is compounded of two or more events, is a repetition of the event which is individually the most probable. This is true *of repetitions:* red being most probable, it is more probable that red should be repeated than that white should be repeated, or that green should be repeated; but that two white ones should be drawn is not so probable as that a white and red should be drawn. The drawing, whatever it may be, is considered independently of succession—and this makes an important difference. If the balls were drawn successively, both red is more probable than white followed by red, or than red followed by white, but not more probable than the chance of one or other, which is the preceding case.

I here also take occasion to notice the common error, that because an event is more probable than any other, it is the one to be looked for. The question ought to be, Is that event more probable than some one or other out of all the other events which may happen? If ten persons engage in a competition, with an equal chance of success, and if two of them, A and B, enter into partnership, it is now more probable that the firm of A and B will win than that C will win, or that D will win, &c. But the chances against the firm are still 8 to 2 or 4 to 1. If a hundred halfpence be tossed up into the air, the result which is more probable than any other, is 50 heads and 50 tails. But common sense will tell us that the chances of this result are very small, *out of the whole,* and calculation would confirm it.

Before laying down any more specific rules, I shall take an instance of a somewhat complicated deduction, in order to show that we do not really require any other principle than is contained in the *measure of probability.* Suppose A and B to play at whist against C and D. A begins, holding ace, king, and queen of trumps, and these trumps only: what right has he to expect that, by playing them out in succession, he will force all the trumps of the other party? That is to say, ten trumps being distributed among B, C, and D, and any single card having the same chance of belonging to either of the three, what is the probability that neither C nor D holds more than three trumps?

Firstly, as to the number of tenures, which are perfectly distinct. To find the number of ways in which any number of distinct objects can be divided among any number of persons, use the following RULE: —

Multiply together numbers equal to the number of persons as often as there are things to be divided among them. Thus, to find in how many different ways ten distinct cards can be divided among three persons, find

$3 \times 3 \times 3$, &c. (ten threes) or 3^{10} which is 59049.

The question now is, how many of these 59049 ways favour the supposition that neither C nor D holds more than three out of ten. Both together they cannot hold more than six: if, then, we pick out *any given six* of the ten trumps, *that set* may be divided among C and D in 2^6 or 64 ways. But of these, there are two ways in which 0 and 6 may be held, twelve ways for 1 and 5, and thirty ways for 2 and 4, none of which must be included. It must be observed, that in the last sentence, the six distinct ways into which 6 may be divided into parcels of 1 and 5 must be doubled, because * each gives two of our cases, ac-

* It is important to observe that no duplication must take place on this account, if the two numbers be the same; for instance, in dividing 6 into parcels of 3.

cording as C holds 1 and D holds 5, or C holds 5 and D holds 1 : and so of the other cases. Consequently, out of a given set of six, there are 64 all but 44, that is, 20 ways, in which, when they happen, A could force all the adversary's trumps. But *every set of six* yields 20 such cases: hence the ways in which C and D can together hold six trumps, are 20 times $C\{6,10\}$ or 20×210, or 4200. Similarly, a given set of 5 trumps may be held by C and D in 2^5 or 32 ways, of which 2 and 10 must be excluded. Hence $20 \times C\{5,10\}$ or 20×252, or 5040 sets are the number favourable to the event in this case. Four trumps can be held in 2^4 or 16 ways, of which 2 must be excluded, or $14 \times C\{4,10\}$ that is, 2940, is the number of favourable cases. On the supposition that C and D together hold only 3, or 2, or 1 trump, no exclusions are necessary, and the number of cases are $2^3 \times C\{3,10\}$ or 960, and $2^2 \times C\{2,10\}$ or 180, and $2 \times C\{1,10\}$ or 20; and there is one case in which C and D hold no trumps. All the favourable cases are, therefore, in number

$$4200 + 5040 + 2940 + 960 + 180 + 20 + 1 \text{ or } 13341.$$

The chance of A being able to force all the adversary's trumps is $\frac{13341}{59049}$, or nearly $3\frac{1}{2}$ to 1 against it.

Given a fraction less than unity, and which has high numbers in its terms, required a set of fractions which shall be very nearly equal to it, and each of which shall be nearer than any other fraction of the same order of simplicity. Required, also, a near estimate of the error committed in each case.

Rule. First perform the process for finding the greatest common measure of the numerator and denominator.

1	13341)59049(4=4
	53364
2	5685)13341(2×4+1=9
	11370
2×2+1=5	1971)5685(2×9+4=22
	3942
5×1+2=7	1743)1971(1×22+9=31
	1743
7×7+5=54	228)1743(7×31+22=239
	1596
54×1+7=61	147)228(1×239+31=270
	147
	81)147(1
	81
	66 &c.

Opposite to the first quotient write 1, and proceed to form columns out of the several quotients, in the following manner: —

1 = 1st Numerator.	1st Quotient = 1st Denominator.
2nd Qu. = 2d Num.	2d Qu. × 1st Qu. + 1 = 2d Den.
2d Num. × 3d Qu. + 1st Num. = 3d Num.	3d Qu. × 2d Den. + 1st Den. = 3d Den.
3d Num. × 4th Qu. + 2d Num. = 4th Num.	4th Qu. × 3d Den. + 2d Den. = 4th Den.
4th Num. × 5th Qu. + 3d Num. = 5th Num.	5th Qu. × 4th Den. + 3d Den. = 5th Den.
Any numerator multiplied by the next quotient, and product increased by preceding numerator, gives succeeding numerator.	Any denominator multiplied by the next quotient, and product increased by preceding denominator, gives succeeding denominator.

Thus, $\frac{1}{4}$, $\frac{2}{9}$, $\frac{5}{22}$, $\frac{7}{31}$, $\frac{54}{239}$, $\frac{61}{270}$, &c. &c., is a set of fractions which approach nearer and nearer to $\frac{13341}{59049}$. The *first* is always too great, the second too small, the third too great, the fourth too small: every odd one too great, every even one too small.

Test of correctness. Take any two successive numerators and denominators, multiplied crosswise; they give products which differ by unity.

54	61		$54 \times 270 = 14580$
239	270		$239 \times 61 = 14579$
			1

Estimation of the error. The error of $\frac{1}{4}$ is less than $\frac{1}{36}$ $(4 \times 9 = 36)$; that of $\frac{2}{9}$ is less than $\frac{1}{198}$ $(9 \times 22 = 198)$; that of $\frac{5}{22}$ is less than $\frac{1}{682}$ $(22 \times 31 = 682)$, &c.

The error of $\frac{\text{any Numerator}}{\text{its Denominator}}$ is less than $\frac{1}{\text{That Den.} \times \text{the next.}}$

I now take the following problem: A die is thrown time after time; in how many times have we an even chance of throwing an ace. The common error attached to this problem is, that since there are six faces, it is most likely all will have come up in six throws.

In the first throw there are six events, five of which are unfavourable. In the first two throws, considered as giving one event, there are 6×6 or 36 possible cases, for every possible case of the first throw may combine with any case of the second. But of these 36 throws, any one of the five unfavourables of the first throw may combine with any one of the second throw, and there are therefore 5×5, or 25 unfavourable compound events. Hence the following table: —

One throw gives	6 cases	; 5 unfavourable	
Two throws give	36 cases	; 25 unfavourable	
Three	216	; 125	(odds still against.)
Four	1296	; 625	(odds turned in favour.)
Five	7776	; 3125 &c.	

Answer: There is not quite an even chance of doing it in three throws, but more than an even chance of doing it in four.

Rule. When the odds against success in one trial are n to 1, then $\frac{7}{10}$ of n. (or the nearest whole number to it) is about the number of trials in which there is an even chance of one success: more correctly $\frac{100}{144}$ of n. Thus, if it be 144 to 1 against a single attempt, there is about an even chance of one success in 100 throws.

A table of the number of trials (odds being *n.* to 1) in which there are various odds of succeeding once : —

1 to 1	69	13 to 1	264	30 to 1	343	1000 to 1	691
2 to 1	110	14 to 1	271	40 to 1	371	2000 to 1	760
3 to 1	139	15 to 1	277	50 to 1	393	3000 to 1	801
4 to 1	161	16 to 1	283	60 to 1	411	4000 to 1	829
5 to 1	179	17 to 1	289	70 to 1	426	5000 to 1	852
6 to 1	195	18 to 1	294	80 to 1	439	6000 to 1	870
7 to 1	208	19 to 1	300	90 to 1	451	7000 to 1	885
8 to 1	220	20 to 1	304	100 to 1	462	8000 to 1	899
9 to 1	230	21 to 1	309	110 to 1	471	9000 to 1	911
10 to 1	240	22 to 1	314	120 to 1	480	10,000 to 1	921
11 to 1	248	23 to 1	318	130 to 1	488		
12 to 1	256	24 to 1	322	140 to 1	495		

The method of using this table is as follows: — Suppose the odds to be 20 to 1 against success in a single trial: required in how many throws it is 100 to 1 there shall be one success (or more). Look in the table opposite to "100 to 1," and we see 462; multiply 20 by 462 *and divide by* 100 (which is always to be done) this gives $92\frac{4}{10}$, or 92 is about the number of throws required.

I now proceed to the method of compounding the probabilities of single events, so as to find those of compound events: that is, the way of methodising the results of actual inspection.

Let there be two events, one of which may happen in 7 ways, and may not happen in 5; and the other of which may happen in 4 ways, and may not happen in 9: whence the probabilities of the two events (A and B) are $\frac{7}{12}$ and $\frac{4}{13}$. Now, the compound event may happen in 12×13 different ways, for any case of the first (12 in all) may come up with any case of the second (13 in all). By similar reason, the compound case in which both A and B happen may come up in 7×4 different ways; hence the probability of A and B both happening, is

$$\frac{7 \times 4}{12 \times 13} \text{ which is the product of } \frac{7}{12} \text{ and } \frac{4}{13}$$

RULE. When events are perfectly independent, so that the happening of one has nothing to do with that of the other, the probability that both will happen is the product of the probabilities that each will happen.

This rule applies to any number of independent events.

It is indifferent whether the events are to happen together, or one after the other: thus the chances of a compound drawing out of two lotteries, one drawing out of each, are the same whether two persons draw simultaneously, or one person first draws out of one, and then out of the other.

Example. Let there be three lotteries, as follows: —

$$\begin{Bmatrix}6\text{ white}\\5\text{ black}\end{Bmatrix}\quad\begin{Bmatrix}7\text{ white}\\2\text{ black}\end{Bmatrix}\quad\begin{Bmatrix}8\text{ white}\\10\text{ black}\end{Bmatrix}$$

What is the chance of drawing from the three, white, black, and white? The probabilities of these events are, $\frac{6}{11}$, $\frac{2}{9}$, and $\frac{8}{18}$, the product of which is $\frac{16}{297}$, or about 17 to 1 against the compound event.

The probability that A will happen, and B will not, is the product of the chance for A, and that against B. The probability that one will happen, and one only, is the sum of the probabilities—1. that A will happen and B will not; 2. That A will not happen and B will happen. The probability that one or both will happen is the remainder when the probability that neither will happen is subtracted from unity. The following are evident results of the measure of probability: —

1. When either P, Q, or R must happen, the sum of their probabilities must be unity, which is always the representative of certainty. Thus, if a lottery contain 7 white, 5 black, and 3 red, either white, black, or red must be drawn, and $\frac{7}{15}$, $\frac{5}{15}$, and $\frac{3}{15}$, together make 1, and the same if the events be more or less than three in number.

2. When of two events, each excludes the other, the probability that one or other of them will happen is the sum of their probabilities. For to say that each excludes the other, is to say that they are connected events, or different possible cases of the same set. Thus,

in the preceding lottery, if I draw white I cannot draw black, and *vice versâ*, there being one drawing only supposed. Hence the probability of drawing black or white is $\frac{7}{15}+\frac{5}{15}$ or $\frac{12}{15}$. But suppose there had been two lotteries, as follows:—

(7 white, 8 red) (5 black, 10 red),

and I draw in one without knowing in which I am drawing. The individual probabilities of white in the two, if the lottery be known, are $\frac{7}{15}$ and $\frac{5}{15}$ as before; but the question, what is the chance of drawing white or black, and not red, is a different one from the preceding. If I draw without knowing in which lottery I am drawing, my position is the same as if one lottery had been thrown into the other, and I had drawn from both in one, giving

(7 white, 5 black, 18 red).

This union, which will readily be admitted to make no alteration in the probabilities, is not admissible unless the total number of balls in both be the same. For, in the mixture, the 8 red balls of the one and the 10 of the other, are considered as severally of equal probability. But this they are not, unless the number of possible cases in the two were the same. Suppose, for instance, I had the following lotteries:—

(10 white, 10 red) (4 black, 5 red);

the probability of each ball of the first is $\frac{1}{20}$, but of each of the second $\frac{1}{9}$. Before I can draw a white ball, two events must happen. 1. I must happen to select the first lottery. 2. I must happen to draw a white ball rather than a red. The chances of these are $\frac{1}{2}$ and $\frac{10}{20}$ or $\frac{1}{2}$; consequently, $\frac{1}{2}\times\frac{1}{2}$ or $\frac{1}{4}$ is my chance of a white ball. But if I mix the two lotteries, that chance will be $\frac{10}{29}$, which is more than $\frac{1}{4}$. The reason is, that I have mixed together balls which *had* unequal chances of being drawn, and treated them as if they were to have equal chances. But it is evident, from the measure of probability, that I do not alter the chances for an event or the general chances which any set of circumstances affords, if I multiply the chances for an event any number of times, provided I multiply the chances against it as often. Reduce the preceding lotteries to common numbers of balls, by putting 20 balls for one into the second, and 9 for one into the first. We then have —

(90 white, 90 red) (80 black, 100 red).

Now mix the two, and we have —

(90 white, 80 black, 190 red);

and the chance of a white is $\frac{90}{360}$, or $\frac{1}{4}$, that is, the problem is not altered by the mixture. And the same may be shown in any other case.

Let us now suppose that the chance of A is $\frac{2}{3}$ and that of B $\frac{5}{7}$. The chances against A and B are therefore $\frac{1}{3}$ and $\frac{2}{7}$.

The possible cases are	The probability of which is
That A and B shall both happen	$\frac{2}{3}\times\frac{5}{7}$ or $\frac{10}{21}$
That A shall happen, and not B	$\frac{2}{3}\times\frac{2}{7}$ or $\frac{4}{21}$
That B shall happen, and not A	$\frac{1}{3}\times\frac{5}{7}$ or $\frac{5}{21}$
That neither shall happen.	$\frac{1}{3}\times\frac{2}{7}$ or $\frac{2}{21}$

One of these cases must happen, and the sum of the chances is $\frac{21}{21}$ or 1, each compound event being exclusive of the others. Again, we find

The probability of	*Both* or *neither* is	$\frac{12}{21}$		
	One or other, but only one	$\frac{9}{21}$		
	One or other, or both	$\frac{19}{21}$		
	One or neither	$\frac{11}{21}$		
	A or both	$\frac{14}{21}$	A or neither	$\frac{6}{21}$
	B or both	$\frac{15}{21}$	B or neither	$\frac{7}{21}$
	A or neither, or both	$\frac{16}{21}$		
	B or neither, or both	$\frac{17}{21}$		

The latter set of events does not consist of those which are mutually exclusive. Any thing which happens falls under six of them, that is, six of them must happen. Thus, A, and not B, actually arriving, would secure any gain which depended upon either of the following: —

One or other, but only one	A, or both
One or other, or both	A, or neither
One or neither	A, or neither, or both,

If we add together all the probabilities of the preceding, we find $\frac{126}{21}$, or 6. This is the indication that every possible event enters in six different ways into the contingencies whose probabilities united amount to six.

Example. Twelve halfpence, A_1 $A_2 \ldots A_{12}$, are thrown up, required the probability of all the cases which can happen, and which we shall symbolise thus: ($H_3 T_9$) means that there are three heads and nine tails. The chance of H or T in any one piece is $\frac{1}{2}$; consequently, the chance of the several pieces A_1, A_2, &c., yielding each any particular letter is $\frac{1}{2} \times \frac{1}{2} \times \frac{1}{2} \times$ —— (twelve factors), or $\frac{1}{4096}$. Now, the 4096 possible cases are thus classified: —

H_0 T_{12} and H_{12} T_0 happen in	. 1 case each.
H_1 T_{11} ...... H_{11} T_1	12 cases each.
H_2 T_{10} ...... H_{10} T_2 .... $C\left\{2, 12\right\}$	or 66 ....
H_3 T_9 ...... H_9 T_3 .... $C\left\{3, 12\right\}$	or 220 ....
H_4 T_8 ...... H_8 T_4 .... $C\left\{4, 12\right\}$	or 495 ....
H_5 T_7 ...... H_7 T_5 .... $C\left\{5, 12\right\}$	or 792 ....
H_6 T_6 happens in 924 cases.	

It appears, then, that the most likely individual result is H_6 T_6, against which, however, it is about $3\frac{1}{2}$ to 1. But if we ask for the probability either of this or of a single variation on one side or the other, that is, of the following event —

Either H_5 T_7, or H_6 T_6, or H_7 T_5 —

we find $792 \times 2 + 924$ or 2508 (more than half of 4096) cases in which the event arrives. That is, the chances are in favour of one or other of these arriving. As the number of events increases, a given degree of nearness to the most probable event becomes more and more likely. To illustrate this: suppose 24 halfpence thrown up, the notation remaining as before. The total number of cases is now 2^{24} or 16777216: calculating the number of combinations of 1,2,3, &c. out of 24, we have the following summary:—

$H_0\ T_{24}$	and	$H_{24}\ T_0$	happen in	1 case each.
$H_1\ T_{23}$	......	$H_{23}\ T_1$	....	24 cases each.
$H_2\ T_{22}$	......	$H_{22}\ T_2$	....	276 ....
$H_3\ T_{21}$	......	$H_{21}\ T_3$	....	2024 ....
$H_4\ T_{20}$	......	$H_{20}\ T_4$	....	10626 ....
$H_5\ T_{19}$	.....	$H_{19}\ T_5$	....	42504 ....
$H_6\ T_{18}$	......	$H_{18}\ T_6$	....	134596 ....
$H_7\ T_{17}$	.....	$H_{17}\ T_7$	....	346104 ....
$H_8\ T_{16}$	......	$H_{16}\ T_8$	....	735471 ....
$H_9\ T_{15}$	......	$H_{15}\ T_9$	....	1307504 ....
$H_{10}\ T_{14}$	......	$H_{14}\ T_{10}$	....	1961256 ....
$H_{11}\ T_{13}$	......	$H_{13}\ T_{11}$	....	2496144 ....

$H_{12}\ T_{12}$ happens in 2704156 cases.

The odds are now about 6 to 1 against the even division of the pieces into heads and tails. But let us consider the same degree of departure from the most probable case as we took before. An alteration in one piece out of twelve answers to that of 2 pieces out of 24. Now the number of cases in which either $H_{10}\ T_{14}$, or $H_{11}T_{13}$, or $H_{12}T_{12}$, or $H_{13}T_{11}$, or $H_{14}T_{10}$ arrives, and the odds of one or other of them, is computed as follows:—

$H_{10}\ T_{14}$ or $H_{14}\ T_{10}$	3922512	16777216	whole No. of cases
$H_{11}\ T_{13}$ or $H_{13}\ T_{11}$	4992288	11618956	No. favourable
$H_6\ T_6$	2704156		
	11618956	5158260	No. unfavourable

or it is now more than 2 to 1 in favour of the heads lying between 10 and 14, both inclusive.

I now put together the principles on which we have hitherto gone, adding two more, the first of which (Principle III.) is obviously a consequence of the preceding, and the second of which will be presently explained.

Principle I. When all the ways in which an event may happen are equally probable, the chance of its happening is the number of ways in which it may happen, divided by all the number of ways in which it may happen and fail.

Principle II. The probability of any number of independent events all happening together, is the product of their several probabilities.

Principle III. The probability of two events arriving together being known; and also, that of one of them: the probability of the other is found by dividing the first mentioned probability by the second.

Principle IV. When an event may happen in several ways, whether equally probable or not, the probability of the event is the sum of the probabilities of its happening in the several different ways.

The best way of illustrating the last principle is by beginning with one of the numerous errors into which we may fall, either in proceeding towards it, or applying it. Let there be an event which may happen in two different ways, for each of which there is an even chance. Then, according to the principle, it is certain ($\frac{1}{2} + \frac{1}{2} = 1$) that the event must happen. In this there is nothing inconsistent. Let there be a lottery containing ten white balls, and let them be sub-divided into two sets of five by a mark. Then there are two ways of drawing a white ball, for each of which there is an even chance; namely, I may choose a ball of one mark, or of the other. But it is evidently certain that in this case a white ball must be drawn. Now, suppose an event can happen in three different ways, for each of which there is an even chance. Then the event is *more than certain* ($\frac{1}{2} + \frac{1}{2} + \frac{1}{2} = 1\frac{1}{2}$); which is absurd. But the absurdity is in the supposition: an event can only have an even chance of happening in one particular way, when that way involves half of the total number of individual cases of the event; and it is impossible that three different ways of arriving can each contain half of the whole number of possible cases. Consequently, when we have made a calculation of the probabilities which different ways of arriving give to an event, there has certainly been an error if the sum of the probabilities exceed unity.

But when we throw three half-pence into the air, are there not three different ways of throwing head, for each of which there is an even chance? If by H we here mean a single H, the three events by which we propose to attain it are not H, H, H, but H T T, T H T, and TTH, the probability of each of which is $\frac{1}{8}$ and, by the application of the principle, $\frac{1}{8}+\frac{1}{8}+\frac{1}{8}$, or $\frac{3}{8}$, is the probability of the event,—one single head. If by throwing a head we mean one head or more, the ways under which it may be brought about are—one head only (involving the cases H T T, T H T, T T H)—two heads only (involving H H T, H T H, and T H H)—and three heads (involving H H H.) The probabilities of these are $\frac{3}{8}$, $\frac{3}{8}$, and $\frac{1}{8}$, or $\frac{7}{8}$ is the chance required.

Another error to which we are liable, is the wrong estimation of the probability of the different cases. For instance, a person is to go on until he throws H, and is to win if he do it in less than five throws. There are then five possible cases; namely, H, T H, T T H, T T T H, T T T T. In four of these cases he wins; in the fifth he loses. His chance of winning then appears to be $\frac{4}{5}$. But this supposes the five cases to be equally likely, which is not true. Their several probabilities are $\frac{1}{2}$, $\frac{1}{4}$, $\frac{1}{8}$, $\frac{1}{16}$ and $\frac{1}{16}$: not $\frac{1}{5}$, $\frac{1}{5}$, &c., as supposed. Consequently the sum of the probabilities which the several winning cases actually have is $\frac{15}{16}$, or it is 15 to 1 that he wins. If we put together all the cases, which four throws present, thus,—

1. HHHH	5. HTHH	9. THHH	13. TTHH
2. HHHT	6. HTHT	10. THHT	14. TTHT
3. HHTH	7. HTTH	11. THTH	15. TTTH
4. HHTT	8. HTTT	12. THTT	16. TTTT;

all those which begin with H are 8 in number, those which begin with T H are 4, with T T H, 2, and with T T T H, one. Hence 8 + 4 + 2 + 1, or 15, is the number of winning cases. But the argument against us, is this: most of the preceding cases are impossible, for the condition is, that the play shall stop as soon as H occurs: so that, in fact, the only possible cases are, H, T H, T T H, T T T H, T T T T. Let it be so; we must then represent the several events as follows:—

1st event; certainly happens, and gives either H or T. Probability of H, $\frac{1}{2}$.

2nd event; does not certainly happen, but is contingent upon the first throw being T; and gives either H or T. Probability $\frac{1}{2}$ that the throw is made, $\frac{1}{2}$ that if it be made it gives H; probability that *the throw is made*, and that, being made, it gives H, $\frac{1}{2} \times \frac{1}{2} = \frac{1}{4}$.

3rd event; contingent upon the two first throws giving T and T, of which the chance is $\frac{1}{4}$. Probability of winning at this throw, $\frac{1}{4} \times \frac{1}{2} = \frac{1}{8}$.

4th event; contingent upon the three first throws giving T, T, T. Probability of winning at this throw, $\frac{1}{8} \times \frac{1}{2} = \frac{1}{16}$.

It is obvious enough, when stated, that every contingency must enter into the consideration of a question; so that if some of the circumstances depend upon the manner in which preceding contingencies arrive, this circumstance itself influences the method of proceeding. If we wish to avoid the necessity of considering a contingency the trial of which is itself contingent, that is, if we wish to make a contingency certain, we must introduce all the new events which such change of contingency into certainty brings with it. The whole problem is exactly the same as if we made the four throws certain, and made the gain dependent upon one head or more being thrown: but we revert again to the former state of the question, if we agree to mark the *first* H as the winner. In this point of view, the distinction between the two is evidently immaterial.

Example. There are seven lotteries, as follows (W means white, B black): —

I. (2 W, 3 B)	II. (3 W, 2 P)	(1 W, 1 B) IV.
		(2 W, 1 B) V.
	III. (1 W, 3 B)	(1 W, 5 B) VI.
		(4 W, 1 B) VII.;

and the conditions of drawing are the following. I. is drawn, and then II. or III., according as I. gives W or B. If II. be drawn, then IV. or V. is to be drawn, according as II. gives W or B. But if III. be drawn, then VI. or

VII. is to be drawn, according as III. gives W or B. What are the probabilities of the several possible drawings?

If from I. we draw W (of which the chance is $\frac{2}{5}$), we proceed to II. If we still draw W (chance, $\frac{3}{5}$), we proceed to IV. And here the chance of W is $\frac{1}{2}$. Hence,

$$\text{the chance of WWW is } \tfrac{2}{5} \times \tfrac{3}{5} \times \tfrac{1}{2} = \tfrac{3}{25}$$

Computing all the other chances in the same way, we get the following:—

1. WWW	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{2} = \frac{3}{25}$		5. BWW	$\frac{3}{5}$	$\frac{1}{4}$	$\frac{1}{6} = \frac{1}{40}$
2. WWB	$\frac{2}{5}$	$\frac{3}{5}$	$\frac{1}{2} = \frac{3}{25}$		6. BWB	$\frac{3}{5}$	$\frac{1}{4}$	$\frac{5}{6} = \frac{1}{8}$
3. WBW	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{2}{3} = \frac{8}{75}$		7. BBW	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{4}{5} = \frac{9}{25}$
4. WBB	$\frac{2}{5}$	$\frac{2}{5}$	$\frac{1}{3} = \frac{4}{75}$		8. BBB	$\frac{3}{5}$	$\frac{3}{4}$	$\frac{1}{5} = \frac{9}{100}$

The sum of all these is equal to unity, as it should be, since one or other of these cases must happen. And by reducing all to the common denominator 5. 5. 4. 6, or 600, we have the following chances:—

1. WWW $\frac{72}{600}$	3. WBW $\frac{64}{600}$	5. BWW $\frac{15}{600}$	7. BBW $\frac{216}{600}$
2. WWB $\frac{72}{600}$	4. WBB $\frac{32}{600}$	6. BWB $\frac{75}{600}$	8. BBB $\frac{54}{600}$

That all shall be white,	7 to 1 against (nearly).
Two white and one black,	3 to 1 against (very nearly)
Two black and one white,	an even chance (nearly)
All black,	10 to 1 against (nearly).

From what has been said in this chapter, no great difficulty will be found in ordinary questions. The circumstances are supposed to be fully known, and the probabilities will be found, of the strength which it follows they must have, to those who admit the axioms on which the measure of probability is founded.

CHAPTER III.

ON INVERSE PROBABILITIES.

In the preceding chapter, we have calculated the chances of an event, knowing the circumstances under which it is to happen or fail. We are now to place ourselves in an inverted position: we know the event, and ask what is the probability which results from the event in favour of any set of circumstances under which the same might have happened. This problem is frequently enunciated as follows: — An event has happened, such as might have arisen from different causes: what is the probability that any one specified cause did produce the event, to the exclusion of the other causes? By a *cause*, is to be understood simply a state of things antecedent to the happening of an event, without the introduction of any notion of agency, physical or moral.

In order that we may secure a problem of sufficient simplicity, we must limit the number of possible antecedent states. Let us suppose that there is an urn, of which we know that it contains balls, three in number, and either white or black, all cases being equally probable: that is, before any drawing takes place, all we can say is, that we are going to draw out of one of the following, having no reason for supposing one in preference to another · —

	A	B C	D E F
I. (• • •)	II. (o • •)	III. (o o •)	IV. (o o o)

A drawing takes place, and a white ball is produced, consequently I. is immediately excluded; for from it the observed event could not have been produced. This much is certain; but we are also tempted to say that

II. is rendered unlikely, because, from such an antecedent state of things, a black ball would have been more likely than a white one. On the same principle III. is more likely than II., and IV. the most likely of all. We have then to decide the relative probabilities of II., III., and IV.

Before the drawing took place, the probability of each set of circumstances was $\frac{1}{4}$; and, the lottery being given, the probability of any one ball in it was $\frac{1}{3}$. Thus the chance of III. being the lottery, and the second white ball being drawn from it, was $\frac{1}{4} \times \frac{1}{3}$, or $\frac{1}{12}$. The same of other balls: so that, in fact, our primitive position was that of having to draw from 12 balls, 6 white and 6 black, all equally probable. But the observed event changed that position; a white ball was drawn: was it a given ball (namely, the white ball in II.), or was it one of two given balls (those in III.), or was it one of three (those in IV.)? There are *six* cases in question, namely, A, B, C, D, E, F, and one of them happened,—we do not know which. We have used all the knowledge we have (namely, that a white ball was drawn,) in excluding the black balls.

Hence the chance that A was drawn, or that II. was the lottery - - - is $\frac{1}{6}$:
That either B or C was drawn, or that III. was the lottery - - - is $\frac{2}{6}$:
That either D, E, or F was drawn, or that IV. was the lottery - - - is $\frac{3}{6}$.

In the preceding instance, owing to the number of balls being the same in every lottery, the antecedent probability of each ball was the same. Previous to deducing a rule, I take an instance in which this is not the case.

Problem. A white ball has been drawn, and from one or other of the two following urns:

(2 white, 5 black) (3 white, 1 black).

What are the probabilities in favour of each urn?

The case is not now that of a lottery of 5 white and 6 black balls; for the chance of our going to the first urn (which is $\frac{1}{2}$), and thence drawing a *given* white ball (chance $\frac{1}{7}$), is $\frac{1}{2} \times \frac{1}{7}$ or $\frac{1}{14}$; while our chance of going to the second urn (which is $\frac{1}{2}$), and thence drawing a given white ball (chance $\frac{1}{4}$), is $\frac{1}{2} \times \frac{1}{4}$ or $\frac{1}{8}$. But since we do not alter the chance of producing a white ball from either urn, if we double, or treble, &c. the number of white balls, provided we at the same time double, or treble, &c. the number of black balls, let us put four times as many balls into the first, and seven times as many into the second, as there are already. Thus we have:

(8 white, 20 black)   (21 white, 7 black).

There are now 28 balls in each: every individual ball has the antecedent probability $\frac{1}{2} \times \frac{1}{28}$; and since our knowledge of the event (a white ball was drawn) excludes the black balls, the question is simply this: — Out of 29 possible, and equally probable cases, was the event which did happen one out of a certain 8, or one out of the remaining 21? The chances of these are $\frac{8}{29}$ and $\frac{21}{29}$; consequently it is 21 to 8 that the second lottery was that which was drawn, and not the first.

On looking at the resulting chance for the first urn, namely, $\frac{8}{29}$, or the $(21 + 8)$th part of 8, we see that 8 and 21 are in proportion to the two chances for a white ball being drawn, when we know that we are drawing from the first urn, or from the second. For these chances are $\frac{2}{7}$ and $\frac{3}{4}$, which, reduced to a common denominator, are $\frac{8}{28}$ and $\frac{21}{28}$, which are in the proportion of 8 to 21. The same reasoning may be applied to any other cases, and the result is as follows: —

Principle V. — When an event has happened, and the state of things under which it happened must have been one out of the set A, B, C, D, &c., take the different states for granted, one after the other, and ascertain the probability that, such state existing, the event which did happen would have happened. Divide the probability thus deduced from A by the sum of the probabilities deduced from all, and the result is the probability that A was the state which produced the event: and similarly for the rest. [Or, reduce the results of the first part of the rule to a common denominator, and use the numerators only in the second part of the rule.]

Example I. There is a lottery which is one or other of the two following:

(3 white, 7 black) (all white).

A ball is drawn, and restored; this takes place five times, and the result is always a white ball. What are the chances for each lottery?

Upon the supposition that the first lottery was that in question, the chance of the observed event is the product of $\frac{3}{10}$, $\frac{3}{10}$, $\frac{3}{10}$, $\frac{3}{10}$, and $\frac{3}{10}$, or $\frac{243}{100000}$. When the second is the lottery, the observed event is certain, and its probability is 1 or $\frac{100000}{100000}$. Consequently, the probability for the second lottery is $\frac{100000}{100243}$, or the second has the odds 100000 to 243, or more than 411 to 1 in its favour.

Example II. Two witnesses, on each of whom it is 3 to 1 that he speaks truth, unite in affirming that an event did happen, which of itself is equally likely to have happened or not to have happened. What is the probability that the event did happen?

The fact observed is the agreement of the two witnesses in asserting the event: the two possible antecedents (equally likely) are,—1. The event did happen. 2. It did not happen. If it did happen, the probability that both witnesses should state its happening is that of their both telling the truth, which is $\frac{3}{4} \times \frac{3}{4}$, or $\frac{9}{16}$. If it did not happen, then the probability that both witnesses should assert its happening is that of their both speaking falsely, which is $\frac{1}{4} \times \frac{1}{4}$, or $\frac{1}{16}$. Consequently, the probability that the event did happen is the $(9 + 1)$th part of 9, or $\frac{9}{10}$; that is, it is 9 to 1 in favour of the event having happened.

Example III. There are two urns, having certainly 3 and 2 white balls; and in one or other, but which is not known, is a black ball. A ball is drawn and replaced; and this process is repeated, but whether out of the same urn as before is not known. Both drawings give a white ball: what is the probability of the several cases from which this result might have happened?

Since the black ball is as likely to be in one as in the other, the antecedent state of things is (so far as a single drawing is concerned,) the same as if there were four urns, as follows:

I. (3 white) II. (3 white, 1 black) III. (2 white) IV. (2 white, 1 black).

There are 16 possible cases, PP [2, 4,] numbered in the first columns following, described in the second, and having the probability which each would give to the observed event (both drawings white) registered in the third, together with the numerator, when all the fractions are reduced to a common denominator 144.

1	I. I.	1, 144	5	II. I.	$\frac{3}{4}$, 108
2	I. II.	$\frac{3}{4}$, 108	6	II. II.	$\frac{9}{16}$, 81
3	I. III.	1, 144	7	II. III.	$\frac{3}{4}$, 108
4	I. IV.	$\frac{2}{3}$, 96	8	II. IV.	$\frac{6}{12}$, 72
9	III. I.	1, 144	13	IV. I.	$\frac{2}{3}$, 96
10	III. II.	$\frac{3}{4}$, 108	14	IV. II.	$\frac{6}{12}$, 72
11	III. III.	1, 144	15	IV. III.	$\frac{2}{3}$, 96
12	III. IV.	$\frac{2}{3}$, 96	16	IV. IV.	$\frac{4}{9}$, 64

That is to say, if (case 8), II. and IV. were the urns of the first and second drawings, the chance of the observed event is $\frac{1}{2}$ or $\frac{72}{144}$. But, it must be remembered, that we do not suppose the black ball may have been removed from one urn into the other before the second drawing takes place. Most of the preceding cases are, therefore, to be rejected; in fact, I. can combine with nothing but I. or IV., and II. with nothing but II. or III. Reject, therefore, cases 2, 3, 5, 8, 9, 12, 14, 15, and the sum of the numerators in the rest is 841. Hence the probability (for instance,)

1. That the black ball is with the three white ones. 2. That the first drawing is from the lottery which has the black ball, and the second from the other, is (case 7) $\frac{108}{481}$. To find the total probability that the black ball is with the three white ones, we must add the probabilities of all the cases (as to drawings) which can take place under this arrangement, namely $\frac{81}{841}$, $\frac{108}{841}$, $\frac{108}{841}$, $\frac{144}{841}$, giving $\frac{441}{841}$. Consequently, from the observed event, it is slightly more probable that the black ball is with the *three* white ones, than with the *two*.

The principle which we have illustrated, though a mathematical consequence of those which precede, is nevertheless received in common life upon its own evidence. When an event happens, we immediately look to that cause or antecedent which such event most often follows. When it rains, we suspect the barometer must have fallen; because, when the barometer falls, it usually rains.

Our next step is to inquire, what is the probability which an event gives to its several possible antecedents, upon the supposition that they are not all equally likely beforehand; as in the following instance.

PROBLEM. A white ball is drawn, and from one or other of the following urns:

(3 white, 4 black) (2 white, 7 black):

but before the drawing was made, it was three to one that the drawer should go to the first urn, and not to the second. What is the chance that it was the first urn from which the drawing was made?

We may immediately reduce the preceding to the case where all the antecedent circumstances are equally probable, by introducing urns enough of the first kind to make it 3 to 1 that the drawing is made from one or other of them. Let us suppose the urns to be as follows:

(3 white, 4 black) (3 white, 4 black) (3 white, 4 black)
(2 white, 7 black):

these urns being equally probable, the hypothesis of the problem exists. If we number the urns A_1, A_2, A_3, B, the chances which they severally give to the observed event are $\frac{3}{7}$, $\frac{3}{7}$, $\frac{3}{7}$, and $\frac{2}{9}$, the numerators of which, reduced to a common denominator, are 27, 27, 27, and 14. Consequently, the probability that A_1 was chosen, is $\frac{27}{95}$; and the same for A_2 and A_3. Therefore, the chance that one or other of the three, A_1, A_2, and A_3, was chosen, is $\frac{81}{95}$; which is the probability of the ball having been drawn from the urn (3 white, 4 black,) in the first statement of the problem.

The rule to which the preceding reasoning conducts us is as follows: When the different states under which an event may have happened are not equally likely to have existed, then having found the probability which each state would give to the observed event, multiply each by the probability of the state itself before using the rule in page 55. The following is another example.

An event has happened, the possible preceding states of which are represented by A, B, and C. The chances of the existence of these different states (independently of all knowledge of the observed event) are, say, $\frac{4}{9}$, $\frac{3}{9}$, and $\frac{2}{9}$: the probabilities that the observed event would have happened are $\frac{5}{11}$, $\frac{4}{11}$, and $\frac{2}{11}$, if A, B, or C were certainly existing. Form the three products

$$\text{Probability of A} \times \left\{ \begin{array}{l} \text{Probability that the event would} \\ \text{happen if A were known to exist,} \end{array} \right\} \text{\&c.}$$

there are

$$\tfrac{4}{9} \times \tfrac{5}{11},\ \tfrac{3}{9} \times \tfrac{4}{11},\ \text{and}\ \tfrac{2}{9} \times \tfrac{2}{11};$$

the numerators of which (the denominators being common) are 20, 12, and 4. Then the probability that A was the state under which the event happened, is 20 divided by 20 + 12 + 4, or $\frac{20}{36}$; those of B and C are $\frac{12}{36}$ and $\frac{4}{36}$.

Let us now suppose, that having only a first event by which to judge of the preceding state of things, we ask what is the probability of a second event yet to come. For instance, an urn contains two balls, but whether white or black is not known; the first drawing gives a *white* ball, and the ball is replaced. What is the chance that a second drawing shall give a *black* ball?

The preceding states under which the first event may have happened, are —

(2 white) (1 white, 1 black);

and 1 and $\frac{1}{2}$ are the chances of a white ball, if one or other state were absolutely known to exist. Hence, by the last principle, $\frac{2}{3}$ and $\frac{1}{3}$ are the chances which the observed event gives to the two states; that is, it is two to one that both balls were white. Now, the black ball can only appear at the second drawing, upon the supposition of the second state existing; and this supposition being made, the chance of a black ball at the second drawing is $\frac{1}{2}$. Hence, page 43., the second event depending upon two contingencies, of which the chances are $\frac{1}{3}$ and $\frac{1}{2}$, its chance is $\frac{1}{6}$, or it is five to one against the second drawing being black. But let us now ask what is the chance of a *white* ball at the second drawing? Either of the preceding states admit of such an event, and, in fact, the event proposed — a white ball at the second drawing — means

One or other of these two combinations { (2 white) and white drawn.
(1 white, 1 black) and white drawn.

In the first combination, the first contingency (the chance of which is $\frac{1}{3}$) ensures the second: so that $\frac{1}{3} \times 1$ is the chance of a white ball being drawn, and of (2 white) being the lottery from which it is drawn. In the second combination, the chances of the two contingencies are $\frac{1}{3}$ and $\frac{1}{2}$, whence $\frac{1}{6}$ is the chance of a white ball being drawn, and being drawn from (1 white, 1 black). But the event proposed happens if either of these cases occur; therefore, $\frac{2}{3} + \frac{1}{6}$, or $\frac{5}{6}$, is the chance of a white ball at the second drawing, as might have been inferred from the probability already obtained for a black ball. By such reasoning as the preceding, the following principle is established:

Principle VI. Having given an observed event A, to find the probability which it affords to the supposition that a coming event shall be B, find the probability which A gives to every possible preceding state; multiply each probability thus obtained by the chance which B would have from that state, and add the results together.

Problem. There is a lottery of 10 balls, each one white or black, but which is not known: drawings are made, *after each of which the ball is replaced.* The first five drawings are white; what chance is there that the next two drawings shall be white?

Let S (20) denote the sum of all numbers up to 20; S (20^2) the sum of the squares of all numbers up to 20^2 or 400; and so on. The possible preceding states are —

(1 W, 9 B) (2 W, 8 B.) . . . . (10 W, 0 B);

and the probabilities of W five times running from each, are

$$\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10}\cdot\frac{1}{10},\quad \frac{2}{10}\cdot\frac{2}{10}\cdot\frac{2}{10}\cdot\frac{2}{10}\cdot\frac{2}{10},\ \&c.$$

up to $\frac{10}{10}\cdot\frac{10}{10}\cdot\frac{10}{10}\cdot\frac{10}{10}\cdot\frac{10}{10}$, or 1, the event being certain, if the last state existed. The numerators of these products (the common denominator being 10^5,) are 1^5, 2^5, ——— 10^5: whence, page 55., the probabilities of the several states are —

$$\frac{1^5}{S\ 10^5},\quad \frac{2^5}{S\ 10^5},\quad \ldots\ldots\ldots\quad \frac{10^5}{S\ 10^5}$$

By the same reasoning, the probabilities of the proposed events (two more white balls,) are —

$$\frac{1^2}{10^2}\quad \frac{2^2}{10^2}\ \ldots\ldots\ldots\ \frac{10^2}{10^2};$$

the different preceding states being successively supposed to exist; whence the actual chance which the observed event gives to the proposed is —

$$\frac{1^5}{S\ 10^5}\times\frac{1^2}{10^2}+\frac{2^5}{S\ 10^5}\times\frac{2^2}{10^2}+\ldots\ldots+\frac{10^5}{S\ 10^5}\times\frac{10^2}{10^2};$$

which is $\frac{S\ 10^7}{10^2\ S\ 10^5}$.

By precisely the same reasoning, if there had been 1000 balls in the lottery, and if 157 had been drawn white, the probability that 27 more drawings would have given white balls, would have been

$$(157 + 27 = 184) \qquad \frac{S\ 1000^{184}}{1000^{27}\quad S\ 1000^{157}}$$

The difficulty of calculating S 1000^{184} is insuperable: but a mathematical theorem which we shall proceed to explain, makes it very easy to find a near approximation to the preceding result, and the nearer the greater the numbers in question.

Take the sums of the powers of the different numbers, as follows:

First powers	1 + 2 + 3 + 4 + 5 + 6 + ...	$\frac{1}{2}$
Squares	1 + 4 + 9 + 16 + 25 + 36 + ...	$\frac{1}{3}$
Cubes	1 + 8 + 27 + 64 + 125 + 216 + ...	$\frac{1}{4}$
Fourth powers	1 + 16 + 81 + 256 + 625 + 1296 + ...	$\frac{1}{5}$
Fifth powers	1 + 32 + 243 + 1024 + 3125 + 7776 + ...	$\frac{1}{6}$

and examine the sum of any number of terms in any line, as compared with the term immediately below the last in the sum; thus: —

$$1+2+3+4 \qquad\qquad 1+16+81+256+625$$
$$16 \qquad\qquad\qquad\qquad 3125$$

Form fractions with such sums as numerators, and their compared terms as denominators, and observe how much each fraction, so formed, differs from the fraction written in the last column, as follows: —

$$\frac{1+2}{4}=\frac{3}{4}=\frac{1}{2}+\frac{1}{4} \qquad \frac{1+2+3}{9}=\frac{6}{9}=\frac{1}{2}+\frac{1}{6}$$

$$\frac{S4}{16}=\frac{10}{16}=\frac{1}{2}+\frac{1}{8} \qquad \frac{S5}{25}=\frac{15}{25}=\frac{1}{2}+\frac{1}{10}, \text{ and so on,}$$

$$\text{whence } \frac{\text{S (any number)}}{\text{square of that number}}=\frac{1}{2}+\frac{1}{\text{twice that number}}$$

Hence it follows, that when the number is large, the preceding fraction is very nearly one half, or 1 + 2 + 3 + ... up to a large number, is very nearly one half the square of that number.

Again :— $\frac{1+4}{8}=\frac{5}{8}=\frac{1}{3}+\frac{7}{24}$ $\qquad$ $\frac{1+4+9}{27}=\frac{1}{3}+\frac{5}{27}$

$\frac{S\ 4^2}{64 \text{ or } 4^3}=\frac{30}{64}=\frac{1}{3}+\frac{13}{96}$ $\qquad$ $\frac{S\ 5^2}{125 \text{ or } 5^3}=\frac{1}{3}+\frac{8}{75}$

$\frac{S\ 10^2}{10^3}=\frac{385}{1000}=\frac{1}{3}+\frac{155}{3000}$; and so on.

In this way it appears that the sum of all the squares of numbers is nearly one third of the cube of the last number, and that the greater the number of squares taken, the greater the proximity in question. This proposition is general, namely, that the sum of the nth powers of numbers is nearly the $(n+1)$th part of the $(n+1)$th power of the last of the numbers: thus, the sum of all the 13th powers $1^{13}+2^{13}+\ldots$ up to 1000^{13}, is very nearly the 14th part of 1000^{14}. This proposition, never absolutely true, may be made as near the truth as we please, by taking the number of terms sufficiently great; and the error made by the substitution, is nearly such a fraction of the whole as has one more than the index of the power for its numerator, and twice the number stopped at for its denominator. Thus, if the tenth powers of all numbers were summed up to $10{,}000^{10}$, the substitute for this sum given by the theorem, namely, $\frac{1}{11}$ of $10{,}000^{11}$, would be too small by about

$\frac{10+1}{2\times 10{,}000}$ or $\frac{11}{20{,}000}$ of the whole.

Problem. A lottery contains 10,000 balls, each of which may be white or black. A ball is drawn and then replaced, and 100 such drawings give nothing but white balls: what is the chance that the five next drawings shall all be white?

This chance, by what precedes, is $\frac{S\ 10{,}000^{100+5}}{10{,}000^5\ S\ 10{,}000^{100}}$

But $S\ 10{,}000^{100+5}=\frac{1}{106}\times 10{,}000^{106}$ very nearly

$S\ 10{,}000^{100} = \frac{1}{101} \times 10{,}000^{101}$........

Consequently, the chance is $\frac{\frac{1}{106} \text{ of } 10{,}000^{106}}{\frac{1}{101} \text{ of } 10{,}000^{106}}$, or $\frac{101}{106}$, nearly.

If the number of balls had been a million, instead of 10,000, the preceding odds, namely, 101 to 5, would still more nearly have represented the chance that after 100 drawings, all white, the next five should be white also. If the number of balls had been absolutely unlimited, the preceding odds will *correctly* express that same chance. But a lottery with an unlimited number of balls, each of which may be either white or black, is a lottery which may be anything whatever. For instance: [3 W, 7 B] is equivalent to an unlimited lottery, in which for every 7 black balls there are 3 white ones. Again, an unlimited lottery in which any number of balls may be black, and the rest white, is one in which the chance of drawing a white ball *may be any whatever, and is absolutely unknown.* The drawing of a white ball from such a lottery may be likened to the occurrence of an event, about the preceding chances of which we are in total ignorance. The preceding process furnishes us with the following theorem. When an event which may, for any thing we can see to the contrary *beforehand,* happen in either of two different ways, happens one way m times in succession, it is $m + 1$ to n that it shall happen n times more in the same way, if it happen n times more at all. Thus, suppose a person on the bank of a river, not knowing in what country he is, and not having the smallest reason to know whether the vessels which come up the river carry flags or not: the first ten ships which come up all carry flags ($m = 10$); then it is $10 + 1$ to 3, or 11 to 3 that the next three ships shall carry them, and $10 + 1$ to 1, or 11 to 1 that the next ship shall carry a flag. And it is always $m + 1$ to 1, that an event which has occurred in one out of two possible ways m times in succession, shall happen the same way on the next occasion.

The preceding affords some view of the way in which chances are obtained, in cases where the antecedent probability of the events stated may be any whatever. The following are conclusions upon the same subject, obtained by a more complicated reasoning of the same kind.

If an event, each repetition of which may be either A or B, have happened $m+n$ times, and if A have occurred m times, and B n times; then it is $m+1$ to $n+1$ that the next event shall produce A, and not B. And in the same case the chance that out of $p+q$ events to come, p shall produce A, and q shall produce B, is (see pages 15 and 16, for explanation of [ ]).

$$\frac{[p+q] \times [m+1, m+p] \times [n+1, n+q]}{[p] \times [q] \times [m+n+2, m+n+p+q+1]}:$$

and $\dfrac{[m+1, m+p]}{[m+n+2, m+n+p+1]}$, and $\dfrac{[n+1, n+q]}{[m+n+2, m+n+q+1]}$

are the chances that in p new events, all shall give A and that in q new events all shall give B.

Example. In a lottery containing an unlimited number of balls, in which the proportion of black and white is absolutely unknown, six drawings give four white and two black; what are the chances that four drawings more shall give all white, or one only black, or two only black, &c.

Let us first take the case of three white and one black: here $m = 4$, $n = 2$, $p = 3$, $q = 1$.

$$[p+q] = 1.2.3.4, [m+1, m+p] = 5.6.7,$$
$$[n+1, n+q] = 3\ [p] = 1.2.3. \quad [q] = 1,$$
$$[m+n+2, m+n+p+q+1] = 8.\ 9.\ 10.11.$$

Chance required is $\dfrac{1.2.3.4. \times 5.6.7. \times 3.}{1.2.3. \times 1. \times 8.9.10.11.} = \dfrac{7}{22}$

That two shall be white and two black ($m = 4$, $n = 2$, $p = 2$, $q = 2$), the chance is

$$\frac{1.2.3.4. \times 5.6. \times 3.4.}{1.2. \times 1.2. \times 8.9.10.11.}, \text{ or } \frac{3}{11}$$

That one shall be white and three black ($p = 1$, $q = 3$), the chance is

$$\frac{1.2.3.4 \times 5 \times 3.4.5}{1 \times 1.2.3 \times 8.9.10.11}, \text{ or } \frac{5}{33}$$

That all shall be white, or all black ($p = 4$, $q = 4$, second and third formulæ), the chances are

$$\frac{5.6.7.8}{8.9.10.11}, \text{ or } \frac{7}{33} \text{ and } \frac{3.4.5.6}{8.9.10.11}, \text{ or } \frac{1}{22}$$

and the verification of the whole is

$$\frac{7}{22} + \frac{3}{11} + \frac{5}{33} + \frac{7}{33} + \frac{1}{22} = 1,$$

which must be, since one or other of the cases considered must happen.

When it is known beforehand that either A or B *must* happen, and out of $m + n$ times A has happened m times, and B n times, then (page 65.) it is $m+1$ to $n+1$ that A will happen the next time. But suppose we have no reason, except what we gather from the observed event, to know that A or B must happen; that is, suppose C or D, or E, &c. might have happened: then the next event may be either A or B, or a new species, of which it can be found that the respective probabilities are proportional to $m + 1$, $n+1$, and 1; so that though the odds remain $m+1$ to $n + 1$ for A rather than B, yet it is now $m + 1$ to $n + 2$ for A against either B or the new event. Thus, suppose a game at which one party or the other must win, and suppose that out of 20 games A has won 13 and B 7: and this is all we know of the game or of the players. Then, it is $13+1$ to $7+1$, or 14 to 8, or 7 to 4, that A shall win the 21st game. But suppose that it is possible to have a drawn game; then there is some chance that the 21st may be a drawn game, though but a small one, as might be inferred from such a thing never happening in 20 trials. The 21st game may be either A's or B's, or drawn: of which the chances are as $13+1$, $7+1$, and 1; or as 14, 8, and 1. Consequently, though in the preceding case it was 14 out of 22 in favour of A's winning, it is now 14 out of 23, and 1 chance out of 23 remains for the next game being drawn.

When a number of different events have happened, A, B, C, &c., write down each number increased by 1, and the results will express the several relative probabilities, on the supposition that no events can happen except those which have happened. But if new events may happen, write down 1 for the relative probability of such an occurrence at the next trial. Thus, if out of a box, and in 100 drawings, there have appeared 49 white balls, 37 red, and 14 black; then if it be known that nothing but white, red, or black can appear, consider the chances of these to be as 50, 38, and 15; that is, $\frac{50}{103}$ is the chance of drawing a white ball at the 101st trial. But if another sort of ball may appear, then the chances of the four cases being as 50, 38, 15, and 1, it follows that $\frac{50}{104}$ is the chance of a white ball at the 101st trial.

In judging of future events by those which have passed, we must be extremely cautious always to preserve the same method of considering the event proposed. If, for instance, in 100 trials, A has appeared 49 times, B 37 times, and C 14 times, we know that there is one chance out of 104 that the 101st drawing shall give neither A, B, nor C, but something else. What the new character may possibly be is left unknown; it may be another letter, or it may be a number, a picture, or a blank. Are we to understand that all these are equally probable? Common sense tells us the contrary; experience makes us feel it much more likely that the letter D should appear at the 101st trial, than any stated number or picture. But we have now changed the question, and, dropping the distinction between A, B, and C, have considered them merely as letters. Having drawn a letter 100 times running, we are to infer (page 64.) that it is 101 to 1 in favour of our drawing a letter at the 101st trial; or that $\frac{101}{102}$ is the chance of this. But it is already 103 to 1 that the next drawing shall be, not merely a letter, but one of the letters A, B, and C: that is, to all appearance, we have this strange result; — the chances of drawing one of the three, A, B or C, are greater than those of drawing one of the set, A or B, or C or D, &c. &c. up to Z. This paradox will afford me a good opportunity of again inculcating the maxim, that the probability of an event is the presumption drawn from certain obvious principles, as to what the state of our minds ought to be with regard to belief in the happening of that event, as influenced by our knowledge of previous events. Consequently, if John know that A, B, and C have been drawn 49, 37, and 14 times, and nothing more, he has reasonable ground, *with his knowledge*, for assenting to the proposition that the 101st trial shall give either A, B, or C, as to a proposition which has $\frac{103}{104}$ of probability, or 103 to 1 in its favour. But if Thomas only know that 100 drawings have all given letters, then he, *with his knowledge*, has no ground of inference with respect to A, B, and C, in particular, but may reasonably assent to the proposition, "the 101st drawing will also give a letter," as having a probability of $\frac{101}{102}$, or 101 to 1 in its favour. But the paradox in question requires that John should make believe he knows no more than Thomas, and then be surprised that a discordance should arise from his using that knowledge in *reconsidering* a result, which he has suppressed in the method of *attaining* it.

It very rarely happens that we meet with a case in which we can so distinctly specify the antecedent circumstances which influence our assent or dissent, as to enable us to apply direct calculation to the determination of the rational probability of an event to come. And in most of the practicable cases, the largeness of the numbers employed would check our progress, if it were not for the approximative methods of the higher mathematics, and the tables at the end of this work. I now proceed to the method of using those tables.

CHAPTER IV.

USE OF THE TABLES AT THE END OF THIS WORK.

I HAVE endeavoured to accumulate in this chapter a considerable part of the uninteresting details of computation which accompany the solution of complicated problems. It is at the readers' pleasure to omit the whole of it, referring to it afterwards in cases where its assistance may be necessary.

In Table I. we see — (I.) a column headed t, containing the series ·00, ·01 ..... ·99, 1·00, ... 2, or every hundredth of a unit from 0 to 2 —(II.) a column headed H, deduced in the manner pointed out in page 17, — (III.) columns headed Δ and Δ^2, which are only the differences of the numbers in column H (marked Δ), and the differences of those differences, (marked Δ^2). The following is an extract from the table: —

t	H	Δ	Δ^2
·47	·49374 52	900 46	8 61
·48	·50274 98	891 85	......
·49	·51166 83	....	......
	....	....	......

The columns Δ and Δ^2 must be made up to 7 places of decimals * by means of ciphers: thus, 90046 means ·0090046; and 861 means ·0000861. The formation of Δ and Δ is then as follows: —

	·5027498	·5116683	·0090046
	·4937452	·5027498	·0089185
Subt.	·0090046	·0089185	·0000861

* A little practice will show how to dispense with the decimal points altogether till the end of the process.

Since it is very seldom necessary to use more than five places of the table marked t, the sixth and seventh places, and those which arise from them in the differences, are separated from the rest by a blank space. The sixth and seventh places are allowed to remain, on account of the use which will hereafter be made of the differences derived from them.

In Table II. we find to five places only a column marked K, and another marked Δ, containing the differences of the former column. This is a modification of the former table, the reason of which will hereafter appear. In the meanwhile, however, observe that we can directly find the value of H and K by these tables only, when t is ·00 or ·01, &c.; that is, when t is a given decimal of two places. But supposing it required to find H when t lies between two of the values in the table; suppose, for instance, we ask what is H when t = ·47694? The method is as follows:

Question. What is the value of H (Table I.) when t = ·47694, correct to five places of decimals?

Rule.	Exemplification.
Take out of Table I. the value of H answering to the two first decimal places and the whole number preceding them, if there be one. Retain only five places of decimals.	Opposite to ·47 we find ·49375 *
Take the figures of the first difference (as far as the blank space), and multiply them by the remaining figures in the value of t, and cut away as many places from the result as there were remaining figures.	Three figures remaining, 694. 900 x 694 = 624600. Cut away three figures, 625. *
Add the figures in the last result to the right hand of the first, and the sum is the answer required.	·49375 625 ·50000 When t = ·47694, H = ·50000.

* Whenever decimal figures are rejected, if the first rejected be five or upwards, the last retained is increased by a unit.

As another example, suppose the value of t to be 1·51209.

When t=1·51,	H=.96728	Δ=114
	24	209
		1026
When t=1·51209,	H=·96752	228
		23,826

We shall now take the inverse problem, and supposing H to be intermediate between two values in the table, require the value of t. For instance, let H = ·93972.

Rule.	Exemplification.
Find in the table the value of H next below the given value; note the corresponding value of t, and subtract the nearest value of H from the given value.	H ·93972, next below in the table } ·93807 t=1·32 165 Tabular diff. 195.
Annex three ciphers to the difference just found, and divide by the figures of the difference in the table which come before the blank space, rejecting fractions, and taking the nearest whole number.	195)165000(846 1560 900 780 1200 1170 30
The quotient cannot have more than three places: if three, annex to the value of t already found; if less than three, place ciphers at the beginning to make up the deficiency, and annex.	t=1·32 846

Let H = ·97169; required the value of t?

```
             H = ·97169
Nearest below = ·97162  ........... t = 1·55
                ------
                101)7000(69        t = 1·55069  Answer.
                     606
                     ---
                      940
                      909
                      ---
                       31
```

The Table II. is used in exactly the same way, except towards the end, from t = 3·40 upwards; in which case the cipher at the end of t must be neglected, and only one decimal place taken out of the table in the value of t. For instance, to find the value of t answering to K = ·98222.

```
             K = ·98222
Next below       ·98176  ........... t = 3·5
                 ------
                 306)46000(150       t = 3·5150
                      306
                      ----
                      1540
                      1530
                      ----
                       100
```

In the first table, there is another result which will frequently be wanted, and which I shall call H′. It arises from adding half the second difference to the first difference*, if the value of t be in the tables, and making five decimal places. But if the value of t be not in the tables, then H′ must be formed for the values of t immediately above and below; and by means of the first and the difference of the two, H′ must be found in exactly the same manner as H is found in the first of the preceding rules, page 70., *remembering to subtract at the last step instead of adding*, if the second H′ thus previously determined be less than the first.

* Meaning the whole differences; not the parts which precede the blank space, as in the preceding rules.

EXAMPLE 1. When t is 1·56, what is H′?

$$\Delta = 9745$$
$$\tfrac{1}{2}\Delta^2 = 151$$
$$H' = \cdot 09896$$

EXAMPLE 2. When t = 1·23412. what is H′?

t = 1·23	H′ = ·24852	Diff. 607
t = 1·24	H′ = ·24245	

412	·24852
607	250
2884	Subt. ·24602 = H′ when t = 1·23412
2472	
250,084	

We have already seen that when two events, A and B, one of which must happen at every trial, have severally happened m times and n times in $m + n$ trials, it is $m + 1$ to $n + 1$ that A shall happen at the next trial. But $m + 1$ to $n + 1$ is very nearly m to n, when m and n are considerable numbers: for instance, 248 to 117 is very nearly 247 to 116. That is, when a great many trials have been made, the numbers of times which A and B have happened express very nearly the odds (relative probabilities) for A against B; or, inverted, for B against A. Let us convert the problem, and supposing that we know beforehand the chances of A and B, are we to suppose that in a great many trials A and B will happen in proportion to their respective probabilities? Common sense tells us that such will always be *nearly* the case, but that the odds are great against an exact result. Suppose 3000 drawings to be made from a lottery containing two As and one B. We must then, it seems clear, draw A twice as often as B, *in the long run.* Our reason convinces us thus. Let one of the As be distinguished from the other by an accent, so that we have A, A′, and B. If the urn be well shaken before each drawing, it is impossible to believe that, in the whole result of 3000 trials, we shall have drawn the three in very unequal numbers; so that, destroying the distinction between A and A′, we feel secure of drawing A twice as often as B; and it is obviously two to one in favour of A at each trial. The following phrases seem to common sense to mean the same thing.

It is two to one that A shall happen, and not B.	In the long run, A will happen twice as often as B.
It is an even chance for head or tail.	In a large number of tosses, the heads and tails will occur in nearly equal numbers.
It is more than a hundred to one, that a ship at sea will not be lost.	Of all the ships which sail, the number which is not lost exceeds that which is lost more than a hundred times.

I now proceed to some problems, which will exhibit the method of applying the tables, and will illustrate and confirm the preceding notions.

PROBLEM I. The odds for A against B being a to b, to find the chance that in n times $a + b$ trials, A shall happen exactly $n \times a$ times, and B $n \times b$ times.

RULE. Divide the H′ belonging to $t = 0$ (page 72) by the square root of the following: 8 times the product of n, a, and b, divided by $a + b$.

Suppose, for instance, a die is thrown 6000 times; what is the chance that exactly 1000 of the throws shall give an ace? Here it is 1 to 5 that an ace shall be thrown in any one trial, and 6000 is 1000 times $1 + 5$. Hence $a = 1$, $b = 5$, $n = 1000$: 8 times the product of n, a, and b is 40,000, the sixth part of which is 6667 (sufficiently near), and the square root of this is 81·65. Again, when $t = 0$, we have in Table I.

$$\Delta = 112833,\ \tfrac{1}{2}\Delta^2 = 11;\ \text{whence H}' \text{ is } 1{\cdot}12844$$

and 1·12814 divided by 81·65 gives ·014 very nearly. This is very near the real probability that 6000 throws with a die shall give exactly 1000 aces: for such an event there are only 14 chances out of a thousand; and it is 1000 − 14 to 14, or about 70½ to 1, against the event. This result is rather above what we should have expected; we might have imagined it to be more than 71 to 1 against 6000 throws giving exactly 1000 aces.

As another example, let us find the probability that, out of 200 tosses with a halfpenny, there shall be exactly 100 heads and 100 tails. Here $a = 1$, $b = 1$, $n = 100$, and $8n\,a\,b$* is 800, which divided by $a + b$, or 2, gives 400, the square root of which is 20. And H′, when $t = 0$ (or 1·12844) divided by 20, gives ·056. It is therefore about 944 to 56, or 17 to 1, against the proposed event; and (page 42.) we must repeat 200 throws 12 times to have an even chance of the equality of heads and tails happening once.

Generally speaking, the rules in this chapter are very accurate only when the number of trials is considerable. Suppose only 12 tosses; required the chances of 6 heads and 6 tails. Here $a = 1$, $b = 1$, $n = 6$, $8nab = 48$, which divided by 2 gives 24, whose square root is 4·9 very nearly. And 1·12844 divided by 4·9 gives ·23, or 77 to 23, that is $3\frac{8}{23}$ to 1 against the event. That is (page 47), this rule is not very inaccurate, even when the number of trials is as low as 12.

We shall call the event whose chance is sought in the preceding problem, the *probable mean;* understanding by that term the event which is more likely to happen than any other. Thus, when 12 halfpence are thrown up, 6 heads and 6 tails is the probable mean, being the event which is more likely than any other, though not in itself more likely than not. When 6000 throws are made with a die, the probable mean is 1000 aces, 1000 deuces, &c.

Problem II. The odds for A against B being a to b, required the chance that in n times $a + b$ trials, the As shall fall short of the probable mean by a given

* Juxtaposition of numbers, in algebra, stands for their product.

number l: *l being small, compared with the whole number of As in the probable mean.*

Rule. Divide twice l by the square root obtained in the last example; and find the value of H′ from Table I., taking the preceding quotient for t. Divide H′, so found by the square root just used, and the quotient is the answer required.

N.B. This rule also applies when the number of As is to *exceed* the probable mean by l.

Example I. In 6000 throws with a die, what is the chance that the aces shall fall short of (or exceed) 1000 by exactly 50? Here $a = 1$, $b = 5$, $n = 1000$, and the square root is 81·65, as before. And twice l, or 100, divided by 81·65, gives 1·22: to which the value of H′ is 25162 + $\frac{1}{2}$ of 611, with five decimal places, or ·25468. This last, divided by 81·65, gives ·0031; so that it is about 997 to 3, or 332 to 1, against the proposed event: and the 6000 throws must be repeated 232 times to give an even chance of succeeding once.

Example II. What is the chance that in 200 tosses, there shall be exactly 95 heads? Here $a = 1$, $b = 1$, $n = 100$, $l = 5$, and the square root, as before, is 20. And twice l, or 10, divided by 20, gives ·50, which being t, the value of H′ is ·87882, which divided by 20 gives ·044 very nearly. It is, then, 956 to 44. or about 22 to 1, against the proposed event.

Example III. In 12 tosses, what is the chance of exactly 7 heads? Here $a=1$, $b=1$, $n=6$, $l=1$, the square root, as before, is 4·9, and 2 divided by 4·9 is ·41 nearly; which being t, H′ is ·95384, which divided by 4·9 gives ·194. It is therefore 806 to 194, or $4\frac{2}{13}$ to 1, against the proposed event. In page 47. it is 3304 to 792 against this event, or $4\frac{1}{6}$ nearly. Hence the incorrectness of our rule is very small.

Problem III. The odds for A against B being a to b, required the chance that in n times $a + b$ throws, the number of As shall not differ from the probable mean by more than l.

Rule. Divide one more than twice l by the square root already mentioned, and the quotient being made t, the value of H in Table I. is the probability required.

Example I. In 6000 throws with a die, what is the chance that the number of aces shall not differ from 1000 by more than 50; that is, shall lie between 950 and 1050, both inclusive. Here $a = 1$, $b = 5$, $n = 1000$, $l = 50$, and the square root as before is 81·65. Divide $2l + 1$, or 101, by 81·65, which gives 1·237, which being t, H is ·91977. Hence it is 920 to 80 in favour of the proposed event, or about $11\frac{1}{2}$ to 1.

Example II. In 200 tosses, what is the chance that the number of heads shall lie between 97 and 103, both inclusive? Here $a = 1$, $b = 1$, $n = 100$, $l = 3$, and the square root, as before, is 20. And $2l + 1$, or 7, divided by 20, gives ·35, which being t, H is ·379. Hence it is 621 to 379, or about 31 to 19, against the proposed event.

Example III. In 12 tosses, what is the chance of the heads being either 5, 6, or 7 in number? Here $a = 1$, $b = 1$, $n = 6$, $l = 1$, and the square root, as before, is 4·9. And $2l + 1$, or 3, divided by 4·9, gives ·61, which being t, H is ·6117. Hence it is about 612 to 388, or 153 to 97, in favour of the proposed event. In page 47 the chance of this event is

$$\frac{792+924+792}{4096} \text{ or } \frac{2508}{4096} \text{ or } \cdot 612$$

Problem IV. The odds for A against B being a to b, and n times $a + b$ trials being to be made, for what number is there a given probability H that the As shall not differ from the probable mean by more than that number?

Rule. Find in Table I. the value of t answering to that of H (page 71), multiply it by the square root already described, subtract 1, and divide by 2: the quotient, or its nearest whole number, is the answer required.

Example. In 6000 throws with a die, within what limits is it two to one that the aces shall be contained? The square root is 81·65, and H is $\frac{2}{3}$ or ·66667, to which the value of t is ·68409, found as follows (page 71): —

	H = ·66667		
Nearest below	·66378	t = ·68	
Tab. Diff. 706)	289000(409	t = ·68409 say	·6841
	2824		81·65
	6600		34205
	6354		41046
	246		6841
			54728
			55·856765
			1·
			2)54·86
			27·43 = *l*.

Answer. It is a little more than 2 to 1, that the aces shall lie between 1000 − 28 and 1000 + 28, and a little less than 2 to 1 that they shall lie between 1000 — 27 and 1000 + 27.

But the most convenient way of solving this problem is by first finding for what degree of departure from the probable mean there is *an even chance.* In this case, since H = ·5 (page 70), t is = ·476936, which the method in page 41, will show to be very nearly $\frac{31}{65}$. It will be worth while to re-state the whole process.

The odds for A against B being *a* to *b*, and the proposed number of trials being *n* times *a* + *b*, required the limits of departure from the probable mean *n a*, within which it is an even chance that the number of As shall be contained.

Rule. Multiply together 8, *n*, *a*, and *b*, and divide by

$a + b$: extract the square root of the quotient, and multiply it by 31 : subtract 65, and divide by 130 : the nearest whole number is the answer required. Thus in the preceding instance, where the square root is 81·65, multiply this by 31, which gives 2531·15 ; and 65 less is 2466·15, which divided by 130 gives 18·97. Hence it is very little less than an even chance that the aces in 6000 throws shall be between 1000 + 19 and 1000 − 19, or 1019 and 981.

Having found the limits of departure for which there is an even chance, we can now use Table II. as follows. The values of t in Table II. are the proportions of various departures (each increased by ·5) to that departure which has an even chance, as just ascertained, and also increased by ·5 : the values of K are the probabilities of the departures answering to those of t. Having then ascertained 18·97 to be the departure for which there is an even chance, suppose I ask what is that limit of departure within which it is two to one that the aces shall be contained. Two to one gives $\frac{2}{3}$ for the chance, or ·66667 : I look into Table II., and find that when K is ·66667, t is 1·43433, found as follows:—

```
             K=      ·66667 t=1·43
    Next below       ·66521
                    -------
 Tab. Diff. 337)  146000(433 t=1·43433
                  1348
                  ----
                   1120
                   1011
                   ----
                    1090
                    1011
                    ----
                      79
```

This is the proportion which the departure in question, increased by ·5, bears to 18·97 increased by ·5 or 19·47. Multiply 1·43433 by 19·47, giving 27·93 ; from which subtract ·5, giving 27·43 for the limit of departure, the same as in page 78.

Suppose the question to be that of page 77, namely, what is the probability that the number of aces in 6000 throws shall lie within 50, one way or the other, of the probable mean 1000? Now, 18·97 + ·5 is 19·47, and 50 + ·5 is 50·5, and 50·5 divided by 19·47 gives 2·594, which being t (in Table II.), K is ·91981, extremely near to the result in page 77.

There are, therefore, two distinct methods of treating these problems, connected with the two tables: and this is a great advantage, since it is a very strong presumption of a correct answer, when the results of the tables agree. The problems III. and IV. being of great importance, I shall now recapitulate their details, with the addition of some new phraseology. Let the instance be 6000 throws of a die, and the event A the arrival of an ace, and B the arrival of some other face. The most probable number of aces is 1000, though the arrival of that exact number is not probable in the common sense of the word. There will then most likely be a departure from the number 1000 in the number of aces thrown; of which departure we are now entitled to say, that it is very improbable it should be considerable. Let the term *neutral* departure mean that degree of departure for which it is just an even chance that the actual event shall be contained within its limits: in the present instance it is 18·97. We may explain the fraction as follows: suppose a person to receive 100*l.* for every unit by which the number of aces falls short of or exceeds 1000. Then, supposing him to try this stake a great many times, he will in the long run receive less than 1897*l.* at a trial, as often as he receives more. But his receipts will oftener exceed than fall short of 1800*l.*; while they will oftener fall short of than exceed 1900*l.* Roughly speaking, there is here the same probability that the aces shall *not* lie between 1000 − 19 and 1000 + 19 (both inclusive), and that they *shall* lie between these numbers.

In all these problems there is a square root to be found, which we call *the* square root, as there is no other

The odds are a to b, for A against B, and $n\ (a + b)$ trials are contemplated. Though we have only instanced whole values of n, yet it may be a fraction: thus, if the odds are 3 to 2, and 96 trials are contemplated, $n\ (3 + 2)$ must be 96, or n must be $19\frac{1}{5}$. In this case, the probable mean is that A shall happen $57\frac{3}{5}$, and B $38\frac{2}{5}$; by which it must be understood, that a person who should repeat 96 throws a great many times, receiving 1*l.* for every A, would, in the long run, gain on the average $57\frac{3}{5}$*l.* per trial of 96 throws.

The square root in question, represented algebraically, is

$$\sqrt{\frac{8nab}{a+b}}$$

or the square root of the product of 8, n, a, and b divided by $a + b$. I now subjoin the two principal problems, with the two rules in parallel columns.

PROBLEM. What is the chance that the number of As in $n(a+b)$ trials shall lie between $na + l$ and $na - l$, both inclusive? or what is the chance that the departure from the probable mean shall not exceed l?

By Table I.	By Table II.
Find *the* square root, and divide one more than twice l by it; call the result t, and find H answering to t in the table. (Use the rule in p. 70. if necessary.) This H is the probability required.	Find the square root, and multiply it by 31; then divide by 130. To l add ·5, and divide by the preceding quotient; call the result t, and find the value of K answering to t: this is the probability required. N. B. The neutral departure is ·5 less than the quotient first found.

PROBLEM. What is that degree of departure within which it is p to q that the number of As in $n\ (a + b)$ trials shall lie?

By Table I.	By Table II.
Divide p by $p+q$, and calling the result H, find the corresponding value of t in the table. Multiply it by the square root, subtract 1 and divide by 2; the quotient being called l, it is then p to q that the As in $n\,(a+b)$ trials shall be contained between $na-l$ and $na+l$, both inclusive.	Find the square root; multiply by 31, and divide by 130. Divide p by $p+q$, and calling the quotient K, find the corresponding value of t in the table; multiply t by the preceding quotient, subtract ·5 from the product, and l being the remainder, it is then p to q that the As in $n\,(a+b)$ trials shall be contained between $na-l$ and $na+l$, both inclusive.

I shall conclude this problem with an example of each case, worked by both methods, without explanation. There is a lottery containing 3 white and 2 black balls: what is the chance that in 50,000 drawings the number of white balls shall be between 30,000 + 100 and 30,000 - 100?

$a=3$ $b=2$, $n=10{,}000$, $l=100$	309·84
$\dfrac{8\times3\times2\times10{,}000}{5}=96{,}000$	31
$\sqrt{96{,}000}=309{\cdot}84$	130)9605·04(73·885
309·84)201(·64873 = t	73·885)100·5(1·3602 = t
H = ·64109	K = ·64109

This question shows how nearly a great many trials may be expected to agree with the probable mean: in 50,000 trials, it is nearly two to one against the number of white balls differing from 30,000 by more than a hundred.

In 100,000 tosses, between what limits is it 99 to 1 that the heads shall be contained?

$a=1$, $b=1$, $n=50{,}000$, $p=99$, $q=1$	447·21
100)99(·99 = H	31
t = 1·8215	130)13863·51(106·64
$\dfrac{8\times1\times1\times50{,}000}{2}=200{,}000$	100)99(·99 = K

$\sqrt{200,000} = 447{\cdot}21$

$$\begin{array}{r} 1{\cdot}8215 \\ 447{\cdot}21 \\ \hline 814{\cdot}6 \\ 1 \\ \hline 2)813{\cdot}6 \\ 406{\cdot}8 = l \end{array}$$

$t = 3{\cdot}821$

$$\begin{array}{r} 106{\cdot}64 \\ 3{\cdot}821 \\ \hline 407{\cdot}47 \\ {\cdot}5 \\ \hline 406{\cdot}97 = \end{array}$$

Answer. Between 50,000 − 407 and 50,000 + 407.

Now for the inverse method attached to the preceding problem. If I be totally unacquainted with the nature of the events A and B, except only that one or other, and not both, must happen every time, it is then clear that, as the matter stands, it is to me 1 to 1 for A against B, with a very great chance that, if I were better informed, I should form a different opinion. At the same time (page 10), I choose 1 to 1 as my rule of action, because, though coming events may not justify my prediction, I know of nothing to warrant my assuming that the odds are in favour of A, rather than in favour of B. A trial takes place, and A happens; it becomes immediately most safe to assume that the odds for A against B are 2 to 1, but still that safety is not very decided. But if 1000 trials be made, and if A have happened 520 and B 480 times, I can then confidently say, that the odds for A against B are very nearly, if not exactly 520 + 1 to 480 + 1, which is nearly 520 to 480. The notion then formed has a strong presumption that it is nearly correct.

Problem. In $a + b$ trials A has happened a times and B b times: from which, if a and b be considerable numbers, it is safe to infer that it is a to b nearly for A against B. What is the presumption that the odds for A against B really lie between $a - k$ to $b + k$ and $a + k$ to $b - k$?

RULE. (TABLE I.)

Divide twice the product of a and b by their sum, and extract the square root of the quotient, by which divide k. Then the last quotient being t, the H of the table is the probability required.

RULE. (TABLE II.)

Having found the square root, and divided k by it, as opposite, from seven times the quotient, take the hundredth part of the quotient, and take three tenths of the remainder. Make the result t, and K in the table is the probability required.

Suppose that in a thousand trials, A has happened exactly 600 times, and B 400 times; what is the presumption that the odds for A against B lie between 570 to 430 and 630 to 370?

$a=600,\ b=400,\ k=30.$

$$2\times 600\times 400=480{,}000$$

$$\frac{480{,}000}{1000}=480$$

$$\sqrt{480}=21{\cdot}91$$

$$\frac{30}{21{\cdot}91}=1{\cdot}369=\text{t}$$

$$\text{H}={\cdot}94713$$

$$\begin{array}{r} 1{\cdot}369 \\ 7 \\ \hline 9{\cdot}583 \\ {\cdot}014 \\ \hline 9{\cdot}569 \\ 3 \\ \hline 28{\cdot}707 \end{array} \quad \text{t}=2{\cdot}871 \quad \text{K}={\cdot}94719$$

Answer. About 95 to 5, or 19 to 1 in favour of the odds being between the limits specified.

In the preceding problem, A and B have happened a and b times; whence the most likely of all individual cases is, that the odds for A against B are a to b; or, in other words, the result which has the strongest presumption in its favour is, that

Probability of A was $\frac{a}{a+b}$. Probability of B was $\frac{b}{a+b}$.

Now we have found, in the preceding problem, the presumption that

Probability of A lies between $\frac{a-k}{a+b}$ and $\frac{a+k}{a+b}$;

or, which is the same thing, that

$$\text{Probability of B lies between } \frac{b+k}{a+b} \text{ and } \frac{b-k}{a+b}$$

For since it is our hypothesis, that either A or B must happen at every trial, whatever presumption there is that the chance of A is x, there is the same presumption that the chance of B is $1-x$.

But we might ask the following questions: A and B having happened a and b times in $a+b$ trials, what are the values of the following presumptions?

1. That the probability of A lies between $\frac{a+k}{a+b}$ and $\frac{a}{a+b}$; or its equivalent, that the probability of B lies between $\frac{b-k}{a+b}$ and $\frac{b}{a+b}$.

2. That the probability of A lies between $\frac{a}{a+b}$ and $\frac{a+k}{a+b}$, or that the probability of B lies between $\frac{b}{a+b}$ and $\frac{b-k}{a+b}$.

To solve these by the help of the following rule, remember that, if a be greater than b, it is more likely that the chance of A falls short of $a \div (a+b)$ than exceeds it: and if a be less than b, then it is more likely that the chance of A exceeds $a \div (a+b)$ than falls short of it.

Rule. First find the result of the preceding problem, and find from Table I. the H′ (p. 72) belonging to the value of t. Subtract this from the H′ derived from 0 in the table (which is 1·12844); multiply by the difference between b and a, and divide by the product of the square root used in the preceding problem and three times the whole number of trials: call the result V. To one half of the result of the preceding problem add V; and from it subtract V: and call these results $\frac{1}{2}H+V$ and $\frac{1}{2}H-V$.

Then, if B have happened most times, $\frac{1}{2}H+V$ is the presumption that the chance of A lies between $\frac{a}{a+b}$ and $\frac{a+k}{a+b}$ or that the odds for A against B lie between a to b and $a+k$ to $b-k$. But in this case, $\frac{1}{2}H-V$ is the presumption that the chance of A lies between $\frac{a-k}{a+b}$ and $\frac{a}{a+b}$, or that the odds for A against B lie between $a-k$ to $b+k$ and a to b.

But if A have happened most times, make $\frac{1}{2}H+V$ and $\frac{1}{2}H-V$ change places in the preceding paragraph, every thing else remaining the same.

Example. In a thousand trials, A has happened 600 times, and B 400 times. What is the presumption, 1. that the odds for A against B lie between 600 to 400 and 630 to 370; 2. that the odds for A against B lie between 570 to 430 and 600 to 400?

From the preceding problem t=1·369, say=1·37; the square root is 21·91, and H is ·9471.

$$
\begin{array}{rrl}
t=1{\cdot}37\ \Delta\ 17036 & 1{\cdot}12844 & b-a=200 \\
\frac{1}{2}\Delta^2\ \ 232 & {\cdot}17268 & \\
\hline
H'={\cdot}17268 & {\cdot}95576 & 3000\times 21{\cdot}91=65730 \\
 & 200 &
\end{array}
$$

$$
\begin{array}{l}
65730)191{\cdot}152({\cdot}0029 = V \\
\qquad\qquad\qquad {\cdot}4736=\tfrac{1}{2}H \\
\qquad\qquad\qquad \overline{{\cdot}4765 =\tfrac{1}{2}H+V} \\
\qquad\qquad\qquad {\cdot}4707 =\tfrac{1}{2}H-V
\end{array}
$$

Hence (since A has happened most times), it is ·4707 to ·5293, or about 47 to 53, that the odds for A against B lie between 600 to 400 and 630 to 370. And it is ·4765 to ·5235, or about 48 to 52, that the odds for A against B lie between 570 to 430 and 600 to 400.

The problems which the preceding part of this chapter has enabled us to solve, are the determination of the chance which exists (under known circumstances) for the happening of an event a number of times which lies between certain limits, and its converse. The latter problem contains a consideration of some difficulty, namely, the *probability of a probability*, or, as we have called it, the presumption of a probability. To make this idea more clear, remember that any state of probability may be immediately made the expression of the result of a set of circumstances, which being introduced into the question, the difficulty disappears. Thus, suppose a large number of urns containing various proportions of black and white balls. Let there be 100 urns, and let one of them only contain equal numbers of black and white balls. If then I lay my hand upon one of these urns with the intention of drawing, it is, before the drawing, 99 to 1 against my having placed my hand upon an urn from which, in the long run, equal numbers of both sorts of balls will be produced: the presumption that black and white balls have an even chance is only $\frac{1}{100}$; the presumption that the probability of a white ball is $\frac{1}{2}$, is $\frac{1}{100}$.

In speaking of compound probabilities, writers have employed six words synonymously—probability, chance, presumption, possibility, facility, and expectation. Rejecting only the word possibility, as indicating a thing of which there cannot be different degrees, the five remaining terms have their advantages, each one pointing out a peculiar and useful view of the main idea. Thus the word presumption refers distinctly to an act of the mind, or a state of the mind, while in the word probability we feel disposed rather to think of the external arrangements on the knowledge of which the strength of our presumption ought to depend, than of the presumption itself. When, therefore, having observed an event, we want to know how strongly we are to suppose that the observed event was preceded by a given arrangement of circumstances, the term presumption of probability is very appropriate. The word facility applies particularly to the notion which we form when we see one event happen more often than another, namely, that it is easier to produce the first than the second. In our problems, however, the facility is not that arising from art, but from previous (it may be accidental) distribution of means. The word expectation will be applied throughout this work to that state of things for the production of which there is an even chance. If (p. 79), 6000 throws be made with a die, it is an even chance that the number of aces lies between 981 and 1019: the odds are against any smaller amount of departure on both sides of the probable mean, and against any greater amount; this is then our *expectation* of the number of aces.

When one of two possible events happens oftener than the other, it being understood that one, and only one, can happen each time, we are led to suppose that the excess of one event is the consequence of some arrangement which would, had we known it, have made us count that event more probable than the other. If A or B must happen, and if in a thousand trials the As outnumber the Bs very much, we feel perfectly certain that such must have been the case. The theory of probabilities confirms this impression, as will appear by the solution of the following

Problem. In $a+b$ trials, the number of As was a, and that of Bs was b. If a exceed b considerably *, required the presumption that there was at the outset a greater probability of drawing A than of drawing B, in any one single trial?

Rule. Divide the difference of a and b by the square root of twice their sum, and let the result be t. Find (page 72) the H′ corresponding to t: multiply the result by the sum of a and b, and divide by the product of 8, t, and the square root of the product of a and b. The result subtracted from unity gives the answer required. Suppose, for instance, that out of 50 trials A occurs 32 times, and B 18 times. Then,

* In order that the result may be *very* correct, a must exceed b so much that the excess of a above b, multiplied by itself, may considerably exceed the sum of a and b.

$$50 \times 2 = 100, \quad \sqrt{100} = 10 \quad \frac{32 - 18}{10} = 1 \cdot 40 =$$

$$H' = \cdot 15891 \quad 15891 \times 50 = 7 \cdot 9455$$

$$\sqrt{32 \times 18} = 24, \ 8 \times 24 \times 1 \cdot 40 = 268 \cdot 8$$

$$7 \cdot 9 \text{ divided by } 268 \cdot 8 \text{ is } \tfrac{8}{269} \text{ nearly: } 1 - \tfrac{8}{269} \text{ is } \tfrac{261}{269}$$

Hence it is about 261 to 8 that A was more probable than B.

Additional Rule. When a and b are nearly equal, find t, as in the last rule; find H (not H′) corresponding to t, add 1, and divide by 2: the result is the probability required.

The additional rule belongs to the more important case of the two, namely, that in which A has not happened so much oftener than B as to justify an immediate conclusion that it was the more probable event of the two. Suppose, for instance, that A has occurred 10,100 times out of 20,000 trials, and B 9,900 times: then t = 200 divided by 200, or 1; to which H is ·843, and this increased by 1, and the result divided by 2, gives ·922. It is, therefore, about 11½ to 1 that A was the more probable.

The preceding solution can be applied to various species of observations; of which we shall see more hereafter. The following may be considered as closely connected with it. If we make two different sets of trials, in circumstances which we suppose to be the same, it will generally happen that the As will not bear the same proportion to the Bs in both sets. If, for instance, we find 1000 As arrive in 2000 trials, the odds are very much against the arrival of exactly 5000 As in a new set of 10,000 trials, though the expectation is that something near that number of As will arrive. Suppose that the first and second sets of trials give—1st, 50 As, 30 Bs; 2nd, 112 As, 61 Bs.

In the second set the As bear a larger proportion to the whole than in the first: and our present question is what presumption thence arises that there is some difference of circumstances between the two sets, which gives A a greater facility in the second than in the first, or a greater probability of being drawn at any one trial? Or, if in a first set of $a + b$ trials, A happen a times and B happen b times; and if in a second set of $a' + b'$ trials, A happen a' times, and B happen b' times; and if a' be a larger proportion of $a' + b'$ than a is of $a + b$; required the presumption that there was a greater chance of drawing A at a single trial in the second set than in the first?

Rule. Divide the cube of the sum of a and b by twice their product: do the same with a' and b': multiply the two results together, and add them together: divide the product by the sum, and extract the square root of the quotient.

Divide a' by $a' + b'$, and a by $a + b$, and subtract the less result from the greater. Multiply the difference by the square root previously found, and let the product be t. Then the H corresponding to t, increased by 1, and divided by 2, is the presumption required.

In the example $a = 50$, $b = 30$, $a' = 112$, $b' = 61$.

$80 \times 80 \times 80 = 512000 \qquad 2 \times 50 \times 30 = 3000, \quad \frac{512000}{3000} = 170{\cdot}7$

$173 \times 173 \times 173 = 5177717 \quad 2 \times 112 \times 61 = 13664 \quad \frac{5177717}{13664} = 378{\cdot}9$

$170{\cdot}7 \times 378{\cdot}9 = 64678{\cdot}23, \; 170{\cdot}7 + 378{\cdot}9 = 549{\cdot}6$

$\frac{64678{\cdot}23}{549{\cdot}6} = 117{\cdot}7, \; \sqrt{117{\cdot}7} = 10{\cdot}85$

$\frac{112}{173} - \frac{50}{80} = {\cdot}6474 - {\cdot}6250 = {\cdot}0224$

${\cdot}0224 \times 10{\cdot}85 = {\cdot}24304 = t, \; H = {\cdot}2689$

$\frac{1}{2}(H + 1) = {\cdot}635$, the probability required; and it is therefore about 16 to 9 in favour of the excess of As at the second set of trials not being accidental fluctuation, but arising from some new circumstance or different arrangement of the old ones.

If, in 1000 trials, A should happen 520 times, and B 480 times, there is strong presumption that in any future number of trials the whole number will be divided among As and Bs nearly in the proportion of 520 to

480. But this is not the same set of circumstances as that of the problem in page 77. We are there supposed to know exactly in what proportion As and Bs are contained in an urn; and with this positive knowledge we can ascertain the probability of drawing any given number of As in a given number of trials. In the present instance we do not know the contents of the urn, but only the result of a certain number of drawings, from which we can draw presumptions, as in page 53. about the whole contents. The determination of chances relative to a new set of trials depends upon two risks in the latter case, and upon one only in the former. The latter problem is therefore more complicated in its principles though not so in its results.

Let us suppose two different persons, John and Thomas, thus situated with respect to the contents of an urn: John knows that there are as many As as Bs; Thomas has observed a hundred successive drawings, of which (so let it have happened) fifty have given A, and as many have given B. That which John knows is rendered not improbable to Thomas by the result of the trials, while the same result would have been thought not unlikely beforehand by John. But there is this difference between their degrees of knowledge, that John has the certainty of a fact (the equality of As and Bs), of which Thomas can only say that the fact, or something near it, is extremely probable. No one could argue with John against any particular venture in such a lottery upon the ground of the *possibility* of the As much exceeding the Bs; while with Thomas it might be urged as possible, though not probable, that the former might exceed the latter a hundred-fold. Again, suppose John and Thomas, having equal fortunes, are disposed to venture as far as produce would warrant, upon the results of a hundred (to Thomas a second hundred) trials. It is obvious to common sense that Thomas must not venture so much as John; for he runs a larger risk, seeing that he assumes as an average result what possibly may have been a rare occurrence.

The following is the rule pointed out by the theory of probabilities:—The expectation of fluctuation should be greater to a person who proposes to try q new instances, upon the assumption that p preceding instances have fairly represented the *long run*, than it should be to another person, who knows in what proportions the As and Bs really exist; and greater in the proportion of the square root of p augmented by q to the square root of p. Thus, if in the preceding case, John and Thomas propose to embark in a matter which depends on 300 more trials, the proportion of the square root of $100+300$ to that of 100, being that of 2 to 1, it follows that, whatever reason John may have to guard against the possibility of 300 drawings giving x more than 150 As, Thomas has as much reason to guard against $2x$ more than the same number.

Problem (to be compared with that in page 77.). When $a+b$ trials have happened to give a As and b Bs, required the chance that in n times $a+b$ *new* throws, the number of As shall not differ from na by more than l.

Rule. Divide one more than twice l by a square root to be immediately mentioned, and the quotient being made t, the value of H in Table I. is the probability required.

The square root in the former problem was that of the product of 8, n, a, and b divided by $a+b$.

The square root in the present problem, is that of the product of 8, n, $n+1$, a, and b divided by $a+b$.

The additional rule in page 81. may also be applied verbatim, the square root now meaning the second square root above given; and the inverse rule (p. 82) may be applied in exactly the same way.

Example. In 600 drawings A occurred 100 times, and B 500 times; what presumption thence arises that in 6000 more drawings A would occur somewhere between $1000-50$, and $1000+50$, or 950 and 1050 inclusive? (See page 77 for the corresponding problem.)

$$n=10,\ a=100,\ b=500;\ n(n+1)\,ab\div(a+b)=73333$$

$$\sqrt{73333}=270{\cdot}8;\quad 2l+1=101;\quad \frac{101}{270{\cdot}8}={\cdot}373=t$$

H = ·4 very nearly, or about two to three against the proposed event.

Having thus shown the use of the tables at the end of this work, in the solution of complicated questions, I now proceed to the application of the theory to questions involving loss and gain.

CHAPTER V.

ON THE RISKS OF LOSS OR GAIN.

The proverb which advises us to throw a sprat to catch a whale, shows that mankind consider a chance of a gain to be a benefit for which it is worth while to give up a proportionate certainty. The principle on which depends the determination of the amount which it is safe to hazard, must vary with the circumstances of the person who runs the risk. A man should not hazard his all on any terms; but in ventures the loss of one of which would not be felt, we may suppose the venturer able to make a large number of the same kind; in which case, the common notions of mankind, reinforced by the results of theory, tell us that the sum risked must be only such a proportion of the possible gain as the mathematical probability of gaining it is of unity. For instance: suppose I am to receive a shilling if a die, yet to be thrown, give an ace; in the long run, an ace will occur one time out of six, or I shall lose five times for every time which I gain. I must therefore make one gain compensate the outlay of six ventures, or one sixth of a shilling is what I may give for the prospect, one time with another. But $\frac{1}{6}$ is the probability of throwing the ace.

PRINCIPLE. Multiply the sum to be gained by the fraction which expresses the chance of gaining it, and the result is the greatest sum which should be given for the chance.

PROBLEM. I am to gain by the throw of a die as many pounds as there are spots on the face which is thrown, with this exception, that I am to lose as many pounds as there are spots on both the face in question and on that of another die which is thrown at the same time, if the two give doublets. What should I give for the chance in question?

Call these dice P and Q: then the chance of an ace from P, and something not an ace from Q, is the product of $\frac{1}{6}$ and $\frac{5}{6}$, or $\frac{5}{36}$; and the same for any given throw from P, and not the same from Q. Consequently, the chance of winning, independently of the chance of losing, is worth

$$\frac{5}{36} \times 1 + \frac{5}{36} \times 2 + \frac{5}{36} \times 3 + \frac{5}{36} \times 4 + \frac{5}{36} \times 5 + \frac{5}{36} \times 6,$$

or $\frac{5}{36} \times 21$, or £$2\frac{11}{12}$. But if the chance of winning must be paid for, the chance of losing must be paid. Now the chance of throwing any one pair of doublets is the product of $\frac{1}{6}$ and $\frac{1}{6}$, or $\frac{1}{36}$: whence the sum I must receive if I pay £$2\frac{11}{12}$ (or not pay out of the £$2\frac{11}{12}$, which is the same thing) is

$$\frac{1}{36} \times 2 + \frac{1}{36} \times 4 + \frac{1}{36} \times 6 + \frac{1}{36} \times 8 + \frac{1}{36} \times 10 + \frac{1}{36} \times 12,$$

or £$1\frac{1}{6}$. Consequently, I must pay only £$1\frac{3}{4}$. If I were to stand a million of such hazards, paying £$1\frac{3}{4}$ for each, I might expect my ultimate gain or loss to be extremely small, *compared with the whole sum risked.* If I had besides a very small profit upon each, I might be sure of winning.

We have in the last chapter considered the amount of fluctuation for which there is a given probability: we will now look at the *percentage* of fluctuation, that is, at *the proportion which the fluctuation bears to the whole.* It is for want of consideration on this point that many notions prevail which are utterly at variance both with the daily evidence of facts, and the results of exact investigation.

Name any sum of money as a probable gain or loss of a commercial transaction, and we immediately proceed to compare it with the whole capital by which it is to be borne, if lost. A hundred pounds is nothing in one case, because it is only a shilling per cent. on the outlay; in another, the same sum is important, because it is ten pounds per cent. The importance of a gain or loss, then, depends upon the relative, and not on the absolute, value of the sum in question. Similarly, in a large number of transactions of the same sort, the number of them which may go against previous calculation is only important as compared with the whole number in question. The following is the correct principle, namely, that the percentage of fluctuation for which there is a given chance, varies inversely as the square root of the whole number of trials. Suppose, for instance, I find that on a certain hundred risks it is an even chance that one out of twenty goes against calculations made by the preceding methods; then if I take four times as many trials, it is an even chance that one out of forty only will disappoint the calculations; if nine times as many, one out of sixty, and so on. Thus it appears that the chances may be made to be against any named percentage of fluctuation, however small, by sufficiently multiplying the number of risks. The probable amount of fluctuation increases, but not so fast as the number of risks, in such manner that the proportion of the probable fluctuation to the whole diminishes.

Gambling, according to the common notion of the word, means the habit of risking considerable sums in hazardous transactions. The gambler, properly so called, at cards and dice, and the jobber at the Royal Exchange, are called by the same name. Moral considerations must induce us to regard with an evil eye the person, whatever be the name of his occupation, who thrives by the actual losses of others, in transactions which are not commercially beneficial to the community. I am not prepared to say that stock-jobbers deserve to be included in this class; it may be that the acquisition or sale of *bonâ fide* investments may by their means be rendered more easy to be obtained, and that they themselves may be a useful circulating medium between those who really wish to buy and sell for their own purposes. However this may be, the character of a gambler has no claim, when he is skilful, to the sort of respect which we pay to those who risk. The man of cards and dice, if he be cool and stick to his principles, secures a certainty to himself, and throws all the hazard upon his opponents: taking care never to risk too large a proportion of his means, and thus always enabling himself to take the benefit of the long run, he manages to play upon terms slightly unequal, and, of course, in his own favour. He runs a greater risk in games of mixed skill and chance than in games of pure chance alone. The latter he does not play at, unless he know the chances to be in his own favour; while in the former it is next to impossible that he should always be more than a match for every opponent. The keeper of a gambling house has the surest game imaginable: the play is so managed that there shall be some chances more for the bank, as it is called, than for those who play against it; care is taken to provide a sum sufficient to stand a considerable succession of losses; and the preceding principles will enable us to show that no individual resources can stand against a large fund thus used. The following problems will illustrate this.

Problem. An indefinite number of successive hazards are tried of the following kind:—One of the events, A, B, C, &c. must happen at every trial, and each event brings with it a specified gain or loss. What may be expected to be the result of continuing to play a very great number of trials; or what, in the long run, may be expected to be the average produce or loss per game?

Rule. Multiply each gain or loss by the probability of the event on which it depends; compare the total result of the gains with that of the losses: the balance is the average required, and is known by the name of the *mathematical* expectation. When the balance is nothing, then the play is equal.

For example, a person plays against a bank in the following manner:—Equal stakes of one shilling each are laid down, and if head be thrown twice successively he wins, or if tails be thrown twice he loses. But if head and tail be thrown, then another throw is to be made, by which, if it be head, the player only recovers his stake, but wins nothing: while, if it be tail, he loses his stake. In such problems, the stake is a mere evidence of solvency, with which the mathematical question has nothing to do. Let us suppose, then, that both parties keep their money until it is called for by the result. The events on which the player gains or loses are as follows (H stands for head, T for tail):—

H H gains him a shilling.
T T, H T T, T H T lose him a shilling.
H T H and T H H give no result.

The chance of H H is $\frac{1}{4}$, and the same of T T; while for each of the set H T T, T H T, H T H, and T H H, the chance is $\frac{1}{8}$. Hence the value of the prospect of gain is $\frac{1}{4}$ of one shilling; that of loss $\frac{1}{4} + \frac{1}{8} + \frac{1}{8}$, or $\frac{1}{2}$ of a shilling: consequently the balance against the player is $\frac{1}{4}$ of a shilling each time, and this he would certainly lose in the long run; that is to say, the number of times he loses would eventually so far exceed the number of times he wins as to make the play cost him three pence per game.

Every risk of loss must then be compensated by an equal chance of gain, when the play is equal. But it does not follow that equal play means prudent play;

and for the following reason. Prudence requires that no one should expose himself to great risks of loss, and does not accept it as an excuse that there was a remote chance of enormous gain, or that another had the same chance of the same gain or loss. Again, the mathematical expectation is derived from the result which, as can be shown, will be produced in the long run: consequently no one can prudently play, even with the play in his favour, unless he continue the occupation through such a number of trials that he may reasonably expect an average of all sorts of fortune. And though I have hitherto appeared to speak only of games of chance, yet precisely the same considerations apply to mercantile speculations, and to every species of affair in which no absolute certainty exists. If any possible event enter into the play which, from the nature of the game, cannot often occur, and if a stake be made upon the arrival of that event, proportionate to some enormous benefit which it is agreed that event shall secure, then prudence requires that the game shall be very often repeated, or, if that cannot be done, that it shall not be played at all. There is a celebrated case known by the name of the Petersburgh problem, which is one of the most instructive lessons in this subject, both on account of its paradoxical appearance, and also because very eminent writers have considered it as a sort of stumbling-block, and have endeavoured to evade the conclusion. Condorcet and others have taken a proper view of the subject; while among those who have considered the problem as an anomaly, we may instance D'Alembert.

If p, q, and r be three fractions whose sum is unity, it follows that we may suppose three events, one of which must happen, and neither of the others, and of which the chances are p, q, and r: and similarly of more fractions than three, whose sum is unity; or of any number of fractions, however great, provided their sum be unity. Let us, then, take an infinite number of fractions whose sum is unity: either of the following series will do.

$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$, &c.	|	$\frac{9}{10}$	$\frac{9}{100}$	$\frac{9}{1000}$	$\frac{9}{10000}$, &c.
$\frac{1}{4}$	$\frac{2}{8}$	$\frac{3}{16}$	$\frac{4}{32}$	$\frac{5}{64}$, &c.	|	$\frac{1}{8}$	$\frac{3}{16}$	$\frac{6}{32}$	$\frac{10}{64}$, &c.

Let the game be as follows: — The events which may happen at every trial are E_1, of which the chance is $\frac{1}{2}$; E_2; of which the chance is $\frac{1}{4}$; E_3, of which the chance is $\frac{1}{8}$; and so on, *ad infinitum*. And one of these must occur. The bank engages to give *2l.* if E should turn up, *4l.* for E_2, *8l.* for E_3, *16l.* for E_4, and so on, *ad infinitum.* — What should the player give to the bank for one trial? Write the several possible gains in a row, and underneath each the chance of its being won, as follows: —

£2	4	8	16	32	64	128, &c.
$\frac{1}{2}$	$\frac{1}{4}$	$\frac{1}{8}$	$\frac{1}{16}$	$\frac{1}{32}$	$\frac{1}{64}$	$\frac{1}{128}$, &c.

Multiply each gain by the chance of gaining it, and each result is 1; consequently the mathematical expectation of the player is unity repeated an infinite number of times, or an infinite amount. No sum, then, however great, can compensate the bank for its risk. The Petersburgh problem realises the preceding supposition as follows: — A halfpenny is tossed up until a head arrives, which is the event in question. If this happen at the first toss, the player receives *2l.*; if not till the second, *4l.*; if not till the third, *8l.*, and so on. Now, H standing for head and T for tail, the chance of H is $\frac{1}{2}$; of TH, $\frac{1}{4}$; of TTH, $\frac{1}{8}$; of TTTH, $\frac{1}{16}$, and so on. But can it be believed, that if I am only to throw until head arrives, and to receive *2l.*, or *4l.*, or *8l.*, &c. according as this happens at the first, second, third, &c. throw — can it be believed, you will say, that this prospect is even worth *100l.*; and is it not altogether monstrous to say that an infinite amount of money ought to be given for it?

Firstly, I will advert to a large number of trials which was actually made. Buffon tried 2048 experiments, or sets of tosses, the results of which were as follows: — In 1061, H appeared at the first toss; in

494, at the second; in 232, at the third; in 137, at the fourth; in 56, at the fifth; in 29, at the sixth; in 25, at the seventh; in 8, at the eighth; and in 6, at the ninth. Let us, then, compute the amount which he would have received if he had *bonâ fide* played all these games on the preceding terms.

1061 × 2 = 2122	The 2048 games would have given 20,114*l.* or nearly 10*l.* per game, one game with another.
494 × 4 = 1976	
232 × 8 = 1856	
137 × 16 = 2192	
56 × 32 = 1792	
29 × 64 = 1856	
25 × 128 = 3200	
8 × 256 = 2048	
6 × 512 = 3072	

No person would stake at this game for a single trial, upon the prospect of head being deferred till the ninth throw. Nevertheless, in this instance, it appeared that out of 2048 trials, such a rare occurrence happened often enough to realise more than any other, with one exception. If Buffon had tried a thousand times as many games, the results would not only have given more, *but more per game.* A larger net would have caught, not only more fish, but more varieties of fish; and in two millions of sets, we might have expected to have seen cases in which head did not appear till the twentieth throw. Let us turn back to page 43, and inquire, by the rule there given, in how many trials it is 10,000 to 1 that head will be deferred till the twentieth throw. Out of 2^{20}, or 1,048,576 cases, representing the number of different arrangements which may happen in 20 throws, the arrangement in question is but one; it is then 1,048,575 to 1 against its arrival in any one given trial. Look in the Table opposite to "10,000 to 1," and we find 921: multiply 1,048,575 by 921, and divide by 100, which gives 9,657,375. It is then more than 10,000 to 1 that head is deferred till the twentieth throw somewhere in ten millions of trials, and more than an even chance that it is found to occur in seventy thousand trials. Thus the reader may readily conceive that with unlimited license of proceeding in this play, the player might continue until he had realised not only any given sum, but any given sum *per game:* a result which is indicated by the application of our rule, when it tells us that the mathematical expectation of the player upon a single game is infinite.

The result of all which precedes shows us that great risks should not be run, unless for sums so small that the venturer can afford to repeat them often enough to secure an average. But it should seem as if we were thus told either not to gamble at all, or else to play incessantly. With a little reservation, this is true; the stake must be lowered, and more games played, instead of risking a large fraction of the whole upon one game. It is better to buy the sixteenth of sixteen different tickets than to stake all upon one ticket; and this even though it should be better than either not to buy at all. It is more prudent to play twenty games, staking one shilling upon each, than to stake a sovereign upon one game. Lay a proper proportion of the whole capital upon any hazard, and stipulate for as many trials as you please, and it will follow that with any mathematical advantage, however trifling, in your favour, you must come off a winner. The mistake committed by those who attempt to gamble with *professional men*, is twofold: firstly, they set out upon unequal terms; secondly, if the terms were equal, their stakes would be too large a proportion of their means. That the terms are unequal may readily be supposed, and will presently appear. No bank or individual gamester can play on fair terms, without losing as much as he wins in the long run. But even in such a case, the player of superior fortune has a great advantage over his antagonist, unless the stake be very small. If A with twenty guineas engage B with forty, all other things being equal, and if they are to play on until one or other has lost all, it is obviously much more likely that A shall lose his money before B, than the converse. If the play be unequally in B's favour, as well as the largeness of the fund, then it is still more against A in any given succession of games. The truth is, that to a young man who is determined to gamble, whether at one of the private receptacles in London, or the (till lately) recognised saloons of Paris, it is of little consequence whether his stakes be high or low, except in this particular, that a longer process of ruination will give him more chances of seeing his error. The play is against him in both cases, and sooner or later he must be ruined. Nor if his means be ever so great, could he make use of them, against the banks in Paris, at least. Those who conducted the play at the Palais Royal were perfectly aware of the necessity of not staking too much, and limited not only the amount of each stake, but also the number of persons whom they would engage at once. The consequence was, that though they played with perfect fairness (inasmuch as the inequality which existed in their favour was known to, and recognised by, their opponents), they gained large returns upon their capital, besides paying a considerable duty to the government.

A gambler (meaning a bold venturer, which the term commonly implies) ceases to be such when he makes his stakes bear a proper proportion to his capital, and takes no hazards which are unduly against him. If, then, a government should attempt to discourage the acquisition of great losses and gains, by limiting the number of hazards which an individual should be allowed to take, it might defeat its own object; and this is the case with our law, as it stands at present. In order to prevent individuals from gambling in life-insurance, the legislature has declared that A shall not insure the life of B, unless he have what is called an *insurable interest* in that life; that is, unless A have some pecuniary interest in B's continuing to live. The insurance offices, for the most part, have virtually, and very wisely, refused to live under this law, by paying all fair claims without questions asked. But supposing that the law were enforced, its effect would be as follows. It is tolerably easy to create a *bonâ fide* insurable interest on a few lives, while it would be difficult and attended with danger of detection to do the same with many lives. Under the system, then, proposed by the law, it would be easy to gamble, but not easy to carry speculation to the extent which would make it cease to be gambling. Allow the venturer to extend his traffic, and he will soon begin to *feel the average,* not to his gain but to his loss. For the mathematical advantage is in favour of the insurance offices, which are sure to gain in the long run. If, then, the law had been intended to save the gambler from certain loss, and to make it a *real toss up,* it would have been rational, considered as means; but if, as I imagine, it was meant to hinder immoral gain, a more futile contrivance can hardly be conceived.

It must be remembered that, in the long run, events will happen in proportion to the chances of their happening in a single trial. We see this result in Buffon's trial of the Petersburgh problem, for which I write down the numbers of cases as they did arise, and underneath as they would have arisen, one time with another, if a great many series of 2048 trials each had been made.

2048	1061	494	232	137	56	29	25	8	6
2048	1024	512	256	128	64	32	16	8	4.

This rule, however, will only apply when so many cases are taken as will produce a great many of every event to which reference is made. For instance, in 2048 trials, one time with another, we can only expect the deferment of head till the seventh throw 16 times; the result gave 25 times, or 56 *per cent.* more than the probable average. But the occurrence of head at the first throw, which, one set with another, would have occurred 1024 times, did really occur 1061 times, or 3 *per cent.* too many times. If we remember that in the long run, and on 2048 trials, we might expect two sets in which head should not appear till the tenth throw, and one in which no such thing should take place till the eleventh, and if we calculate the total amount which would have been realised had the average case occurred, we shall find it to be £11 per game. In the experiment in question, it would have produced £10. In precisely the same way, sets each consisting of 2^n games would have realised £n per game (2048 is the eleventh power of 2).

I now come to the estimation of the chances of fluctuation in loss or gain, meaning by fluctuation any departure from that general average to which the results of more and more trials will continually approach. It has been assumed in what precedes, that the proportions which the fluctuation will bear to the whole will diminish without limit as the number of speculations increase. The following problems are easy deductions from those in the last chapter.

Problem. It is known to be a to b for A against B. A is an event which brings a loss or gain of g pounds; B is another event which brings a loss or gain of h pounds. What is the general average of such trials; and what is the chance that in n times $a + b$ trials, the result as to loss or gain shall differ from the general average by not more than v pounds.

Rule. Find g times a, and h times b, and if g and h be both gains or both losses, take their sum; but if one be a gain and the other a loss, take the difference, counting it gain or loss, according as the term which contained the gain or the loss was the greater. Multiply the result by n, which gives the most probable total result (call this M). The general average is the $n\,(a+b)$th part of this; or, more simply, the $(a+b)$th part of the balance of g times a and h times b. Take the difference of g and h, if of the same name, or their sum, if of opposite names, and by it divide v. Take one more than twice the quotient. Having found this result, divide it by a square root immediately to be described, and let the quotient be t. Then the value of H in

Table I. is the probability that the resulting gain or loss shall lie between $M + v$ and $M - v$ pounds.

If it be absolutely known that the chances are as a to b for A against B, then, as in page 81, the square root is that of the product of 8, n, a, and b, divided by $a + b$. But if all that is known be that in $a + b$ previous spelations, a gave A, and b gave B, then, as in page 92, the square root is that of the product 8, n, $n + 1$, a, and b, divided by $a + b$.

Example. It has been observed, that of 100 speculations, 70 yielded a profit of £20 each, and the remainder a loss of £25 each. What is the probability that in 150 more such speculations the total result shall not differ by more than 100 from its most probable amount?

$a = 70$, $b = 30$, $n = 1\frac{1}{2}$, $g =$ £20 gained, $h =$ £25 lost, $v =$ £100.

g times a = 1400 gain
h times b = 750 loss } $\frac{650}{100}$ = £$6\frac{1}{2}$ general average of gain.

650 gain × $1\frac{1}{2}$ = £975 probable total.

g = £20 gain
h = £25 loss } $\frac{100}{45} = 2\cdot2222$; $2\cdot2222 \times 2 + 1 = 5\cdot4444$

Add £45.

$$8 \times n \times n + 1 \times a \times b = 8 \times 1\tfrac{1}{2} \times 2\tfrac{1}{2} \times 70 \times 30 = 63{,}000.$$

$$\frac{63{,}000}{100} = 630, \ \sqrt{630} = 25\cdot100, \ \frac{5\cdot4444}{25\cdot1} = \cdot217 = t.$$

Table I., if $t = \cdot217$, $H = \cdot241$.

Hence it is more than 3 to 1 against the result lying within the given limits.

Problem. All things remaining as in the last problem, what is the amount of departure from the probable total for being within which there is the given odds p to q?

Rule. Turn p divided by $p + q$ into a decimal fraction, and find it in the column H of Table I., taking out the corresponding value of t. Multiply t by the square root above mentioned, and having subtracted 1, divide the remainder by 2. Multiply the quotient by the difference or sum of g and h, according as they are of the same or different names, and the product is the answer required.

Example. In the preceding example, within what departure from £975 is it 10 to 1 that the result shall be contained?

$$p = 10, q = 1, p \div (p + q) = \cdot 9091, \cdot 9091 \times 25 \cdot 1 = 22 \cdot 8184.$$

$$\tfrac{1}{2}(22 \cdot 8184 - 1) = 10 \cdot 9092 \quad 10 \cdot 9092 \times 45 = 490 \cdot 9l.$$

It is, then, 10 to 1 that the balance of 150 speculations shall lie between 975 + 491 and 975 — 491 pounds, or 1466*l.* and 484*l.* Even such a case shows the effect of multitude in diminishing risks. The possible extremes of the problem (or the result of the problem itself, if we supposed one speculation instead of 150) are a gain of 3000*l.*, and a loss of 3750*l.*

I will now add an example which will tend to show the ultimate effect of gambling against a bank with a slight mathematical advantage in its favour. Suppose the game is such, that at each trial it is 30 to 29* that the bank shall win, the stake on both sides being one sovereign. Here $a + b$ is 59, and making n a whole number for convenience of calculation, let $n = 50$, or let 2950 games be tried. The bank has each time a mathematical advantage (page 97) of $\frac{30}{59} - \frac{29}{59}$, or $\frac{1}{59}$ of a sovereign, and will, in the long run, realise £50 upon 2950 games. What are the chances in favour of the individual fluctuation of this one set of 2950 games leaving the bank without any profit, and with more or less loss? To apply the preceding rule, we must first ask what are the chances that the departure from the probable total of 50*l.* shall not exceed 50*l.*; that is, that the bank shall realise between 0*l.* and 100*l.* Here we have

* This supposition is more in favour of the player than is often the case.

$a = 30$, $b = 29$, $n = 50$, $g = £1$ gained $h = £1$ lost $v = £50$.

$$\left.\begin{array}{l} g \text{ times } a = 30 \text{ gain} \\ h \text{ times } b = 29 \text{ loss} \end{array}\right\} \frac{1}{59} = £\frac{1}{59} \text{ general average of gain.}$$

1 gain × 50 = £ 50 probable total.

$$\left.\begin{array}{l} g = £1 \text{ gain} \\ h = £1 \text{ loss} \end{array}\right\} \frac{50}{2} = 25;\ 25 \times 2 + 1 = 51.$$

Add £ 2.

In this case the probability is absolutely given. The square root is therefore the first one mentioned.

$$8 \times n \times a \times b = 8 \times 50 \times 30 \times 29 = 348{,}000$$

$$\frac{348000}{59} = 5898{\cdot}3,\ \sqrt{5898{\cdot}3} = 76{\cdot}800,\ \frac{51}{76{\cdot}8} = {\cdot}664 = t.$$

Table I., if t = ·664, H = ·652.

Consequently $\frac{652}{1000}$ is the chance that the bank shall not lose, but shall gain something less than £ 100; and consequently the chance that the bank shall either lose, or gain more than £ 100 is $\frac{348}{1000}$. On account of the nearness of 30 and 29, the results last mentioned are nearly equally probable, and it is near enough for our present purpose, to say that $\frac{174}{1000}$ is the chance of the bank gaining more than £ 100, the supposition being against us; for it is more likely that the bank should gain more than £ 100 than that it should lose. Hence it follows, that the chance of the bank gaining on 2950 games is more than $\frac{826}{1000}$, or about *five to one*. If such be the case with a bank much less unfairly constituted than is often the case, against a player who can not only command £ 2950, but who has the prudence to determine that he will only play 2950 games, at a stake of £ 1 for each game, what must be the chances against those who risk a larger proportion of their means at more unequal play, with a determination to win — that is, to go on till they are ruined?

The inequality of means is an important consideration in calculating the chances of two antagonist gamesters. If two persons, with equal means and equal chances, play for equal stakes, it is an even chance whether A shall ruin B, or B shall ruin A; but that one or other will ultimately be ruined is certain. Suppose each party to have a hundred guineas, the stake being one guinea, and suppose two millions of games are to be played. The most probable individual case is that each shall win a million of games; but if the fluctuation amount to 100 in favour of either, the other is ruined. Now, page 81, the probability that the number of games won by A shall lie between a million + 100 and a million − 100 is ·112, which is therefore the chance that neither player shall be ruined. Consequently, it is about nine to one that one player or other is ruined or more than ruined in two millions of games. And the chance is even greater than this: for the preceding method of treating the problem supposes the players not to balance their account till two million of games have been actually played, so that one player or the other may have been repeatedly playing on credit. The same rule may be easily applied to any inequality of play, the fortunes of the players being equal; and the result is, 1. that ultimate ruin to one or other player is certain; 2. that, if the stake be a sufficiently small fraction of the player's income, the number of games which must be played to render probable the ruin of either may be made as large as we please. There are but two conditions under which gambling can be prudently followed as an amusement — *small stakes* and *equal play*. In games of pure chance it is possible to obtain the latter, and almost impossible in games of mixed skill and chance. Unfortunately, the stimulus of gambling, a combination of suspense and hope of large gain, cannot be obtained upon any terms which prudence would sanction.

When two players, of unequal fortunes, play together for the same stake, however equal the play may be, the larger fortune has an unfair advantage. To estimate the amount of the disadvantage, proceed as follows.

Problem. Two players, A and B, having funds of m and n times their stake, play a game, at which it is a to b that A wins, or b to a that B wins. — What is the chance that in the long run B will ruin A, and that A will ruin B?

Rule I. If a and b be equal, it is m to n that A will ruin B, and n to m that B will ruin A.

Rule II. If a and b be unequal, let M represent the difference of the mth powers of a and of b, multiplied by the nth power of a, and let N represent the difference of the nth powers of a and of b, multiplied by the mth power of b. Then it is M to N that A ruins B, or N to M that B ruins A.

N.B. — If m and n be considerable, it is almost impracticable to apply the above rule without the aid of logarithms. When b is many times a, take the nth power of a for M, and the nth power of b for N.

Rule III. When m and n are equal, it is as the mth power of b to the mth power of a that B will ruin A.

Rule IV. If the means of both players be unlimited, then it is certain the player who has odds in his favour on a single game will, in time, gain any sum, however great; if the means of the stronger player be unlimited, then it is certain he must at last ruin the weaker.

Rule V. If the means of the weaker player be unlimited, and those of the stronger limited, the chance that the latter will in time win any sum, however great, from the former, is as follows. Let B be the stronger player (that is, b greater than a), and let him begin with n times his stake, while A has unlimited means: then it is the excess of the nth power of b over that of a to the nth power of a that B gains any sum, in the long run, from A, and *vice versâ* for A ruining B.

Example. A has *5l.* and B *3l.*, and they play at a game for which it is three to two that B shall win any one game. — What are the chances for the ultimate success of each player?

$$a = 2, b = 3, m = 5, n = 3$$

M or $(3^5 - 2^5) \times 2^3$ is $(243 - 32) \times 8$ or 1688

N or $(3^3 - 2^3) \times 3^5$ is $(27 - 8) \times 243$ or 4617:

It is therefore 4617 to 1688 that B shall ruin A.

A gaming bank must be considered as a player of limited means, playing against all who choose to enter, that is, playing against unlimited means. It is, therefore, essential to its existence that some mathematical advantage should be allowed, even more than is necessary to reproduce the expenses of its management. What I have hitherto said on the subject refers to the relation between the bank and the *individual* player against it. but considering the former as the antagonist of all who choose to play, it absolutely requires the protection of a mathematical advantage. But having this advantage, it must, in the long run, ruin its individual opponents; so that bankruptcy to itself, or degradation and suicide to its customers, are the initial conditions of its existence. But since the banks flourish, it is plain that whatever advantage is necessary to their continuance, is really obtained by them; and I shall now inquire how much this advantage must be in several cases.

Example (Rule V.) It is 30 to 29 for the bank upon each game, and the bank stakes the tenth part of its means at every game.—What are the chances of its perpetual continuance? $(b = 30,\ a = 29,\ n = 10)$.

$$30^{10} = 590,490,000,000,000$$
$$29^{10} = 420,707,233,300,201$$
$$169,782,766,699,799.$$

Answer. About 170 to 421, or such a bank would not be likely to last; that is, in the long run, only 170 out of 421 such banks would avoid ruin.

Problem. What is the mathematical advantage which a bank must have, in order that its permanent continuance may have k to 1 in its favour; the supposition being that the bank stakes the nth part of its means at every game?

Rule. The odds in favour of the bank, on a single game, must be the nth root of $1 + k$ to 1. Thus if, judging by the experience of the Parisian* banks, we say

* These banks were open to the public and to the municipal police. Of the gaming-houses in London, those who know them must speak. The

that the permanency of each had 100 to 1 in its favour, we are entitled to conclude that (the tenth root of 101 being 1·59) the games played were each not less than 3 to 2 in favour of the bank, if they staked the 10th of their resources at each throw, or 11 to 10 if they stake one fiftieth.

PROBLEM. The odds in favour of the bank being b to a, required the greatest proportion of the fund which may be staked at one game, in order to insure the chance k to 1 for the permanency of the establishment.

RULE. Divide the logarithm of one more than k by the excess of the logarithm of b over that of a: the quotient is the denominator of the fraction required, 1 being the numerator. Suppose, for instance, that the odds for the bank, on single games, are 30 to 29, then, if 99 to 1 be required to be the chance that the bank shall continue to exist, divide the logarithm of $99 + 1$ (which is 2) by the excess of the logarithm of 30 (or 1·47712) over the logarithm of 29 (or 1·46240), and $2 \div {\cdot}01472$ gives a fraction more than 135; whence the 135th part of the capital is the highest which should be staked.

A merchant, who engages in speculations which must produce a fixed loss or a fixed gain, and who offers to deal with any one in such a manner, is precisely in the position of a bank such as is above described. The reason why neither party need be ruined in this instance is that the produce of the earth and sea is an unlimited fund, upon which the merchant draws. Trade by itself would tend to ruin the many, and accumulate all the stakes in the hands of a few, and the theory of probabilities would enable us to foretell that continual approximation towards the extremes of wealth and poverty which commercial countries always present. We have seen that the poorer player must, to maintain his ground, have a

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protection and encouragement which legal regulation of gambling-houses would appear to give to gambling in general, is a good reason for the state of our own law; otherwise, there can be no doubt that much particular evil would be prevented by allowing a regulated system.

mathematical advantage in his favour. Now, it is the nature of free trade that whatever mathematical advantage can be gained at all, is more accessible to the rich speculator than to the poor one. Consequently, the richer player, for that reason, can make himself the stronger player. If, then, a certain number of persons were to play upon a fixed total of stakes, equally divided at the commencement, with the condition that every stake won should enable the winner to make his next throw with somewhat more (no matter how little) of mathematical advantage than he had before, it is certain that, in the long run, the whole of the stakes would be in the hands of some one of the players. But, in the actual state of things, there is always an accession of new stakes and new players, so that the original players are contending against an unlimited fund. If the continual augmentation of stakes and players be not sufficient to counterbalance the tendency to extremes, a wise government would throw the burthen of taxation more upon the rich and less upon the poor. The mathematical advantage of wealth would be taxed, as well as its power of procuring luxuries. Such a result never can be expected until the public mind is better informed upon the subject of which this work treats.

CHAPTER VI

ON COMMON NOTIONS WITH REGARD TO PROBABILITY.

Those who have not considered this subject with particular attention, seldom fail to think that there must be more or less of fallacy in the attempt to connect its principles with its results. Some, indeed, of the latter are strange and new, and are used as arguments against the validity of the theory. I propose in this chapter to turn those which precede to account, in examining opinions of various kinds, whether on this subject at large, or on particular cases of its application.

The doctrine of probabilities seems to some to assume a sort of power of prophesying, or of predicting the run of events; to others, it appears that unless such a power of prophesying be attained, the theory can be of no use. Both notions are correct in one sense and incorrect in another: there is prophecy, but not of particular events, and derived, not from inspiration, but from observation. The astronomer predicts—and all the world knows that his predictions daily come true. His means of prophecy are aided by deduction from certain notions of which, be the cause what it may, we are as certain as of our own existence. From his very distinct (and therefore often called intuitive) perception, that two straight lines cannot inclose a space, and various other axioms of arithmetic and geometry, he is able to make his observations tell him more as to the future motions of our system than his unassisted perceptions of the past could ever have accomplished. He is a dealer in probabilities of a very high order. But before his prediction appears, it is necessary that he should consider much more doubtful questions of probability. The minute errors of observation, coupled with the various trifling effects which result from yet undiscovered causes, oblige him to have recourse to the principles which we have explained in the preceding chapters.

Again, there is no prophecy of particular events in the theory of probabilities, of which it is the very essence that there should be more or less tendency to falsehood in every one of its assertions. No result is announced, except as having a certain chance in its favour, which implies also a certain chance against it.

With regard to the second class of assertions, namely, that unless the theory of probabilities enable us to predict, it can be of no use—it may be said that, for the purpose contemplated, it *is* of no use. Theory would never enable us to tell what face of a die will be turned up in any one instance, nor would the maxims of our science be worth putting into practice with respect to an event which is to happen only a few times. If a man were determined to run six hazards, and never to gamble afterwards, say if he were determined to wager twice upon a pair of dice giving doublets, I should think it perfectly immaterial whether he accepted an even wager, namely 5 to 1, or not. For though, in the long run, only one throw out of six will give doublets, yet the probability that six throws will give such a pair once at least is not very great. It is as a provider of general rules of conduct that the science is valuable; the adherence to rules being desirable on precisely the same principles as those which obtain in morals or legislation, no maxim of which will be found to meet *every* case which will occur.

It is an assumption of this theory that nothing ever did happen, or ever will happen, without some particular reason why it should have been precisely what it was, and not any thing else. Conceive it possible that a ball which is white might have been black, without the alteration of any action or circumstance which took place in time previous to the moment at which the ball is shown, and the foundations of the theory of probabilities have ceased to exist in the mind which has formed that conception. There is no one but will admit, that out of a box, which contains nothing but two black balls, nothing but black balls can be drawn; and that out of a box which contains only two white balls, no black balls can be drawn. The difficulty lies in a clear perception of the remaining assertion; namely, that when the box contains one white ball and one black ball, a very large number of drawings will give as many white as black nearly, and the more nearly the greater the number. This proposition might be proved in three ways: firstly, by actual experiment; secondly, by showing that out of all the possible cases which can happen, those in which black and white are equal, or nearly equal, much exceed in number all the rest put together; thirdly, by showing that there can be no possible reason for an excess of white, which does not equally, by express condition of the question, apply in favour of an excess of black. The last is more unanswerable than convincing; the second really shows that the event which we propose and treat as one event, namely, "as many white as black, or nearly so," is, in fact, a collection of a large number of events, much exceeding in number all the rest which can happen. It is as if, having a million of possible cases, I separated nine hundred thousand from the rest, called each of them A, and each of the rest B, and then asserted that A would happen more often than B. But, nevertheless, I suspect that to the first mode of demonstration, actual experiment, most persons owe that degree of confidence in the theory, which (often without knowing it) they exhibit in the affairs of life; and I derive such a suspicion from observing that every result of the theory of probabilities which is not of a nature to admit of every-day confirmation, or which would escape an inattentive observer, is looked upon with distrust. In no case is this more obvious than in the prevailing notions with regard to luck.

It is observed that some people always have luck at cards. The order of things seems disturbed at their caprice; if they sit opposite to the dealer at whist, then there is always an undue proportion of trumps among the cards which come second, sixth, tenth, &c., up to the fiftieth; while, when they become before the dealer — *Hocus Pocus* (for to no other spirit, ancient or modern, can the agency be attributed) puts all the good cards, third, seventh, eleventh, &c. The fact is stated as a sort of mystery, and we hear of people who are always lucky at cards and never at dice, or *vice versâ*. The statement implies that the parties who make it believe there is *something in luck* — an assertion which I do not think of questioning; for, as I shall proceed to show, it would be the most improbable thing imaginable that there should be no such lucky people.

Firstly, every question of probabilities stands in precisely the same relation to our faculties, whether we suppose a moral government of the universe, or none at all, *provided* that we have no reason to suppose we know any thing of the plan of that government in the particular case in question. If I am before an urn which contains a black and a white ball, which is all I know, I am then disposed to say the chances are even. The ball which I am to draw is undetermined (by me), and that which we call chance appears to exist. But suppose I draw the ball, and without looking at it hold it in my hand. That which we call chance has ceased to exist — the ball is actually determined, and I am clearly and physically placed in the same position as I should have had before the drawing, if a superintending power, capable of guiding my thoughts and actions without my perceiving it, had predetermined which I should draw. But my position with respect to knowledge of the ball is not in any way changed, either by the predetermination of the superintending power before the drawing or by my own act of drawing, as long as I do not know what I am to draw or have drawn. It tells me nothing, if I hear that the drawing is settled, unless it be in a manner by which I can form some guess as to the nature of the settlement. Consequently we must not, unless some reason be shown for it, consider the question of the luck of individuals in any other light, with reference to calculation, than that in which it would have been placed by the supposition that, all imaginable species of fortune being described on the tickets of a lottery, each individual had one drawing made for him at birth, which should describe his future successes and reverses. To create an analogous question, within reasonable numerical limits, let us suppose a thousand individuals, each of whom is to play two thousand deals at whist, with a given suit as trumps.* Let there be a lottery, containing an enormous number of books, in each of which 2000 deals are described, and let the books be so many in number that among them is one containing every possible set of 2000 deals which can be imagined, the four hands in each deal being described, and that allotted to the drawer of the book being marked as such. Let each individual draw one of these books, replacing it before the next drawer arrives: these individuals are then precisely in the same situation with regard to us, if their hands are to be dealt to them according to the directions laid down in their books, as if the distribution were made by accident (as we call it). Now the question is, which is most likely, that the luck of these individuals shall all be nearly the same, or that some of them shall have a marked predominance over others? To take one simple question: consider only the chance of gaining the ace of trumps. Excluding the dealer's advantage, to simplify the question, the chance of any one individual gaining the ace of trumps at any given deal is $\frac{1}{4}$. Considering 2000 deals, he has a very good chance of gaining it 500 times, or a few more or less. But the probabilities are much in favour of several of these 1000 individuals having a very different lot from the average. Frame a set of circumstances in this respect against which it shall be twenty to one, and (page 43) it is a hundred to one that this fate (or a better) shall be found to be that of some one or other out of any 92 individuals taken at hazard from among the thousand. And when to the chance of holding the ace of trumps we add the various others which constitute a good hand at the game, we thereby much increase the probability of large fluctuations, one way or the other; and though it is certain that uniformity will be found in the average lot of a large number of per-

* This does not alter the question; since the substitution of four possible different sorts of trumps would only multiply every possible case of good and bad fortune four times.

sons, yet the larger the number, the greater will be the extremes of fluctuation.

Now it must be noticed, that this variation is the thing observed — not on one side only, but on both. For every one who is lucky at cards, there is another who is unlucky. It would be, indeed, such a sort of mystery as that which I am endeavouring to explain, if the exceptions to common luck were all on one side, or if there were no such thing at all as uncommon luck, or only in very few instances. This latter would be the same sort of phenomenon as we should see if a halfpenny gave head and tail alternately through an enormous recurrence of throws. The event observed is precisely that which might have been expected beforehand. If by thinking mysteriously of the fluctuations of luck which are observed in comparing the fortunes of individuals, any reader should mean to imply that the alternative, namely, slight individual departure from the average, would *not* have been mysterious, he is in a singular error. The state of things which he would regard with no wonder would be an apparent interference with the material world on the part of its governor, without the intermediate agency of any second causes; that is, something resembling a *miracle.* For though the philosopher, in such a case, would suspect an intermediate cause, and endeavour to discover it, this consideration does not enter into the view of people in general. When the world wonders, whether at one side or another of a question of probabilities, it is at the want of any apparent physical or moral reason: on which account they refer it to the Creator in a manner different from that in which they refer what they call usual occurrences.

The law of individual cases is, that there shall be marked differences; of the masses, that there shall be great approach to uniformity. There are a hundred years in which, and hundreds of diseases by which, any individual who is born may die: a lottery, which should contain one ticket for every disorder, repeated as often as there are years of age in which it has been fatal, would present at least 20,000 chances. Before an individual is born, it is, say 20,000 to 1 against his dying at a given age of a given malady; and yet, even with such imperfect observations as exist at present, it begins to be seen that uniformity is the law of large masses compared with each other. I will illustrate this by some cases. Few things appear more varied than the distribution of maladies, that of criminal acts, and that of the sex of children in different families. I have taken purposely a case of evil, physical and moral, and one which is neither. The experience of any one individual might lead him to say that it is no uncommon thing for three or four times as many persons to die of consumption in one period of five years as in the previous period; but the experience of one large city will show that such is not the case. The bills of mortality in London showed the following results; the upper line denoting the last year of the five in question, and the lower line the average number in every thousand deaths which were caused by consumption, or what was called such.*

1732	1737	1742	1747	1752	1757
135	163	165	180	187	197.

Here is nothing like enormous fluctuation. The gradual increase of the number shows an increasing tendency to the complaints then described under the head consumption, but cannot be called fluctuation, being itself regular.

The number of murders committed in the whole extent of any one country might be supposed liable to very large yearly fluctuation, and still more the comparative numbers committed with different classes of weapons. A few years ago, extreme derision would have followed the assertion that the sword and pistol would be felo-

* The known loose manner in which these Tables were put together does not affect the argument, further than to favour its conclusion. The chances of error in the description increase the probability of fluctuation in the results.

niously used in different years by nearly the same numbers. Let us look at the following Table, extracted from the *Essai de Physique Sociale* of M. Quetelet. The country referred to is France.

YEARS.	1826	1827	1828	1829	1830	1831
Total number of murders brought to justice.	241	234	227	231	207	266
Fire-arms - -	56	64	60	61	57	88
Sharp weapons of war - -	15	7	8	7	12	30
Knives - - -	39	40	34	46	44	34
Clubs or sticks -	23	28	31	24	12	21
Stones - -	20	20	21	21	11	9
Sharp instruments not above described - -	35	40	42	45	46	49
Striking or kicking	28	12	21	23	17	26
Other modes, and unknown - -	25	23	10	4	8	9

I now compare the number of male and female baptisms registered in England in 1821, and the nine following years. For these successive years it is found that for 1000 girls baptised there were 1048, 1047, 1047, 1041, 1049, 1046, 1047, 1043, 1043, and 1034 boys.

Such cases as the preceding tend to establish the law in question; namely, that different large masses of facts, collected under the same circumstances, will present nearly the same averages. I now proceed to another point.

When two circumstances happen to change together, it is frequently presumed that they are connected with each other, when, in truth, there is no reason for any such supposition. In order to justify any notion of necessary connection it is necessary that the two circumstances should always happen together, and that one should never happen without the other. If it should only be observed that one is very frequently accompanied by the other, we must then inquire into the probability that either may happen without the other. If two events are almost always happening, then it is evident that very frequent coincidence is no evidence of connection, so long as exceptions tell us that there is no *necessary* connection. And even if we always observe A to be immediately followed by B, it does not immediately follow that they are necessarily connected. We must remember that the phenomenon is this — our perception of A is immediately followed by our perception of B. This may arise in different ways, as follows.

1. A may make B necessarily follow it; that is, the common deduction from the phenomena may be true.

2. Our *perception* of A may make B follow. This is the case with regard to all effects produced upon our own minds by A.

3. A itself may make our *perception* of B follow. The latter may be always happening, but it may require the arrival of A to make us see it.

4. Our *perception* of A may make our *perception* of B follow; that is, B may be always happening, but it may need both the arrival of A and our knowledge of that arrival to make us see B; or the circumstances which favour our perception of A may be the same, or have necessary connection with, those which do the same for B.

5. It may be B which is the antecedent event; but our perception of A, the consequence of B, may be necessary to our perception of B itself.

We sometimes, for example, note what takes place from the time A happens, and compare it with previous events, merely because the arrival of A suggests the comparison. In many instances we do this correctly, and merely for convenience; but whenever the events which are compared with A are such as to exhibit what we call accidental fluctuations, we are apt to imagine that the difference between the two sequences is a consequence of A. In the mysterious subject of luck, already alluded to, this tendency to error produces superstition. There are many who imagine that the change of seats, or a new pack of cards, changes the luck. They imagine it, because they observe that the luck is not the same before the change as after it, which, for the most part, is true. But it is equally true that, for the most part, no number of games exhibits the same fortune as those which precede it; that is, this change of luck is always taking place, but is usually only perceived when the introduction of some novel circumstance affords a point to date from, on one side and the other. The growth of the superstition is this:—An individual who has been unlucky during several games, happens to begin to win after the introduction of new cards. His fortune changes, as most probably it will do; for if the chances be even, and three games have previously been lost, it is seven to one against the next three games resembling them, and an even chance that he shall win the next game. If he win the next, or, indeed, if he do not go on losing, he notes the circumstance, and the next time a run of ill luck occurs, he takes particular care to repeat the experiment. In this way he soon furnishes himself with a tolerable number of facts in support of his theory. The exceptions are forgotten; for it is the character of negative events to lay less firmly hold of the mind* than positive ones. Thus the theory of the change of the weather with that of the moon receives more confirmation from one fact in its favour than of doubt from two against it. This last notion is another case in point. The weather of any three days, in by far the most instances, differs from that of the preceding three

* The lucky hit of a prophet of the weather, in foretelling the coldest day of January, 1838, did more to establish his infallibility than weeks of succeeding mistakes could destroy.

days. When once the notion is obtained that a change of weather will follow that of the moon, the epoch is watched, and the change which is in most instances observed, is admitted as evidence. If any one would carefully note for a considerable time the weather preceding and following prorogations of parliament, he would perhaps astonish the world with a result which no one has yet dreamt of. These cases fall under the third head above described.

It was frequently supposed, a few years ago, that comets produced hot weather. An examination of the number of comets discovered in years of different average temperature gave it as a result that there were more comets in hot summers than in cold ones. But since hot summers are generally fine, with clear skies, and cold summers cloudy and rainy, it is obvious that the former are more favourable to the discovery of comets than the latter. The fact, then, from which the inference was drawn amounted to this, that the years of heat are those in which we see most comets. With what we know of the matter, there is no more reason to suppose that comets bring heat than that heat brings comets. We must, in all instances of presumed connection, look closely at these two distinct things — the happening of an event, and our perception of it; otherwise, we shall always be liable to suppose that an event may produce the first, when it produces only the second.

There is, however, a class of events which does not appear capable of any of the preceding explanations; namely, the occurrence of what are called runs of luck at play, and repetitions of similar events in common life, which give rise to such proverbs as this — that it never rains but it pours, and misfortunes never come alone. We shall first attempt to destroy the extraordinary character of these occurrences, and shall then show that any other order of things would be indeed extraordinary.

Let there be any event of an unusual character, say the drawing of a black ball out of a lottery of twenty balls in which only one is black. Surely, any one would say, we shall never draw the black ball fifty times running. I answer, continue drawing as long as I direct, and you certainly shall, that is, if you will admit that an event must some time or other arrive, which has ten thousand to one in its favour. Supposing the ball to be replaced immediately after drawing, and the lottery to be shaken, so that every ball has its fair chance, I will calculate the number of drawings in which it is much more than 10,000 to 1 that a black ball shall be drawn five times running, and it will be evident that by the same principles the number might be calculated which would give as great odds for a run of 50, or 5000, or any number, however great.

The chance of drawing the black ball at one trial is $\frac{1}{20}$, and through five successive trials it is $\frac{1}{20} \times \frac{1}{20} \times \frac{1}{20} \times \frac{1}{20} \times \frac{1}{20}$ or $\frac{1}{3200000}$. It is, therefore, 3,199,999 to 1, say 3,200,000 to 1, against five successive black balls. In page 43, look in the Table opposite to 10,000 to 1, and we see 921. Let every succession of five drawings be called a set, then it is 3,200,000 to 1 against any given set being all black. Multiply 3,200,000 by 921, and divide by 100, which gives 29,472,000, and this is the number of sets in which it is 10,000 to 1 that one whole set, at least, shall be black. Much greater are the odds that there shall be a run of five somewhere in all the drawings which make up these sets; for this latter event would arrive, not only when all of a set are black, but when the last of one set, and four of the next, or the two last of one set and three of the next, &c. are black.

Let two players, with equal chances, play a large number of games in succession. It is 63 to 1 against a run of six games in one given manner, and 62 to 2, or 31 to 1 against a run of six games for one or the other. Now, $31 \times 921 \div 100 = 285{\cdot}51$, or it is, as before, more than 10,000 to 1 that in 286 sets of six, a run of six shall occur for one or the other. This gives 1716 games, and yet a sturdy whist player, who plays on an average a dozen deals an evening for 200 evenings in the year, or 2400 deals *per annum*, will look grave when he relates that he had bad cards for six deals together, and will assure you that there is *something in luck*.

But at the same time, the number of trials which makes a run of the unlikely event extremely probable, will give the same probability to a much larger run in favour of the more probable event. To find to what extent this goes, use the following

Rule. If it be a to b for A against B, in any single trial, then subtract the logarithms of a and b separately from that of $a + b$; take any two whole numbers which are very nearly in proportion to these differences, and to each add 1; the sums show the runs of the two events, which have the same probability (be it small or great) of happening in a very large number of throws. Thus, suppose it is 10 to 3 for A against B.

Log. 13	1·11394	log. 13	1·11394	
Log. 10	1·00000	log. 3	·47712	
	0·11394		0·63682	: : 11 : 64 nearly.

Consequently, whatever chance there is for a run of 12 B's, there is as much for a run of 65 A's.

In the last mentioned lottery, when $a = 19$, $b = 1$, we have $\log 20 - \log 19 = {\cdot}02228$, and $\log 20 - \log 1 = 1{\cdot}30103$, which differences are 1 to 65 nearly, and as 6 to 390. Consequently there is as much reason to expect a run of 391 white balls as of 7 black balls.

No person can take a rational view of probabilities until he ceases to recoil from the supposition that an event is never to happen because the odds are very much against his choosing, out of a large number of trials, the one in which it is to happen. The best way to force the mind upon the consideration is to return to the first principles upon which the method of judging is founded. You find it difficult to imagine, that out of twenty balls, one only of which is black, you shall draw the black ball five times running. But yet in 30,000,000 sets of five drawings each, it is asserted that you are what is called "almost sure" of drawing the black ball throughout the whole of one set. Waiving the question of probabilities, I will now state what it is of which mathematical demonstration makes us *quite* sure. Let vol. i. be a book which describes 30,000,000 of sets; vol. ii. another, which describes 30,000,000 more, differing from the preceding in some, many, or all, its sets, and so on until every possible collection of 30,000,000 of sets is described in one volume or another. Now it is *quite certain* that out of the innumerable volumes which will thus be produced, the volumes which somewhere or other describe a set all black will outnumber those which do not describe such a set many thousand times, 10,000 at least. Suppose the black sets when they exist, to be in a frontispiece; the question then is, having an enormous library, with books at the rate of 10,000 with a frontispiece, for one which has none, and taking down a book blindfold, which do you suppose to be most likely, that you shall draw a frontispiece, or none at all? Unquestionably you answer that you are almost mathematically certain of not drawing the latter. But this is (page 124) an exaggerated statement of the case of the chance of a run of five black balls in 30,000,000 of sets.

But it is said that no person ever *does* arrive at such extremely improbable cases as the one just cited. That a given individual should never throw an ace twelve times running on a single die, is by far the most likely; indeed, so remote are the chances of such an event in any twelve trials (more than 2000,000,000 to 1 against it), that it is unlikely the experience of any given country, in any given century, should furnish it. But let us stop for a moment, and ask ourselves to what this argument applies. A person who rarely touches dice will hardly believe that doublets sometimes occur three times running; one who handles them frequently knows that such is sometimes the fact. Every very practised user of those implements has seen still rarer sequences. Now suppose that a society of persons had thrown the dice so often as to secure a run of six aces observed and recorded, the preceding argument would still be used against twelve. And if another society had practised long enough to see twelve aces following each other, they might still employ the same method of doubting as to a run of twenty-four, and so on, *ad infinitum*. The power of imagining cases which contain long combinations so much exceeds that of exhibiting and arranging them, that it is easy to assign a telegraph which should make a separate signal for every grain of sand in a globe as large as the visible universe, upon the hypothesis of the most space-penetrating astronomer. The fallacy of the preceding objection lies in supposing events in number beyond our experience, composed entirely of sequences such as fall within our experience. It makes the past necessarily contain the whole, as to the quality of its components; and judges by samples. Now the least cautious buyer of grain requires to examine a handful before he judges of a bushel, and a bushel before he judges of a load. But relatively to such enormous numbers of combinations as are frequently proposed, our experience does not deserve the title of a handful as compared with a bushel, or even of a single grain.

Further to illustrate this point, let us turn back to page 64, where we see that when an event has happened m times running, it is $m + 1$ to n that it shall happen n times more. This proceeds upon the supposition that the chances of the event were entirely unknown before its happening, and the presumptions drawn are therefore entirely derived from experience. When an event has happened 1000 times one way, it is 1001 to 1 that it happens in the same way next time. But it is only 1001 to 1000 that it happens 1000 times *more* the same way, and only 1001 to 1,000,000 that it happens

1,000,000 of times more in the same way. Hence experience can never, on sound principles, be held as foretelling all that is to come. The order of things, the laws of nature, and all those phrases by which we try to make the past command the future, will be understood by a person who admits the principles of which I treat as of limited application, not giving the highest degree of probability to more than a definite and limited continuance of those things which appear to us most stable. *No finite experience whatsoever can justify us in saying that the future shall coincide with the past in all time to come, or that there is any probability for such a conclusion.*

CHAPTER VII.

ON ERRORS OF OBSERVATION, AND RISKS OF MISTAKE.

In every measurement, as well as in unassisted estimation, the observer is liable to error; the only difference being that the mistakes of careful instrumental measurement are likely to be less than those of estimation. That which we call *estimation* means *guess* formed by a person whose previous habits and experience are such as to make it very likely that he can tell nearly true that which would require instruments to obtain with great approach to accuracy. To illustrate this distinction, imagine a certain small length, say about twelve inches, to be presented to a large number of persons, who are required to write on separate bits of paper how many inches and tenths of inches it appears to them to contain. If these persons had been used to estimate lengths by the unassisted eye, it would be extremely probable, 1. That the average of their guesses would be very near the truth. 2. That their widest limits of error would be small. If their habits have not been accurate, it is still reasonably probable that the average of their guesses would be nearly true; the limits of error would certainly be larger. It is the object of the present chapter to show how the theory of probabilities must be applied to the detection of the most probable result, when various observations are discordant with each other.

Error, as used in this part of the subject, merely means discordance of which the cause is unknown. In the different branches of physics, in their application to the arts, &c. &c., that which we signify by the preceding word may arise from various causes, the chief of which are here enumerated.

1. From some law of nature not known to the observer. Thus before the discovery of the aberration of light, all the small yearly changes which the places of stars receive from that cause, only appeared in the form of embarrassing differences between the observations of different months. Those who used astronomical instruments might suspect the existence of some unknown motion in the heavenly bodies; they might think it extremely improbable that their improvements in the art of observing should permit purely casual discrepancies of so large an amount as those which occurred: but still, so long as no account of the magnitude of these possible results of law could be given, those who observed could in no manner distinguish them from the imperfections of the instruments, or of the human senses. But had it been shown that these discrepancies were always the same at the same time of the year, for any one star, they would then have ceased to be errors, and would have become the objects of prediction, as soon as one year's observations had been registered. The physical cause might or might not have been subsequently discovered, without altering the state of the question: the certainty of a phenomenon is all that is required to remove it from the domain of probability.

2. From the personal constitution of the observer;

by which is meant, not that general facility of misperception which is common to all the human race, but that particular habit or temperament which causes some to differ from persons in general in their method of perceiving. Thus it will frequently happen that when two observers note the time of a phenomenon, by the same watch, one will always see the event, or imagine he sees it, before the other does the same. This *personal* error, which is seldom large, is beginning to receive the attention of observers. It is not perceptible as long as the natural *data* of a science remain imperfectly known, being mixed up and lost in errors of greater magnitude; but it produces discoverable effects so soon as the science approaches towards accuracy.

3. From fixed sources of error peculiar either to the species of apparatus employed, or to the individual instrument with which the observations are made. This answers precisely to the personal error of the observer in its effects: it matters nothing whether the clock be one second too fast, or the observer, in the result of the observation.

4. From the imperfection of human senses and instruments. To note a measurable phenomenon without any error at all, would require sight and touch by which every magnitude, however small, could be perceived and correctly estimated. Such senses belong to no one, and the degree of approach towards perfection not only varies with the observer, but is different at different times with the same observer. Many errors to which instruments are subject ought in strictness to be classed under the first head; if, for instance, an astronomical circle gradually change its form, or undergo daily expansion and contraction by variations of temperature, the diversity of results which such a piece of brass will shew are certainly subject to laws, and might be predicted, if we possessed sufficient knowledge of the constitution of the metal, and the laws which regulate the effect of pressure, temperature, moisture, &c. upon it. But so long as such laws are

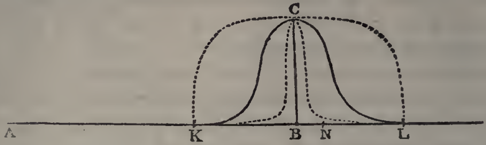
unknown, and the variations do not follow any distinguishable rule, their effect upon general results differs in nothing perceptible from that of the observer's own errors, with which they are mixed up in the particular results of observation.

Before any trials are made, that is, before any thing is known of the character of the observer, of the instrument he uses, or the perceptibility of the phenomenon, we can have no reason to suppose that any one observation is more likely to exceed the truth than to fall short of it. When any observation is greater than the reality, the error is called *positive;* when less, *negative.* The hypothesis, therefore, of an equal presumption for positive and negative errors, is one with which we must commence; and it follows from the supposition that the average is the most probable result of a number of discordant observations. The sum of all the observations will be affected by the balance of all the errors, but will be without error itself if the amount of the positive errors be equal to that of the negative ones. This last supposition, though not probable in itself, is nevertheless more probable than any other, and the odds are very much in favour of its being nearly true. Now whatever may be the error of the sum of observations, say 100 in number, the average, or the hundredth part of the sum, contains only the hundredth part of that error; and the presumption that such an average is very near indeed to the truth, greatly exceeds the probability in favour of any one of the observations.

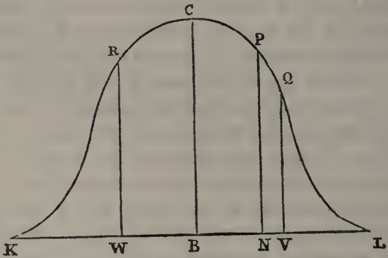
But before we proceed further, it becomes necessary to ask what laws of error can be absolutely determined, and shown in the nature of things to exist? And first, what do we mean by a *law of error?* Let A B be a length to be measured * or estimated, subject to error of observers and instruments; and let the greatest possible errors be B K (negative), and B L (positive); so that the result of measurement may be any thing

* The second figure is an enlargement of part of the first.

between A K and A L. Let K B and B L be equal, and suppose positive and negative errors equally likely

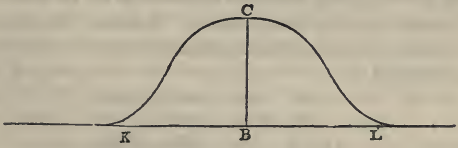


The probability of any one measurement giving *exactly* any predicted result, say A N, must be inappreciably small; since A N is only one out of an infinite number of possible cases. But take any point V, however near to N, and the chance that the result



of an observation shall lie between AN and AV is capable of being imagined to be finite, though small. Whatever the law of errors may be, let a curve KCL be so drawn that the chance of something between AN and AV shall be the proportion which the area NPQV is of the whole area KCL. Or if we call the whole area 1, the area RPNW, for example, is that fraction which expresses the chance of a result lying between AW and AV. The symmetry of the curve on the two sides of CB is an expression of the hypothesis of positive and negative errors being equally likely: and the approach of CK and CL towards the axis is equally an expression of the supposition that large errors are not so likely as small ones. If we wish to express that errors of all magnitudes are very nearly equally likely, we draw such a curve as the upper dotted line: while, if we wish to express that the presumptions are very strong for each measure being very nearly true, we describe the lower dotted line. We can thus figure to the eye a representation of the law of errors, be it what it may; and the description of a law of error is that of a curve.

Generally speaking, it is impossible to commit an error of more than a certain magnitude: but this circumstance is one which embarrasses the mathematical treatment of the subject exceedingly. It is practically the same thing to consider any error, however great, as possible, but errors of more than a certain magnitude as extremely improbable. If, for instance, a case should arise in which an error of more than an inch is impossible, let it be agreed to consider such an error as not impossible, but so improbable that there shall not be an even chance of its happening once in a million of times. The curve which exhibits the law of error must then be of the following kind, never meeting the axis, but continually approaching towards it, so that the whole of its area from and after L, is incomparably small by the side of the area CBL.



The curve here drawn is something like that in p. 17., and we might suspect from the utility of the latter and of the tables derived from it, that it should play an important part in the present subject. This we shall presently see: in the mean while I proceed to explain the sense in which I use the terms *average balance*, *average error*, *mean risk of error* and *probable error*, to which I direct the reader's particular attention. When we consider the errors of different kinds as balancing each other, it follows that positive and negative errors being equally likely, the balance of all the errors will be trivial in a very great number of observations. The average of all the errors will be extremely small: and, in the long run, nothing. In this instance then, it is said that the average balance of error is nothing. But if positive and negative errors were not equally likely, the more probable class would, in the long run, predominate, and the average balance of error would be of a definite magnitude, positive when positive error is more likely than negative, and *vice versâ*.

The preceding has nothing to do with the average of the absolute magnitudes of errors, considered without reference to the distinction of positive and negative. For example, whether the greatest error may be a mile or an inch, it is equally true that the long run will establish a compensation, when positive and negative errors are equally likely. But in the former case, the average magnitude of the errors which occur will, *cæteris paribus*, as much exceed that in the latter, as a mile does an inch. This average magnitude of errors, independently of sign, is an important element of the whole question, because a tolerably probable estimation of its value can be found from the observations. Suppose, for instance, that fifteen observations or estimations gave the following results.*

722	1311	967	1309
933	1089	1344	858
1033	972	1250	1029
917	1294	744	

The average of these is 1051, and assuming this as the true result, the errors are,

329	260	84	258
118	38	293	193
18	79	199	22
134	243	307	

* These are not numbers written at hazard, but actual results of estimation, on a subject which it is not here necessary to explain.

The average of the errors is 172; subject of course to the supposition that the average of the observations is nearly true. But the error of the last assumption must be considerable before it can much affect the present result.

The average of all the errors, taken without reference to sign, will, in the present hypothesis, be the same thing in the long run as the average of positive error, or the average of negative error. The reason is, that the number of positive and negative errors will in the long run be equal, and also their sums. If, in a very large number of observations, there be s positive and s negative errors, and if the sum of each set be S, it is evident that the sth part of S (which is the average positive error, or the average negative error), is the same as the $2s$th part of 2S (which is the average error without reference to sign). But it is customary to prefer to the average error another function of the errors, which may be called the *mean risk* of error, and which differs from the average in the following manner. Every error, positive or negative, is an increase or diminution of the final result; just as every game won or lost at gambling is an increase or diminution of the stock of the player. On precisely the same principles as those explained in chapter V, I may consider an even chance of an error 2 as a thing to be compounded for by a certainty of an error 1. If, in a large number of observations $2s$, there will come a sum of positive errors equal to S, and the same of negative errors, and if, as in taking the average, the sum of the errors be the only material point, it may be considered that every observation will have either a positive error or a negative error, of the value of the sth part of S. The results of such a supposition will, in the long run, and so far as the sum of errors is concerned, represent the actual case under consideration. If, therefore, a person could compound for positive errors alone, leaving the negative ones to chance, he must suppose *every* observation to have the half of $S \div s$, or $S \div 2s$ of positive error, combined with such a negative error as chance may yield. That is, he must suppose all the negative errors altered by the introduction of such an additional positive error, and each of the positive errors increased or reduced to $S \div 2s$. And the same if he would compound for negative errors only: while, to compound for both, he must suppose every observation affected both by a positive error of $S \div 2s$ and by a negative error of the same amount. This latter case supposes every observation to be correct, which is the result in the long run. The use of this consideration is, to keep before the mind the average effect of positive error, *not upon those observations which have positive error, but upon all the observations*; and the same for negative errors. Suppose I have an instrument which makes positive and negative errors in equal numbers and to equal amounts, in the long run; and suppose that it is in my power totally to destroy positive error, leaving the chance of negative error as before. What is the effect upon the whole system of errors, positive and negative, one with another? What must I do with all the errors to reproduce the same amount of absolute error as before? I must affect every observation with one half of the average amount of positive error, or one half the magnitude which the positive errors have, one with another. Or look at it in this way; if I have to pay a shilling for every unit of positive error, for how much should another person take the risk off my hands, that is, for how much *per observation*, whether its error be positive or negative. If the average positive error were 1, I should have in the long run to pay at the rate of one shilling for every two observations, against which I should insure at the rate of sixpence per observation.

The mean risk of positive error, then, is the average positive error, when the errors of this kind are equally distributed over *all* the observations: and the same for negative error. When positive and negative error are equally likely, each is one half the average error, considered without reference to sign.

By the *probable error* I mean that amount of error which is such, that there is an even chance for exceeding or falling short of it. Thus if it be 1 to 1 that the error shall lie between 0 and 10, and of course the same that it shall exceed 10, then 10 is called the probable error. For any number greater than 10, the chances are (no matter how little) in favour of the error being within that number; for any thing less than 10, the chances are against the error falling within that amount.

Problem. The number of observations being n, and positive and negative errors being equally likely, required the probability that the average of the n observations lies within a given quantity k of the truth; or, M being the average, that the truth lies within $M + k$ and $M - k$.

Rule. (By Table I.) Take the average of the observations, find all the errors upon the supposition that the average is the true result, add together the squares of the errors, and divide the square of the number of observations by twice the sum of the squares of the errors. Call the result * the *weight* of the average. Multiply k by the square root of the weight; let the result be t; then the H answering to t in table I. is the probability required.

Rule. (By Table II.) Find the weight as in the last rule, and divide 62 by 130 times the square root of the weight. The result is the *probable error* of the average. Divide k by the probable error, and let the quotient be t; then the K answering to t in table II. is the probability required.

Example. In the preceding instance, what is the probability that the average 1051 lies within 50 of the truth. The squares of the errors are, 108241, 13924, 324, 17956, 67600, 1444, 6241, 59049, 7056,

* The reason of the appellation will be afterwards explained.

85849, 39601, 94249, 66564, 37249, 484, the sum of which is 605831; and twice this is 1211662. Divide 225, the square of 15, by 1211662, which gives ·0001856953, the weight of the average. The square root of the weight is ·013627, which multiplied by 50 gives ·6814, the value of t: that of H is then ·66. So that it is more than 3 to 2 that the true result of the preceding very discordant observations lies between 1001 and 1101

To use the second table, multiply ·013627 by 130 which gives 1·77151, by which divide 62, giving 35 very nearly. This is the probable error, so that it is an even chance the result lies between 1051 — 35 and 1051 + 35, or 1016 and 1086. Divide 50 by 35 giving 1·43 the value of t; for which in table II., K is ·66, as before.

I now proceed to explain the meaning of the term *weight*, as used above. When an observer has made various observations, one or more of which he thinks superior to the rest, as to the favourableness of the circumstances under which they were made, it follows that the good observations should *tell* more in the formation of the most probable result than the indifferent ones. If for example, a remarkably good trial give 10 and an indifferent one 11, it is not reasonable to say that $10\frac{1}{2}$ is the most probable result. If the first observation be remarkably good, it may seem not unfair to give it the force of four observations, or to let the number 10 have the weight which would result from four observations giving 10, 10, 10, 10, a fifth giving 11. On this supposition the average is the fifth part of 51, or $10\frac{1}{5}$, instead of $10\frac{1}{2}$. This was called giving the observations 10 and 11 weights of 4 and 1, and the method of finding an average is this: multiply every observation by its weight and divide the sum of the products by the sum of the weights. Such a method was adopted before the theory of probabilities was applied to the subject, as a direction of common sense. When that theory came into use, it was found that the square of the number of observations divided by twice the sum of the squares of the reputed errors (the average being reputed correct) ought to stand in the place of the weight in the preceding rule, whenever different averages are to be combined together to form one general average. If, for instance, one average of 100 observations gave 10 and another of 50 gave 11, and if the squares of 100 and 50 respectively divided by twice the sums of the squares of the errors gave 1·5 and 1·1, the most probable average of these averages would not be $10\frac{1}{2}$, but the product of 10 and 1·5 increased by that of 11 and 1·1, and divided by the sum of 1·5 and 1·1, which gives 10·4. Hence the term weight is now applied to the quotient above described.

When the law of error is of the kind figured in p. 17., the mean risk of either sort of error, the probable error, and the weight of a single observation, are connected by very simple relations, as follows:—

1. The mean risk is 200 divided by 709 times the square root of the weight; more nearly, ·2820953 divided by the square root of the weight.

2. The probable error is 62 divided by 130 times the square root of the weight; more nearly, ·476936 divided by the square root of the weight.

3. The weight is 113 divided by 1420 times the square of the mean risk.

4. The probable error is $1\frac{7}{10}$ of the mean risk; more nearly 1·690694 of the mean risk.

5. The weight is 5 divided by 22 times the square of the probable error; more nearly, ·227468 divided by the square of the probable error.

6. The mean risk is $\frac{10}{17}$ of the probable error; more nearly, ·591473 multiplied by the probable error.

We can thus find the remaining two, when either of the three is given. Of the three, I apprehend that the probable error refers to the most instructive notion; but the mean risk and the specific weight enter more usefully into formulæ of calculation. The average error, being twice the mean risk, is readily determined when wanted.

Many persons confuse the average and the probable error in their own minds: that is, they imagine it to be as likely that any error should exceed the average as fall short of it. That such cannot be the case is evident from the following consideration.

The average error depends upon the magnitude of the error, as well as upon the proportions in which errors of different magnitudes enter; the probable error depends only upon the latter. If, then, small errors enter in larger numbers than great ones, the probable error is rendered less than it would otherwise be. In determining the probable error, the error 100 entering once, counts no more than the error 1 entering once. But in the average error, the error 100, entering once, counts 100 times as much as the error 1 entering once. Consequently the former must be less than the latter. But whether the probable error exceed or fall short of the mean risk (*half* the average error), must depend on the law of error. In the present case the former considerably exceeds the latter.

It may be asked whether the preceding results are always strictly true. Granting that the probability of an error diminishes with its magnitude, and that positive and negative errors are equally likely (which are the only hypotheses of the preceding question), does it necessarily follow, whatever may be the law of the diminution of probability, that the mean risk of error will, in the long run, be $\frac{10}{17}$ of that error which is as often exceeded as not? The absolute answer to this is, that the assertion is not strictly true, except upon further suppositions as to the law of error. The manner in which this inconsistency is explained, depends upon whether the person asking the question be supposed to be an inquirer seeking methods of disciplining his judgment, or an experimental philosopher requiring only a sufficient practical rule for the treatment of a set of observations.

To one who is looking for sound principles, I observe that he does not want in this matter the exposition of the consequences of any one law of facility of error, but an account of the general character of those laws to which common sense and daily experience assure him that his faculties and means of observation are subject. The facts of which he stands assured are, that the probability of error does not diminish very rapidly at first, but that as the error we consider grows larger, its probability *does* diminish very rapidly, and becomes insensibly small for errors of a certain magnitude and upwards. No curve of comparison, drawn in the manner described in p. 132., will be a true representation of what we know on this subject, unless it have the general form, of which the following varieties are instances. Now though the preceding results are not strictly true for every curve which has such a form, yet there is a class of curves of this form, some variety or other of which will approach tolerably close to any line which can be drawn to resemble one of those in the

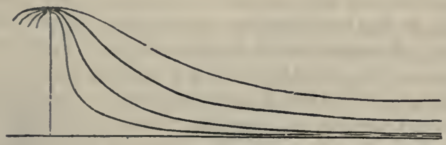


figure. In every one of this standard class of curves, all the preceding relations are strictly true, and therefore are nearly true for all the curves which resemble any one of the standard class. Thus though the average error may not always be $\frac{10}{17}$ of the probable error, yet the former is always some fraction of the latter, not differing very greatly from $\frac{10}{17}$. There is another reason for the adoption of this law of error as a standard, for which the reader may consult the fourth appendix to this work.

The experimenter, looking for a method of treating observations which shall produce trustworthy results, well knows that it matters nothing whether a method be true or false, if demonstration can be given that the consequences of the method are true. That falsehood necessarily produces falsehood is a fallacy, pardonable enough in everything but mathematics. True reasoning on true hypotheses must necessarily produce true results; false reasoning, or false principles, or both, may, and most probably will, lead to false consequences, but *may* lead to the direct reverse. In every part of knowledge, except mathematics, error must be carefully avoided, because there is no method of distinguishing between the cases in which it leads to truth, and the contrary cases. But. in the exact sciences, the knowledge of the consequences of falsehood and of those of truth are equally *exact:* and it is possible to introduce an erroneous addition to the conditions of a problem, to trace the consequences of such error, and to annihilate them at any part of the process. It is possible also to substitute for truth an erroneous supposition, in such manner that the effect of successive lapses of this kind shall be compensatory of each other, or so that the more often the error is repeated, the nearer is the result to the truth. The preceding case affords an instance: let the law of error be what it may (provided only that positive and negative errors are equally likely and that of two errors the larger is always the less probable), and let a moderately large number of observations be in question, and it follows that the results of the real law, and those of the preceding supposition, are nearly identical. Let the number of observations be still larger, and the resemblance is still nearer, and so on without limit. And this is true, even when the law of error, as regards a single observation, or two or three observations, varies to a large amount from that which is used above. Consequently, for a tolerable number of observations, it is absolutely indifferent whether the real law of error be known, or whether the nearest variety of the class under consideration be substituted for it.

Having then asserted, as a result of investigation, the existence of a standard law of facility of error which not only represents or resembles the impressions which unassisted reason would form *à priori*, but the results of which are more than sufficient mathematical approximations to truth, whatever (with some easily admissible limitations) the law of error may be, I proceed to describe it more particularly, calling it in future the *standard law of facility of error.* The sole datum necessary for its specific application, is either of the three, the weight of an observation, the mean risk of error, and the probable error, any two of which may be deduced from the third by the rules in p. 137.

PROBLEM. Given either of the three data, required —(A) the chance that the error of any one observation shall lie between *e* positive and *e* negative, or that the observation shall give something between *e* too much, and *e* too little—(B) the chance that the preceding shall not happen—(C) the chance that the error shall be positive, but not exceeding *e*; or that it shall be negative, not exceeding *e*.

RULE. Multiply *e* by the square root of the weight, and let the product be t; then (A) is the H corresponding to t in table I., and (B) is the remainder of (A) subtracted from unity; each of the chances called (C) is one half of (A).

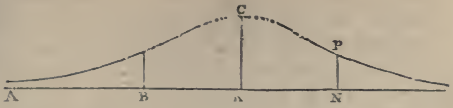
EXAMPLE. The mean risk of error is 10; required the chance of the error lying between + 15 and — 15; that is, between 15 too much, and 15 too little. Since the square root of the weight is 200 divided by 709 times the mean risk of error 10, or $\frac{20}{709}$, and since 15 times this result is $\frac{300}{709}$ or ·42; the probability required is ·45; or 11 to 9 against the event. The probability of a positive error less than 15 is ·225, and the same for a negative error within the same limits.

When all the individual observations are made under the same circumstances, so as to have the same weight, the common average is the most probable truth, and its weight is as many times the weight of one observation as there are observations in all: or, the average of n observations each of which has the weight w, is entitled to the same confidence as one observation made under circumstances which give it a weight nw. Of this we have an example in p. 137., and I now give another, detached from the method there given of finding the weight.

Example. The weight of each observation being 18, what is the probability that the average of 50 observations lies within ·05 of the truth. The weight of this average is 18×50 or 900, the square root of which is 30. Now $\cdot 05 \times 30$ is 1·5, which, being t, H is ·966, so that the probability required is about 23 to 1 in favour of the event. The mean risk of error of such an average is $\frac{200}{709}$ divided by 30, or $\frac{20}{2127}$, less than ·01; by which it is meant that if repeated sets of 50 observations each were made, the errors of these sets, neglecting their signs, would not average so much as $\cdot 01 \times 2$ or ·02 each.

Let us now suppose that positive and negative errors are *not* equally likely. Hitherto, the absolute truth has been the most likely; that is, though the probability of any one observation giving mathematical truth was infinitely small, yet so was that of any given error being exactly attained, and the infinitely small probability of the first case was greater than that of the second.* Let us now suppose that errors equal to or near to P are more probable than any other, and so that, x being the truth, any observation is equally likely to exceed or to fall short of (not x) but $x + \text{P}$. This is equivalent to describing a curve of the following figure in place of that in p. 132. Here AB is to be

* To compare the *proportions* of these indefinitely small probabilities, say of absolute truth and of either the error $+e$, or that of $-e$ exactly, take H′ from table I., corresponding to 0, and to e multiplied by the square root of c. Thus, c being 1, the probabilities of an error 0 and an error 2 are as 1·1 to ·02, or as 55 to 1.



measured, BK is P, and the chance of any error falling between 0 and BN is such a fraction of unity as the area BPN is of the whole area of the curve. The case before us is precisely that of an observer with a personal error equal to P, in addition to casualties. If we imagine a very large number of observations, made under circumstances equally favourable to positive and negative error, and if the error P be added to or taken away from each of the results, according as P is a positive or a negative error, we shall then have such a succession of results as might be looked for on the present hypothesis. For instance, let the quantity P be 10, then whatever may be the prospect of having the error 1, when positive and negative errors are equally likely, we have the same chance for the error 11, in the present case.

For simplification, I shall adopt the algebraical method of signifying positive and negative errors. Thus $+3$ means 3 too much, or 3 added to the truth; -4 means 4 too little. Thus $+4-5$ is -1; meaning that four too much from one cause, and 5 too little from another, gives on the whole 1 too little. Again, we consider -2 less than -1, since the effect of the former is to lessen the result more than would be done by the latter. To show that the supposition now before us is equivalent to that of an observer with a personal error, or an instrument with an individual error, imagine an instrument wrongly graduated, so that for every reading we ought to read 10 less: thus for 125 we ought to read 115. In other respects let the instrument be equally likely to give positive and negative departures from truth. If then an observation give 176 we know immediately that the truth is 176 − 10

$\pm$ a casual error, the effect of which disappears in a large number of observations. The whole error then is $10 \pm$ the casual error, $10 + x$ and $10 - x$ being equally likely. This is precisely the hypothesis in question, in the case in which $P = 10$.

The average of observations, in the case before us, does not necessarily give an approximation to the truth. Calling the quantity P a *fixed* error, meaning by a fixed error one which is as likely to be exceeded as not, we see, in the following theorem, a justification of the term : the most probable result of a large number of observations is the truth, increased or diminished by the fixed error, according as it is positive or negative. This error may either be fixed in the instrument, fixed in the observer, or in different degrees in both.

Let the phenomenon to be observed be perfectly unknown, except by what the instrument tells us. Then it is totally impossible to discover the amount of this error ; which, nevertheless, must be assumed to exist, unless the contrary be shewn. For example, an instrument wrongly graduated throughout could never tell the truth, either in individual or average results. But it is obvious that such an apparatus, incapable as it is of telling absolute truths, might nevertheless detect such results as are obtained by measuring the differences of other results. Thus a clock which goes truly, but is set too fast or too slow, will serve to find the time elapsed between two events, though it will not show the real time of either. Instruments which, on account of some permanent error affecting all their results, can only be used to determine differences, are called *differential* instruments.

All instruments, as well as observers, are subject more or less to this species of error ; how then is it possible to depend upon the results of any observations? The answer to this question will require some detail. Since perfect exactness cannot be attained, either on the part of the instrument, or of the observer, we can only call either good, when positive errors are as likely as negative. The average of a large number of observations will in such a case be extremely near the truth, and provided this condition can be fulfilled, the absolute amount of the tendency to error is comparatively unimportant. Let two observers, A and B, have instruments the average error of the first of which is double of that of the second. A given number of observations made by A is not as likely to be within a given amount of the truth in the average as the same number made by B. But the former may more than make the difference, by taking a larger number of observations. The rule is, that the square roots of the numbers of observations must be in proportion to the average errors of the instruments. That is, if A's instrument have an average error double of that of B, he must make four times B's number of observations before he can place the same reliance upon his own observations which he ought to do upon those of B. And the same is true if for average error we read mean risk, or probable error. But if the weights of the observations be known, the numbers of observations (and not their square roots), must be inversely as the weights.

When, however, there is a fixed error in the instrument, independently of casual errors, or of such as are as likely to be positive as negative, there are two modes of proceeding. The average of the observations will now be too great or too small, according as the fixed error is positive or negative.

1. If the truth can be found by any other means, in any one instance, a large number of observations, such as would be made if the truth in that one instance were the object of inquiry, will serve to detect the fixed error, with a high degree of probability that the result shall be correct. If a result should be 29 and the average of 100 observations give 28, then it must be presumed that instead of errors $+x$ and $-x$ being equally likely $-1+x$ and $-1-x$ are equally likely, or there is a fixed error of -1. If A be the true result, and if P be the fixed error, then $A+P$ is

the result which the instrument would give, in the long run. In p. 137. is shewn the method of determining the chance that in s observations, the average should lie within e of the final average, that is, within e of $A + P$. This ascertains the chance that the final average just mentioned lies between $A + P + e$ and $A + P - e$. Let R be the result shown by the instrument, the truth A being otherwise known. Then if R lies between $A + P - e$ and $A + P + e$, it follows that P lies between $R - A + e$ and $R - A - e$, and the chance of the first is that of the second.

Example. The truth being known to be 30, and the average of 20 observations giving 31, what is the chance that there is in the instrument a fixed error lying between $1 + \frac{1}{4}$ and $1 - \frac{1}{4}$, or between $\frac{3}{4}$ and $\frac{5}{4}$.

The weight of the observations must first be found, which is done by summing the squares of the errors, taking the average given by the instrument as true, precisely in the manner used in p. 137. Suppose this weight to be 10; then e or $\frac{1}{4}$, multiplied by the square root of 10, or 3·162, is ·79, to which in table I., the value of H is ·74. It is therefore about 3 to 1 that the instrument, in the long run, would give a result between $31 + \frac{1}{4}$ and $31 - \frac{1}{4}$; that is, that there is a fixed error in the instrument lying between $1 + \frac{1}{4}$ and $1 - \frac{1}{4}$.

This first method, then, of ascertaining the fixed error of a set of observations, supposes that there are cases in which the result is known beforehand, so that the instrument may first be read by the aid of phenomena, instead of phenomena by the instrument. The first observation is that of the error of the latter, found by comparing its indications with the known truth; the second, the observation of unknown phenomena, follows: accurate results being obtained, not by altering the instrument, but by applying the correction to the observations which the preceding class of observations has rendered necessary.

A reader unused to astronomical works, on opening a book on the practical part of the science, might imagine that no part of the subject pretended even to ordinary accuracy. Nothing appears to be done which is unaffected by serious error; and it seems as if a little more attention to the fabrication of instruments would render nine tenths of what has been written altogether useless. This appearance is the victory of the head over the hands; the means of detecting the errors of instruments are much more powerful than those of correcting them. It is also the victory of astronomy over the other physical sciences, on our knowledge of which the manufacture of utensils depends: we know more of the laws which regulate the changes of the heavens than we do of those on which the stability and fluctuation of instruments depend. Nor does the semblance above mentioned entirely spring from unavoidable error: for it is frequently the most convenient plan to allow an error to subsist which might be corrected at once, but which may be more easily corrected at another stage of the process. It is also sometimes useful to allow an error to remain of a larger magnitude than is physically necessary, if by that means another risk of error may be avoided. For instance, if requisite correction be either an addition or a subtraction, sometimes one and sometimes the other, the most practised calculator will be very liable to confound the two. This may be remedied by allowing to the instrument a fixed error, either additive or subtractive, of such a magnitude that casual fluctuations will never alter its name. The correction, therefore, will always require the same process, and the risk of error arising from taking the wrong method will be avoided.

2. The next plan of eliminating the fixed errors of an instrument is by giving it such a construction that an observation can be made in two different ways, in which the fixed error must necessarily have different signs, and must be of the same amount in both cases. This is in reality a method of making the positive and negative errors of the same amount in

the long run. Suppose, for example, that (as in the transit-instrument) the correctness of the observations depends (partly) upon the line of sight of a telescope being always exactly perpendicular to the axis upon which the telescope turns. Such exact perpendicularity is a mathematical fiction, which was never yet realised: the telescope will incline more or less to the right or to the left. But if the telescope be fixed to its axis, and if the axis itself rest on pivots, from which it can be taken off and the position of the instrument reversed, it is obvious that such a reversal of the ends of the axis will alter the error of the instrument, throwing the line of sight as much to the left as it was before to the right, or *vice versâ*. The average of a large number of observations will now present no signs of fixed error, arising from this cause at least: provided that the numbers of observations made in the two different positions be equal. The chance of the average of s observations in each position lying within a given degree of nearness to the truth is precisely that of twice s observations made with no fixed error, and the same tendency to positive and negative casualties as before. When the result has been obtained by combination of the different sets, the fixed error of the instrument may be ascertained by comparing the combined average with the separate averages of each set. If the observations be numerous, and the reversal of the method of observing introduce no new errors, then the combined average will be an arithmetical mean (or nearly so) between the other averages, and the difference between the former and either of the latter will be the fixed error required.

Independently of one or other of these two methods, the only result directly furnished by the observations (except the average affected by the fixed error) is their weight, which is obtained precisely as in p. 137. It will be seen that either of the preceding methods introduces entirely new elements; in the first we have previous truths for comparison with subsequent observations; in the second, an adaptation of the instrument which reproduces an equal likelihood of positive and negative error, by making the fixed error itself as often positive as negative. What we have called a fixed error is in fact a part of the phenomenon, styled an error because it is not a part of the result we wish to observe. The errors which a simple application of our theory removes are those of which no account whatever can be given, and of which nothing can be previously known. Such is not the case when positive error is more likely than negative, or *vice versâ:* for this very circumstance is itself a phenomenon, which must arise from some unvarying cause.

Having stated that it is indifferent, in a mathematical point of view, whether the law of the facility of error above explained be true or not, because any law whatsoever which falls within the widest permission of common sense leads to the same results as above explained, when the number of observations is considerable—I will now point out, in some simple cases, how different laws of error are to be reduced to the preceding.

1. Let all errors, positive and negative, between + E and − E be equally probable, and all others impossible. Treat large numbers of such observations as in pages 137. and 144., on the supposition that the weight is 3 divided by twice the square of E.

2. Let the probability of error decrease uniformly as the magnitude increases, the greatest possible errors being + E and − E; which implies that the chance of an error lying between $-x$ and $+x$ is the product of x and the remainder of 2 E divided by the square of E. For instance, if E be 10, and this law of facility prevail, the chance of an error lying between − 2 and + 2, is the product of 2 and 18 divided by 10 times 10, or $\frac{36}{100}$. In this case a large number of observations must be treated as if the weight of each observation were 3 divided by the square of E.

3. If the weight of the observations be considered

to be 5 divided by twice E^2, the preceding methods will show the chances of a large number of observations, upon a supposition intermediate between the two last, and coinciding nearly with the first when the errors are small, and with the second when they are considerable.

I now proceed to the method of combining the results of observation, and deducing the mean risks of error. Suppose, for example, that A and B are two results of a large number of observations, of which the product is required. Nothing can be more erroneous than to suppose that the mean risk of this product will be the product of the mean risks of its factors. By the mean risk here is meant the same thing as before: imagine the product of $A \times B$ formed in every possible way from the single results of the several observations. Each error will be as likely to be positive as negative, if the errors of the original observations be the same. Take the average of all the several errors, neglecting their signs, and one half of this average will be (in the long run) what is called the mean risk, explained in the same manner as in p. 135.

The risk of the result will be modified by the manner in which the operation makes one quantity affect the error of the other. Suppose, for example, that observation gives $A = 100$, and $B = 150$. If these were certainly true, the required result would be accurately 100×150, or 15,000. But if 100 be wrong to any amount, the product will be wrong by 150 times that amount, on that account only: while, if B be wrong, its error will be multiplied 100 times in the result; besides which, there will be an additional error, the product of the two errors of A and B. On the other hand, if observation give A too large and B too small, the opposite errors may either compensate each other exactly, and give the product precisely what it ought to be, or may make some approach towards this compensation. The product then may be rendered much more erroneous than the observations, or much less so; both of which possible cases are considered, in all their extent, in the investigations which give the following results. In all of them, except the first, the mean risks are supposed to be small.

1. To find the mean risk of the sum or difference of any number of quantities determined by observation, add together the squares of all their mean risks, and extract the square root of the result. Thus, if the mean risks of two quantities be 3 and 4, that of their sum or difference is the square root of 16 + 4, that is, 5. If the mean risks be all equal, the rule may be simplified into that of multiplying the mean risk of one by the square root of the number of quantities. Thus the mean risk of the sum of 100 observed quantities of equal risk is 10 times that of one of them.

EXAMPLE. Given the mean risks of A, B, and C, namely, 1, 2, and 3, required that of 10 A + 9 B − 4 C. Here every error which can happen in A is made tenfold in 10 A, and the mean risk of 10 A is 10 × 1 or 10. Similarly the mean risks of 9 B and 4 C, are 9 × 2 and 4 × 3 or 18 and 12. The squares of 10, 18, and 12, added together, give 568, the square root of which is 23·8, the mean risk required.

It may seem strange at first sight that, *cæteris paribus*, the mean risk of a sum and difference should be the same. But a little consideration will show that, positive and negative errors being equally likely, the errors of a difference may be as large as those of a sum: and that no combination of errors can affect a sum, without an equal probability of another equally probable combination affecting the difference in the same way.

2. To find the mean risk of the product of any number of quantities A, B, C, &c. Take the fraction which each mean risk is of its quantity: add the squares of these fractions, and multiply the square root of the result by the product itself. This rule is

only to be trusted when the mean risks are small. Let A = 100, B = 150, and let the mean risks of A and B be 1 and 2. Then 1 is ·01 of 100, and 2 is ·0133 of 150. The squares of ·01 and ·0133 are ·0001 and ·00017689, the sum of which is ·00027689, the square root of which is ·0167. This multiplied by 100 × 150, or 15,000, gives 250·5, which is the mean risk of the product 15,000.

4. To find the mean risk of a fraction, or of the quotient of a division, multiply each term (numerator and denominator, or dividend and divisor) by the mean risk of the other, add the squares of these products and extract the square root of the sum: divide this by the square of the denominator or divisor; the result is the mean risk required. But if the fraction be very small, it is sufficient to divide the mean risk of the numerator by the denominator; while if the fraction be very great, it is sufficient to multiply the fraction by the risk of the denominator, and to divide the result by the denominator.

The preceding will serve as specimens of the manner in which complicated results of operation can have those probabilities investigated which depend upon the probabilities of error in their constituent parts. It would be impossible to lay before a reader unacquainted with the differential calculus, any such digest of rules as would enable him to treat all cases with facility. Any one of the mean risks obtained above will serve to determine, as in p. 139., the weight of the result, from which its law of error may be investigated, as in p. 143.

It appears that the chances of error may be considerably multiplied in the course of the operations to which the results of observation are subjected. It must, therefore, be the object of an inquirer not only to make good observations, but also to select such methods of observing, and such methods of treating the observations (the latter generally depending upon the former), as will render the final error the least possible. The considerations necessary for this purpose form a great part of the application of mathematics to the sciences of observation : in which it frequently happens that good methods of observing are rendered useless by the multiplication of error which the methods consequent upon them involve: and conversely, that formulæ good in other respects, are inadmissible from the tendency to error in the observations which they require. And it has happened before now that mistakes of serious amount have arisen from the use of mathematical methods in which the errors of the observations are much multiplied.

I could hardly close such a chapter as the present without some mention of the celebrated *method of least squares*, on which the astronomy of the last thirty years has depended for much of the increase of accuracy which has been its characteristic. But as any development of this very interesting subject is impracticable without recourse to mathematical symbols and reasoning, I content myself with a description of one particular case, which is of very frequent occurrence.

Suppose a number of results to be obtained by observation, from which a consequence is to be drawn by mathematical reasoning. If the observations were all correct, the consequence deduced from any one would be the same as that from any other; but owing to the errors of the observations, such agreement is of course unattainable. It is, therefore, a question what method of combining the several results should be adopted: and mathematical analysis shows that the object is attained by choosing such an intermediate result as shall make the sum of the squares of the errors the least possible. It might seem as if, positive and negative errors being equally probable, the average of results is the most probable truth; and this is the case when the observation is itself made directly upon the result which is required, or when there is only one datum into which the uncertainty of observation is introduced. But even in such a case we have no right to say that the average is preferred to the result of the method of least squares; for the former is then a particular case of the latter. Let three observations give 9, 11, and 16, the average of which is 12. The errors, taking this average as the truth, are 3, 1 and 4, the sum of the squares of which is 26. This is the least possible sum of the squares. To try this, assume 11 as the most probable truth: the errors are then 2, 0, and 5, the sum of the squares of which is 29 : assume 13, and the errors are 4, 2, and 3, the sum of the squares of which is 29. To avoid introducing fractions, I have only assumed whole numbers, but if I take 12·1 or 11·9, I find the sum of the squares of the errors to be in the first case 26·03, and in the second 26·03. So that when one result only is in question, a direction to take the average is equivalent to a direction to make the sum of the squares of the errors the least possible.

Let us now suppose two results of observation, say that we wish to know the fraction which A is of B, where both A and B are subject to errors, the positive and negative being equally likely. Suppose, for example, that we ask for the proportion of the population of a country which is buried in a year. Statistical returns will furnish the population of each year, and the burials, both subject to errors. There are now obviously two ways of taking an average; I may either divide the average burials by the average population, or find the proportion which the first is of the second in each year, and take the average of the proportions. One not versed in mathematics would suppose that these must give the same results, but any simple instance will show the contrary. The average dividend and the average divisor do not give the average quotient. For instance, let dividends be 12, 13, and 17, and divisors 20, 22, and 30. The fractions $\frac{12}{20}$, $\frac{13}{22}$, and $\frac{17}{30}$, are ·6, ·591, and ·567, the average of which is ·586. The average dividend is 14, the average denominator 24 and $\frac{14}{24}$ is not ·586, but ·583. The results nearly coincide, but so do the data, for which reason the most probable result may be very nearly found, or of two results which differ very little, one may be much more probable than another. When observations give magnitudes so nearly coinciding as ·6 ·591 and ·567, it is worth while to examine the relative probabilities of methods which give results so nearly equivalent as ·586 and ·583. Which of the two preceding methods is most entitled to confidence? Analysis points out that this question is useless, because there is a third method which is more safe than any other. The method of least squares in the case before us leads to the following rule; — When both the numerator and denominator of a fraction are to be determined by observation, and various corresponding observations of both are made, multiply each numerator and denominator by the denominator, and divide the sum of the numerators so formed by the sum of the denominators. Thus in the preceding instance, it is

$$\frac{12 \times 20 + 13 \times 22 + 17 \times 30}{20 \times 20 + 22 \times 22 + 30 \times 30} \text{ or } \frac{1036}{1784} \text{ or } \cdot 581$$

which is more probable than either ·586 or ·583.

If the mean risks of all the observations be the same, the mean risk of the preceding result is found by adding 1 to the square of the result obtained (·581) dividing by the denominator which produced it (1784), extracting the square root of the quotient, and multiplying the mean risk of each observation by this square. Thus ·581 × ·581 is ·337561, which divided by 1784 gives ·000189, the square root of which is ·014. The mean risk of the result, therefore, is less than one seventieth part of that of each of the observations.

The method of least squares is an extension of that of taking an average, or rather it indicates the most probable average in cases which, by reason of more results of observation than one being involved, an infinite number of different averages exists. It is not yet introduced into the affairs of common life, though many cases occur in which it might be made useful. But many things which are only demonstrable by the higher branches of mathematics are looked upon as useless by those who do not understand them ; nor is this result of ignorance only to be looked for among the uneducated. While the Reform Bill was in its progress through the House of Commons, a method was suggested by a man of science, with whom the government advised upon the subject, for estimating the relative importance of boroughs by considering their population and contributions to the revenue combinedly. This method, to the efficiency of which most of those who examined it gave strong testimony, was ridiculed by some members of the house, partly because it involved decimal fractions, and partly because another and a more simple (but palpably wrong) method gave, in that particular case, nearly the same results. When legislators are neither able to see that erroneous methods may sometimes lead to truth, being not therefore one bit the less erroneous, nor that the truth of a result is the same, whether decimal or common fractions be employed, it is little to be wondered at if useful applications of abstract reasoning are looked upon with suspicion and introduced with difficulty.

CHAPTER VIII.

ON THE APPLICATION OF PROBABILITIES TO LIFE CONTINGENCIES.

When questions connected with life contingencies were first considered, it was regarded as most deliberate gambling to be in any way concerned in buying or selling such articles as annuities, or any interests depending upon them. Before we can well enter upon the question of the truth or falsehood of the preceding notion, it will be necessary to ask what laws the duration of human life follows, and whether it follow any laws at all? Take two separate hundreds of persons, each aged twenty, is there any reason to conclude that the united lives of all the first hundred will make an amount of years nearly equal to that of the second?

In order to try this point, I shall take another question, yet more unfavourable to the result which I wish to establish. In 100 persons all aged twenty, we know that there is but a very slight chance that any given one of them shall reach the age of eighty; and we may consider it a certainty (or of an extremely high probability), that none of them will see the age of a hundred and twenty. We will consider it therefore as given, that no one shall live to the last-mentioned age, and we will even suppose that all ages of death between 20 and 120 are equally probable. This of course very much increases our chance of fluctuation: but even with this supposition it is not very great.

Let us suppose a lottery in which there are counters marked with every possible number or fraction intermediate between 0 and E: so that the drawing may have any mark whatsoever. If then we draw out 100 counters, the least possible amount of drawings will be 0, the greatest 100 times E: and if all drawings be equally probable, we have no reason to suppose that our amount will exceed 50 times E, which does not equally apply in favour of its falling short of that quantity. That we shall have exactly 50 times E, is an event of which the chance is infinitely small: but that the amount shall lie between limits which are tolerably near 50 times E, is very probable.

Problem. Let there be counters, in equal numbers, with every possible mark between 0 and E. What is the probability that the average of n drawings shall not differ from the half of E, one way or the other, by more than k.

Rule. Multiply k by the square root of six times

n, and divide the product by E. Call the quotient t; then the value of H (Table I.) is the probability required.

Example. In 600 drawings, each of which may be any thing between 0 and 100, required the probability that the average of all the drawings shall lie between $50+5$ and $50-5$.

$n=600$, $E=100$, $k=5$; the square root of 6 times 600 is 60, and 5 times 60 divided by 100 is 3. The first table does not contain values of t higher than 2: an event being almost certain, or of a very high probability when t is equal to 2. Table II., however, furnishes us with an extension of Table I.; the K opposite to any value of t in that table being always nearly the H which belongs to half that value of t. Consequently, the H belonging to $t=3$, is the K belonging to $t=6$. But the second table only goes to $t=5$; in which case K is ·999. It is then more than 999 to 1 that the average of the 600 drawings is within the limits specified. If we take $k=1$, in which case $t=\cdot 6$, we find it is 3 to 2 that the average is contained between 49 and 51.

If then there were 600 infants born, and if it were the law of human life that any individual is as likely to die at one age as another, for any age not exceeding 100 years, even then, and with so much more scope for fluctuation than is actually found, it would be more than 999 to 1 against the average life of the 600 infants exceeding 55, or falling short of 45 years; and more than 3 to 2 that the same average should fall between 49 and 51 years. If such be the case, it is obvious that the chances of fluctuation are much diminished by the superior chances of death happening at some periods of life rather than at others; as well as by the smaller limits of human life, which need not for any practical purpose be supposed to extend as far as one hundred years.

To suppose that the duration of human life is regulated by *no* laws, would be to make an assumption of a most monstrous character, *à priori*, and most evidently false. For it is a law, were it the only one, that no individual shall attain the age of 200 years. So much is known to all; but to those who consider the subject more closely, by the aid of recorded facts, it may be made as evident as the existence of a limit to human life, that the casualties of mortality are distributed among mankind in so uniform a manner, that the average existence of a thousand infants will differ very little from that of another thousand born in the same country and station of life. It is true that differences of race, climate, manner of living, &c., &c., produce marked effects upon the duration of life; which is no more than might be expected: but it is equally true that the notorious *individual* uncertainty of life cannot be discovered in the results of observations made upon masses of individuals.

There are various results of observation, which are called *tables of mortality*, which differ only in the methods of presenting the same sets of facts. Firstly, we have what may be called tables of the numbers living. These show, for a given number born, how many attain each year of age. Thus, in the Carlisle table, opposite to 0 and 50, we find 10,000 and 4397, indicating that, according to observations made at Carlisle, the proportion of those born to those who saw their fiftieth birthday, was that of 10,000 to 4397. Again, opposite to 60, we find 3643, meaning, that of 4397 persons aged 50, 3643 attain the age of 60. Secondly, we have tables of *yearly decrements*, in which the same number of persons are supposed to be alive at every age, and the proportion who die in the next year is set down in the table. Thus in the government annuity tables, opposite to 50 and 60, we find 161 and 315, meaning that, according to the observations from which these tables were constructed, of 10,000 persons aged 50, 161 died before completing the next year of life; and of 10,000 persons aged 60, 315 died before attaining the age of 61. Thirdly, we have tables of *mean duration* of life (commonly called expectation of life), which show the average number of years enjoyed by individuals of every age. This, in another variety of the Carlisle tables, opposite to 50 and 60, we find 21·11 and 14·34; meaning that, according to these tables, persons aged fifty live, one with another, 21·11 years more, and persons aged 60, 14·34 years more.

Until observations of human mortality become more extensive and correct, I prefer the tables of mean duration to all others. The events of single years are subject to considerable error, and generally present such varieties of fluctuation, that it has become usual to take some arbitrary and purely hypothetical mode of introducing regularity. This practice cannot be too strongly condemned, since the tables thereby lose some of their value as representations of physical facts, without any advantage ultimately gained. For if by using the raw result of experiments, tables of annuities were rendered unequal and irregular, it would be as easy, and much more safe, to apply the arbitrary method of correction to the money results themselves, than to introduce it at a previous stage of the process. It is not, however, a matter of much consequence as to the annuities, &c., deduced from the tables: and as yet, the rudeness of the original observations renders the effect of any such alteration not so great as the probable errors of the observations themselves.

The mean duration of life is approximately calculated as follows. Suppose (taking an instance from the Carlisle tables) that 75 persons are alive at the age of 92, of whom are left at the successive birthdays, 54, 40, 30, 23, 18, 14, 11, 9, 7, 5, 3, 1, 0. Consequently, in their 93d year, 54 persons enjoy a complete year of life, and 21 die, whom we may suppose, one with another, to live through half the year, and 54 years and 21 half years make $64\frac{1}{2}$ years, which is the total life of 75 persons for that year. Proceeding in this way, we find that there are,

in the 93d	year	$54 + \frac{1}{2}$ of 21 years.
94th		$40 + \frac{1}{2}$ of 14

95th	30 + ½ of 10
96th	23 + ½ of 7
97th	18 + ½ of 5
98th	14 + ½ of 4
99th	11 + ½ of 3
100th	9 + ½ of 2
101st	7 + ½ of 2
102d	5 + ½ of 2
103d	3 + ½ of 2
104th	1 + ½ of 2
105th	0 + ½ of 1
	Total, 215 + ½ of 75

Hence 75 individuals, aged 92, enjoy 215 + ½ of 75 years, and each has, one with another, the 75th part of this, or 3·37 years.

Rule. To find the mean duration of life from a table of the living at every age out of a given number born, add together the numbers in the table for all the ages above the given age, divide by the number at the given age, and add half a year to the result.

The preceding rule is mathematically incorrect, being only an approximation to the truth, even supposing the tables perfectly correct. The error of computation may be found, nearly, as follows. Divide the number who die in the year next following the given age by twelve times the number in the table at that age, and diminish the result of the preceding rule by the quotient. Thus, in the instance before us, 21 divided by 12 times 75 is ·02, so that 3·35 is nearer the truth. This error, however, is immaterial for practical purposes.

A more important question is that of the degree of confidence which may be placed in tables of mean duration, the errors of observation being supposed to be as likely to be positive as negative. In order to estimate this, we must compute the mean square of the duration of life; that is, multiplying the time which each individual lives by itself, we must add the results together and divide by the whole number of individuals. To make a rough approximation to this in the case before us, remember that

21 individuals live $\frac{1}{2}$ a year	$\frac{1}{2}\times\frac{1}{2}=\frac{1}{4}$	giving	$\frac{21}{4}$
14 — $\frac{3}{2}$ —	$\frac{3}{2}\times\frac{3}{2}=\frac{9}{4}$	—	$\frac{126}{4}$
10 — $\frac{5}{2}$ —	$\frac{5}{2}\times\frac{5}{2}=\frac{25}{4}$	—	$\frac{250}{4}$
7 — $\frac{7}{2}$ —	$\frac{7}{2}\times\frac{7}{2}=\frac{49}{4}$	—	$\frac{343}{4}$
5 — $\frac{9}{2}$ —	$\frac{9}{2}\times\frac{9}{2}=\frac{81}{4}$	—	$\frac{405}{4}$
4 — $\frac{11}{2}$ —	$\frac{11}{2}\times\frac{11}{2}=\frac{121}{4}$	—	$\frac{484}{4}$
3 — $\frac{13}{2}$ —	$\frac{13}{2}\times\frac{13}{2}=\frac{169}{4}$	—	$\frac{507}{4}$
2 — $\frac{15}{2}$ —	$\frac{15}{2}\times\frac{15}{2}=\frac{225}{4}$	—	$\frac{450}{4}$
2 — $\frac{17}{2}$ —	$\frac{17}{2}\times\frac{17}{2}=\frac{289}{4}$	—	$\frac{578}{4}$
2 — $\frac{19}{2}$ —	$\frac{19}{2}\times\frac{19}{2}=\frac{361}{4}$	—	$\frac{722}{4}$
2 — $\frac{21}{2}$ —	$\frac{21}{2}\times\frac{21}{2}=\frac{441}{4}$	—	$\frac{882}{4}$
2 — $\frac{23}{2}$ —	$\frac{23}{2}\times\frac{23}{2}=\frac{529}{4}$	—	$\frac{1058}{4}$
1 — $\frac{25}{2}$ —	$\frac{25}{2}\times\frac{25}{2}=\frac{625}{4}$	—	$\frac{625}{4}$
75	(Average square 21·5)		$\frac{6451}{4}$

RULE. From the mean square of the duration of life at any age, subtract the square of the mean duration at that age: divide the difference by the number of lives of the given age from which the table was made, and extract the square root of the quotient. Take four tenths (more correctly ·39894), of this square root, which gives the mean risk of error, and ·67 of the square root gives the probable error.

Suppose that in the case before us, the number of lives aged 92 was 40 *, from which the preceding table was made. We have then,

Mean square of durations	21·5
Square of 3·37, the mean duration	11·36
	40)10·14
	·254 $\sqrt{\cdot 254}=\cdot 504$

·504 × ·67 = ·33 of a year, the probable error.

The same process may be applied to any other case, and the result of the whole is, that observation of a number of lives which is not very great, will be suffi-

* This is nearly the number of lives at that age among those from which the Carlisle table was formed, but the arbitrary help introduced from other tables at the older ages, on account of presumed insufficiency of data, makes the result of this example of no greater value than a numerical instance arbitrarily chosen.

cient to give the mean duration of life with considerable approach to exactness. This is confirmed by the results of various tables, from which it appears that when the individuals composing an observation are of the same country, and under the same general circumstances, the results of such tables come very near to each other.

The reader who desires to know the history of tables of mortality should consult the articles MORTALITY and ANNUITIES in the new edition of the Encyclopædia Britannica, both from the accurate pen of Mr. Milne, the author of the Carlisle tables. I cannot, in this work, pretend to give more than a slight summary of results connected with life contingencies, such as may guide the reader who understands the main points of the theory of probabilities to safe conclusions.

From some tables made from observations at Breslau, De Moivre concluded that the following hypothesis, namely, that of 86 persons born one dies every year till all are extinct, would very nearly represent the mortality of the greater part of life, and that its errors would nearly compensate one another in the calculation of annuities. The Northampton tables of Dr. Price, which have been used by most of the insurance offices, very nearly represent this hypothesis at all the middle ages. But both give much too large a mortality for the circumstances of the last half century, as is proved by all the tables which have been lately constructed. The greater part of the difference, I have no doubt, is due to the real improvement of life which has taken place, from the introduction of vaccination, more temperate habits of life *, better medical assistance, and greater cleanliness in towns. We may now state, as a much nearer approximation to the mortality of the

* I must be understood, here, as speaking particularly of the middle classes, in English towns and cities. Most of the tables have a majority of this class, and there is not any very precise information on the mortality of the labouring classes, or in the inhabitants of the country as distinguished from those in towns. With regard to the point on which this note is written, all old persons remember the time when what we should now call hard drinking was almost universal.

middle classes, that from the age of 15 to that of 65, the *average* may be represented as follows: — of 100 persons aged 15, one dies every year till the age of 65. But the mean duration of life will serve to give a more precise idea, and a simple rule may be given, which will, for rough purposes, represent the Carlisle table between the ages of 10 and 60. Of persons aged 10 years, the average remaining life is 49 years, with a diminution of 7 years for every 10 years elapsed; thus of persons aged 20 years, the average remaining life is 49–7 or 42 years; at 30 years of age, 35 years. The following list of tables will be followed by some notice of each.

Years of age.	De Moivre's hypothesis.	Northampton.	Amicable.	Carlisle.	Equitable.	Government, males.	Government, females.	Years of age.
0	43	25·2	- -	38·7	- -	- -	- -	0
5	40·5	40·8	- -	51·3	- -	48·9	54·2	5
10	38	39·8	- -	48·8	48·3	45·6	51·1	10
15	35·5	36·5	- -	45·0	45·0	41·8	47·2	15
20	33	33·4	36·6	41·5	41·7	38.4	44·0	20
25	30·5	30·9	34·1	37·9	38·1	35·9	40·8	25
30	28	28·3	31·1	34·3	34·5	33·2	37·6	30
35	25·5	25·7	27·7	31·0	30·9	30·2	34·3	35
40	23	23·1	24·4	27·6	27·4	27·0	31·1	40
45	20·5	20·5	21·1	24·5	23·9	23·8	27·8	45
50	18	18·0	17·9	21·1	20·4	20·3	24·4	50
55	15·5	15·6	15·1	17·6	17·0	17·2	20·8	55
60	13	13·2	12·5	14·3	13·9	14·4	17·3	60
65	10·5	10·9	9·9	11·8	11·1	11·6	14·0	65
70	8	8·6	7·8	9·2	8·7	9·2	11.0	70
75	5·5	6·5	6·2	7·0	6·6	7·1	8·5	75
80	3	4·8	5·0	5·5	4·8	4·9	6·5	80
85	0·5	3·4	4·0	4·1	3·4	3·1	4·8	85
90	- -	2·4	2·9	3·3	2·6	2·0	2·8	90
95	- -	0·8	1·4	3·5	1·1	1·2	1·6	95
100	- -	-	.. -	2·3	- -	- -	- -	100

1. *De Moivre's hypothesis* was suggested by Halley's Breslau tables, made from observations of the mortality of that town in the years 1687—1691. It confessedly errs considerably at the beginning and end of life.

2. The *Northampton* tables were constructed by Dr. Price from the mortality of that town, in the years 1741—1780, the numbers of male and female deaths being very nearly equal. These tables were, and are, almost universally used by the assurance offices, and are those by which legacy duties are estimated in the act of parliament, 36 Geo. III. cap. 52.

3. The *Amicable Society*'s table was formed some years ago by Mr. Finlaison, at the suggestion of that gentleman and myself to the directors, and as a means of furnishing information upon points * as to which they had consulted us. The Amicable society was founded in 1705, and the table is formed from the experience of more than half the subsequent period ending in 1831.

4. The *Carlisle* table, formed by Mr. Milne from the observations of Dr. Heysham upon the mortality of that town, in the years 1779—1787. They are to be considered the best existing tables of healthy life which have been constructed in England. The relative proportions of the sexes are 9 females to 8 males.

5. The *Equitable* table (published by the Equitable society in 1834) gives the results of the experience of that society from 1762 to 1829. The total number of deaths recorded is upwards of 5000.

6. The *Government* tables (male and female life separately). These tables were constructed by Mr. Finlaison, actuary of the national debt office, from various tontines, &c., of which the records are in the possession of the government. Each table contains about 5000

* In mentioning this subject, I may be allowed to state my full approval of the plan subsequently adopted by the society, and my conviction that the errors of their ancient system have entirely disappeared.

The mean durations above given were computed by myself, from the tables of decrements circulated by the directors among the members.

deaths. These are the tables on which the commissioners for the reduction of the national debt grant life annuities in lieu of stock.

I will now add some deductions made by myself from the tables contained in the *Recherches sur la Reproduction, &c., &c.* Brussels, 1832, by M. Quetelet and Smits; republished in the treatise *Sur l'Homme, &c.* of the former. They are founded upon the statistical returns of the whole of Belgium, made in three successive years, and distinguish not only the sexes but the residences of the parties, whether in towns or in the country. The middle table is the general average of the whole country, whether male or female, in town or country.

Age.	Towns.		Both.	Country.		Age.
	Males.	Females.	Both.	Males.	Females.	
0	29·2	33·3	32·2	32·0	32·9	0
5	45·0	47·1	45·7	46·1	44·8	5
10	42·9	45·0	43·9	44·4	42·9	10
15	39·0	41.3	40·5	41·2	40·0	15
20	35·4	38·0	37.3	38·1	37·0	20
25	33·1	35·0	34·7	35·7	34·2	25
30	30·4	32·1	32·0	33·0	31·5	30
35	27·5	29·2	28·9	29·7	28·7	35
40	24·4	26·5	25·8	26·0	25·9	40
45	21·5	23·3	22·7	22·5	23·2	45
50	18·3	20·1	19·5	19·1	20·0	50
55	15·5	17·1	16·4	16·2	16·9	55
60	12·8	14·0	13·4	13·3	13·7	60
65	10·4	11·2	10·8	10·6	10·9	65
70	8·2	8·6	8·4	8·2	8·5	70
75	6·3	6·6	6·4	6·3	6·5	75
80	4·8	5·1	5·0	5·0	5·1	80
85	3·7	4·0	3.8	3·8	3·8	85
90	2·9	3·0	3·1	3·1	3·2	90
95	1·8	2·0	2·1	2·2	1·9	95
100	0·0	0·5	1·3	0·5	0·5	100

Most tables in which the sexes are distinguished unite in presenting this result, that female life is materially better than male life. But this fact is much more distinctly apparent in towns than in the country, and in the Belgian tables the phenomenon is reversed, so that while female life is decidedly better than male life in the towns, it is not so good in the country. Mr. Milne has remarked that in Stockholm the difference between male and female mortality was three times as large a per centage of the whole as it was in all Sweden. The probable reason for this discordance is the different employment of women in town and country; all the tables yet constructed which distinguish the sexes, and include rural life, having been made from a great preponderance of the working classes. The only tables which separate the sexes, and which are formed from the middle classes, are those of Mr. Finlaison; and here the difference is greatest of all.

This consideration is very material in comparing the tables which I have given. If a table of male life should fall short of one of female life, all other circumstances remaining the same, it is no more than we might expect; while at the same time the true proportions of male and female life, as well as the manner in which they depend on local or other circumstances, are very imperfectly known. But if a table of male life only should present the same results as one of mixed lives, we are then sure that the former represents a longer duration of existence. For instance, the table of the Equitable insurance office, which is almost entirely composed of males, is almost identical with the Carlisle table in which there are more females than males. This shows that the select male lives of the office are much better than the *male* lives of the Carlisle table: but that the male lives of the office, constantly recruited as they have been with selected lives of all ages, are *no better* than the mixed lives of the Carlisle table. Similarly, the males of the Amicable table are very much better than those of the Northampton table. The male and female lives of the latter are nearly equal in number; the former is almost entirely founded upon male lives: while the former, with its male lives only, gives a longer duration of life than the latter. For old lives, however, the Northampton table gives a somewhat longer duration than the Amicable. This is only one fact out of many which show that the Northampton table, while it gives much too great a mortality to the younger class of lives, errs in the other extreme as to the older. Of thirty-five tables, made in different countries and at different times, and including all of any celebrity which had appeared before 1830, I find that the Northampton table is the eighth from the lowest at the ages of 10—25, and the tenth from the highest at the age of 65. The same may be said of De Moivre's hypothesis, which the Northampton table closely follows.

The Northampton and Amicable tables are decidedly older as to the period at which their members lived, than the Carlisle and Equitable. Life is shorter in the former pair than in the latter, while both of the former agree in presenting the older lives comparatively better than the younger ones, as compared with the latter pair. By the Northampton table, the duration at 65 years is about a third part of that at 25; while the same proportion is decidedly less in the same ages of the Carlisle table: a similar result appears in the Amicable and Equitable tables. I remember remarking the same phenomenon in the results of a comparison of the lives of naval officers. There can be little doubt that the reason is as follows: in circumstances which create a large mortality at the younger ages, all the feeble constitutions are prevented from attaining old age, so that the lives which really arrive at advanced years are the remains of the very best lives. I saw the ocurrence of the same disproportion in the lives of officers of the Anglo-Indian army; in which, however, it was probably increased by the residence of many of the officers in question in England during the latter years of their lives.

The Equitable society has the character of having been much more careful in the selection of its lives than was the Amicable society during the earlier part of its existence. This, together with the gradual improvement of human life, serves to explain the very great difference between the results of their experience. The latter years of the Amicable society do not exihibit any very decided difference of the sort.

The state in which we stand with respect to tables of human life is singular, considering the enormous amounts which daily pass from hand to hand in the purchase of life interests. I may have occasion to speak more at length on this subject in the sequel: in the mean time let the reader observe the difference between the various tables, and remember that each has its votaries. If the late Mr. Morgan (whose name stands very high as an authority on such matters) had been requested to state the value of an annuity on the life of a female aged 40, and the same for a male of the same age, he would have replied that there was no material difference between male and female life, and that both belong to a class whose average existence is 23 years; and he would accordingly have used the Northampton tables of annuities. At the national debt office, it would have been answered by Mr. Finlaison that the male and female life are two very distinct cases, and that the two different classes to which they belong have severally the average lives of 27 and 31 years. That such differences should exist, is a proof of insufficiency of information upon the subject: a want which nothing but the government can supply, but which no government ever will attempt to supply until increasing knowledge among the community at large creates an influential body of remonstrants.

Having given a table of mean durations, it is easy to find the proportion who die in one of the intermediate periods, on the supposition that the deaths are equally distributed through the period. This supposition is not actually true, though for a long course of ages the amount of mortality does not vary much from year to year. The main feature of De Moivre's hypothesis, equal decrements, appears in some measure at the adult and middle ages of life in all tables. I do not, however, know of any observations in which the numbers dying at every age are large enough to produce much confidence in the details of the tables of decrements, though the fluctuations may compensate each other in the determination of the mean durations of life. Tables which agree in the latter point may differ materially as to the former. As an instance, I give the following comparison of the Carlisle and Equitable tables, which agree more closely than any others in their mean durations. The first column shows the common age of 10,000 persons, the second and third the number who die in the following year in the Carlisle (C.) and Equitable (E.) tables.

A.	C.	E.	A.	C.	E.	A.	C.	E.	A.	C.	E.
10	45	72	35	103	92	60	335	315	85	1753	2210
15	62	75	40	130	110	65	411	428	90	2606	2686
20	71	73	45	148	127	70	516	639	95	2333	5566
25	73	76	50	134	150	75	955	931			
30	101	81	55	179	208	80	1217	1329			

According to the Carlisle table, of 10,000 persons aged 30, 101 die before attaining the next birthday; while one fifth less die in the Equitable table. And yet, one with another, the average lives of two sets of 10,000 do not differ by more than their 170th part. I now compare the actual tables of decrements, writing opposite to each age the survivors of 10,000 births who attain that age. (A, age; C, Carlisle; E, Equitable.)

A.	C.	E.	A.	C.	E.	A.	C.	E.
0	10,000		35	5362	5292	70	2401	2310
5	6797		40	5075	5034	75	1675	1572
10	6460	6460	45	4727	4751	80	953	898
15	6300	6192	50	4397	4441	85	445	354
20	6090	5956	55	4073	4069	90	142	86
25	5879	5733	60	3643	3588	95	30	21
30	5642	5524	65	3018	3002	100	9	

In the Equitable table, 5000 persons are supposed, each aged 10 years; this I have altered to 6460, to make the two tables agree at their outset.

The successive quinquennial decrements of the Carlisle table from the age of 20 are 211, 237, 280, 287, 348, 330, 324, 430, &c. If these deaths be supposed to take place at equal or nearly equal intervals during the five years,—if, in fact, we may suppose each of the individuals who die in a period to enjoy, one with another, half that period of existence,—we may ascertain the law of mortality from the table of mean durations in the following manner.

Rule. To the mean duration at the end of the period add the term elapsed and subtract the mean duration at the beginning; divide by the smaller duration increased by half the term, and the quotient is the fraction which expresses the proportion dying during the term. For example: the mean durations of life at 25 and 30 in the Carlisle table are 37·9 and 34·3; and $34{\cdot}3 + 5 - 37{\cdot}9$ is 1·4; which, divided by $34{\cdot}3 + 2{\cdot}5$ or 36·8, gives $\frac{14}{368}$ or $\frac{7}{184}$; so that of 184 persons aged 25, 7 die before attaining 30 years. In the table, we have $\frac{237}{5879}$, while $\frac{7}{184}$ is about $\frac{224}{5879}$.

If we were to take any table now existing on English lives, and ask, (as in p. 92.), what is the probability that a large number of lives, say 1000, should drop nearly in the same manner as those from which the table was formed, we should find the resulting chance not strong enough to make it prudent to risk much money in such contingencies. Nevertheless, the application of this theory to pecuniary risks has always been in a more forward state than the physical theory of human life. The reasons will be explained when we come to treat on the grounds of the confidence to which a contingency office is entitled. In the mean while, supposing a table to represent perfectly the average of a large number of the lives of the class to which an individual belongs, I proceed to show the method of using such a table

Persons who are desirous of using tables of life on a larger scale, are referred to the standard works of Messrs. Morgan, Baily, and Milne, on life insurance. In the present work I assume that it will be sufficient to be within two years and a half of any age which may be named, and I have given the several tables for intervals of five years. The extremes which are used by actuaries generally being contained in the Carlisle and Northampton tables, and having given the former, I now add the latter. The first column contains the age, the second the table of decrements, the third the number out of 10,000 who die in one year after completing the age in the first column.

Age.			Age.			Age.		
0	10,000		35	3437	187	70	1056	649
5	5356		40	3116	209	75	713	962
10	4864	92	45	2784	240	80	402	1343
15	4648	92	50	2449	284	85	159	2204
20	4399	140	55	2098	335	90	39	2609
25	4080	158	60	1747	402	95	3	7500
30	3759	171	65	1399	490			

Supposing the tables perfectly accurate, the following simple questions will show the nature of the first steps which occur in their application. The Carlisle tables are used throughout.

Question 1. What is the chance that an individual aged 35 will live to the age of 50? Of 5362 persons aged 35, 4397 live to be 50; hence the chance in question is $\frac{4397}{5362}$ or ·82. Answer, 41 to 9 for the event.

Question 2. What is the chance that A aged 45 and B aged 50, shall both be alive in ten years? The chance for A, by the last question, is $\frac{4073}{4727}$, or ·862 and that for B $\frac{3643}{4397}$ or ·829; the product of these (p. 43.), or ·715, is the chance required. Again, the chances of A and B dying during the ten years are 1—·862 and 1—·829, or ·138 and ·171; whence,

	The chance is
That both shall live	·862 × ·829
That A shall live and B die	·862 × ·171
That A shall die and B live	·138 × ·829
That both shall die	·138 × ·171

Question 3. What is the chance that A aged 25, shall die between the ages of 60 and 65? Of 5879 persons aged 25, 3643—3018 or 625, die between the ages of 60 and 65; hence $\frac{625}{5879}$ is the chance required.

Questions of this kind are readily solved, the only impediment being the arithmetical operation. It frequently happens, however, that the probability of one individual surviving another is required, which though an even chance when the individuals are of the same age, is a matter of considerable calculation when one is older than the other. Suppose, for example, that the chance of A (aged 25) surviving B (aged 30) is required. The *survivorship*, as it is called, meaning the period during which A lives after the death of B, may begin in any one year of A's age. For each year the probability of the survivorship beginning in that year must be calculated. To make this calculation for one individual year, say that in which A is between 49 and 50, two cases must be considered: either B may die between 54 and 55, and A may attain 50 complete years (of which the chance may be found as in the preceding questions), or both may die in the same year (that is A between 49 and 50, and B between 54 and 55), but B may die first. If the chance of both dying in that year be, say, ·012, it is sufficiently correct to consider the half of this chance, or ·006, as being that which expresses the chance of A's survivorship both beginning and ending in that year: a supposition which is quite correct only when the deaths of the year are equally distributed through it. The result of this calculation is arranged in tables, of which I here give a brief abstract.

Older.	Younger.	Northampton.	Carlisle.
15	5	·447	·400
20	10	·415	·383
25	15	·420	·381
30	20	·423	·375
35	25	·417	·372
40	30	·409	·366
45	35	·402	·360
50	40	·394	·350
55	45	·385	·329
60	50	·376	·315
65	55	·360	·323
70	60	·339	·322
75	65	·317	·303
80	70	·300	·320
85	75	·292	·332
90	80	·302	·329
95	85	·161	·462
25	5	·377	.307
30	10	·344	·283
35	15	·345	·279
40	20	·343	·270
45	25	·331	·263
50	30	·317	·251
55	35	·303	·231
60	40	·288	·212
65	45	·269	·194
70	50	·246	·177
75	55	·218	·177
80	60	·189	·190
85	65	·166	·174
90	70	·157	·191
95	75	·072	·300
35	5	·314	·235
40	10	·273	·207
45	15	·271	·202
50	20	·265	·189
55	25	·249	·174
60	30	·230	·158
65	35	·209	·143
70	40	·186	·126
75	45	·160	·104

Older.	Younger.	Northampton.	Carlisle.
80	50	·136	·093
85	55	·113	·094
90	60	·097	·113
95	65	·037	·147
45	5	·252	·177
50	10	·206	·146
55	15	·201	·135
60	20	·193	·119
65	25	·172	·110
70	30	·148	·097
75	35	·124	·081
80	40	·102	·075
85	45	·082	·059
90	50	·069	·052
95	55	·025	·078
55	5	·195	·125
60	10	·144	·091
65	15	·136	·085
70	20	·125	.071
75	25	·103	·061
80	30	·082	·056
85	35	·064	·046
90	40	·051	·044
95	45	·018	·049
65	5	·143	·086
70	10	·087	·054
75	15	·078	·048
80	20	·069	·040
85	25	·053	·034
90	30	·041	·033
95	35	·014	·039
75	5	·098	·057
80	10	·044	·028
85	15	·037	·024
90	20	·035	·012
95	25	·012	·029
85	5	·063	·040
90	10	·022	·017
95	15	·007	·023
95	5	·021	·036

The quantity found in this table is the probability of an elder life surviving the younger. The difference of ages differs in the various compartments of the table; in the first it is ten years, in the second twenty years, and so on. The two results accompanying each pair of ages are those of the Northampton and Carlisle tables. Thus. according to the Northampton table, the chance of a life of sixty surviving one of thirty is ·23; that of the younger surviving the elder is therefore 1 — ·23 or ·77. According to the Carlisle table, the same chances are ·158 and ·842.

Almost universally, the Northampton table gives a greater chance of the elder life beating the younger than the Carlisle. This is a consequence of that undue degree of comparative goodness which the former table gives to older lives, and to which I have already adverted.

If De Moivre's hypothesis were correct, it would be sufficient to divide the mean duration of B's life by twice that of A, and the result would be the chance which B has of surviving A, B being the elder of the two lives. This process, applied to the Northampton table, will give results very near the truth, when neither of the lives is very young. The same rule would give comparatively but a very rough guess at the result of the Carlisle table. If, however, the chance be calculated which the younger life possesses of dying in the average term of the elder, the result will be an approximation to the probability of the elder surviving the younger, when neither of the lives is very young, and when their ages are not nearly equal. Thus, the mean duration of a life of 50 being 21 years, and the chance of a life of 30 years surviving 21 years being ·769, the chance of the same life not surviving 21 years is ·231; while in the table, the chance of a life of 50 surviving one of 30 is ·251.

I shall, in the next chapter, consider the application of the tables to pecuniary questions, and shall now proceed to point out the connexion between a table of mortality and one of population.

The whole number of persons inhabiting any country is in continual state of increase from births and immigration, and of decrease from deaths and emigration. There are few countries in which immigration and emigration produce any serious effect upon the population, and, in times of very moderate quiet and prosperity, the births always exceed the deaths: so that, generally speaking, the number of people alive in a given country is yearly augmented by the excess of the births over the deaths. If accurate registers of births and deaths (with the ages at death) were kept for a a century and a half, accompanied, if need were, by a register of incomers and outgoers, with their ages, the community would be in possession of a complete history of its statistical changes, from which the law of mortality might be deduced, and its fluctuations noted, if any.

Again, if in any one year a complete census were made, registering the age of every individual, and if the deaths which took place in the 365 days next following the day of the census were noted, the law of mortality could be deduced. In such a case, the numbers of the living at every age would be so large that the proportion of deaths among them in a single year could be safely depended on for pointing out, with great nearness, the law which regulates the mortality of large masses of people.

No such statistical means exist in this country, partly from the defective manner in which the censuses of population are made, partly from the circumstance of the registries of births and deaths having been, almost up to the time of writing this work, connected with the religious ceremonies of the established church, which has had the effect of excluding many dissenters from registration. In the absence of all specific information, recourse was had to the registers of burials, which are usually accompanied by a statement of the age of the parties, though without any sufficient guarantee for the accuracy of the information. The hypothesis upon which alone registers of burials will give a correct law of mortality, requires that one of two alternatives should exist: either a permanent law of mortality, with a knowledge of the population in every year, and of the number of emigrants and immigrants, with their ages; or a stationary population, with the same number of births and of deaths in each year, and a permanent law of mortality. This latter supposition is never exactly true; but, as many societies have made a near approach to it, and as many tables have been constructed by its means, it will be worth while to explain the consequences of the supposition.

If the Carlisle law of mortality remained in uninterupted operation for a century, and if 10,000 infants were born alive in every year, the time would come when the number of the living at any age in that table would express the number alive at that age in the society in question. Thus the number of persons aged 25 would be the 5879 survivors of those who were born 25 years ago; and the number of the living at every age and upwards would be found by multiplying the number alive at that age by the mean duration of life in the table in p. 166. If, then, the law of mortality of such a society were required, it would be found written in the burial registers of any one year. For the numbers of births and deaths being equal, there would be found for each year 10,000 burials; which, if the law of mortality were permanent, would be found distributed among the different ages according to the table. Hence the number, out of 10,000 born, who attain a given age, would be found by adding the number buried after that age.

But let us now suppose a population uniformly increasing from year to year, say at the rate of 2 per cent. per annum; such a population would double itself in 35 years; and the younger lives would always exist in a greater proportion to the older ones than would be indicated by a correct table of mortality. The burials at the younger of two ages would therefore occur in too large a proportion to those at the older. Suppose, for instance, that 350 deaths take place at the age of 40—41, and 1200 at the age of 5—6; we are not therefore to conclude, that out of 10,000 individuals born, the deaths at 40 and 5 would be as 350 to 1200: for since the population doubles itself in 35 years, those who now die aged 5, are part of twice as great a number of such lives as were of the same age 35 years ago: consequently, of the set from whom 350 died at the age of 40, 600 died at the age of 5. If, then, a table were constructed from burials alone, without paying any attention to the rate of increase of the population, the older lives would appear too good of their kind; that is, relatively to the younger ones of the same society. This, as already observed, is the case in the Northampton table; whereas, in the formation of the Carlisle table, proper attention was paid to the variation in question. The difference is very perceptible in comparing each of these tables with that of the insurance office which it most resembles. At 25 years of age, the mean duration of the Northampton table is 30·9, and that of the Amicable 34·1. If the proportions of the mean durations remained nearly the same, (as generally happens,) then the Amicable table at 60 giving 12·5, the Northampton table should give 11·4; instead of which it gives 13·2.

The preceding supposes, that while the population changes, the law of mortality remains stationary. It is very unlikely that such should be the case; and observation, so far as it goes, tends to confirm the *à priori* suspicion. When provisions are cheap, or wages high,—when, in fact, it is easy to maintain a family,—marriages are more frequent, and are contracted at earlier ages. The same abundance of nourishment which tends to production, also tends to preservation, both of parents and children; the consequence of which is, that a rapid increase of population is often accompanied by a diminution of the proportionate mortality. On the other hand, and from contrary causes, a diminution of the rate of population may be attended by an increase of the mortality.

As this work does not profess to enter further into statistics than is necessary to exemplify the principles of the theory of probabilities, I shall here close what I have to say on the rate of mortality, considered independently of the most important pecuniary applications. The next chapter will point out in what way money calculations are made.

CHAPTER IX.

ON ANNUITIES AND OTHER MONEY CONTINGENCIES.

If money could make no interest, the principles of this chapter would be simplified, and the details of calculation connected with it would be somewhat reduced in amount. It will first be requisite to point out the effect of compound interest, and to show how to make computations connected with it. The fundamental calculation may be saved, for all such purposes as this work is intended to answer, by the following table, which may be described as follows. Opposite to any year in the column headed Y, and under the rate of interest in question (which is in numerals at the head), will be found, *within ten shillings,* the number of pounds which will, in such number of years, at such rate of interest, *produce a thousand pounds.* Thus, opposite to 23 years, in the column headed 3, we see 507; that is, 507*l.* (or, more strictly, something between 516*l.* 10*s.* 0*d.* and 517*l.* 10*s.* 0*d.*,) will, when improved at 3 per cent. for 23 years, produce 1000*l.*

Y.	2	2½	3	3½	4	4½	Y.
1	980	976	971	966	962	957	1
2	961	952	943	934	925	916	2
3	942	929	915	902	889	876	3
4	924	906	888	871	855	839	4
5	906	884	863	842	822	802	5
6	888	862	837	814	790	768	6
7	871	841	813	786	760	735	7
8	853	821	789	759	731	703	8
9	837	801	766	734	703	673	9
10	820	781	744	709	676	644	10
11	804	762	722	685	650	616	11
12	788	744	701	662	625	590	12
13	773	725	681	639	601	564	13
14	758	708	661	618	577	540	14
15	743	690	642	597	555	517	15
16	728	674	623	577	534	494	16
17	714	657	605	557	513	473	17
18	700	641	587	538	494	453	18
19	686	626	570	520	475	433	19
20	673	610	554	503	456	415	20
21	660	595	538	486	439	397	21
22	647	581	522	469	422	380	22
23	634	567	507	453	406	363	23
24	622	553	492	438	390	348	24
25	610	539	478	423	375	333	25
26	598	526	464	409	361	318	26
27	586	513	450	395	347	305	27
28	574	501	437	382	333	292	28
29	563	489	424	369	321	279	29
30	552	477	412	356	308	267	30
35	500	421	355	300	253	214	35
40	453	372	307	253	208	172	40
45	410	329	264	213	171	138	45
50	372	291	228	179	141	111	50
55	337	257	197	151	116	089	55
60	305	227	170	127	095	071	60
65	276	201	146	107	078	057	65
70	250	178	126	090	064	046	70
75	226	157	109	076	053	037	75
80	205	139	094	064	043	030	80
85	186	123	081	054	036	024	85
90	168	108	070	045	029	019	90
95	152	096	060	038	024	015	95
100	138	085	052	032	020	012	100

Y.	5	6	7	8	9	10	Y.
1	952	943	935	926	917	909	1
2	907	890	873	857	842	826	2
3	864	840	816	794	772	751	3
4	823	792	763	735	708	683	4
5	784	747	713	681	650	621	5
6	746	705	666	630	596	564	6
7	711	665	623	583	547	513	7
8	677	627	582	540	502	467	8
9	645	592	544	500	460	424	9
10	614	558	508	463	422	386	10
11	585	527	475	429	388	350	11
12	557	497	444	397	356	319	12
13	530	469	415	368	326	290	13
14	505	442	388	340	299	263	14
15	481	417	362	315	275	239	15
16	458	394	339	292	252	218	16
17	436	371	317	270	231	198	17
18	416	350	296	250	212	180	18
19	396	331	277	232	194	164	19
20	377	312	258	215	178	149	20
21	359	294	242	199	164	135	21
22	342	278	226	184	150	123	22
23	326	262	211	170	138	112	23
24	310	247	197	158	126	102	24
25	295	233	184	146	116	092	25
26	281	220	172	135	106	084	26
27	268	207	161	125	098	076	27
28	255	196	150	116	090	069	28
29	243	185	141	107	082	063	29
30	231	174	131	099	075	057	30
35	181	130	094	068	049	036	35
40	142	097	067	046	032	022	40
45	111	073	048	031	021	014	45
50	087	054	034	021	013	009	50
55	068	041	024	015	009	005	55
60	054	030	017	010	006	003	60
65	042	023	012	007	004	002	65
70	033	017	009	005	002	001	70
75	026	013	006	003	002	001	75
80	020	009	004	002	001	000	80
85	016	007	003	001	001	000	85
90	012	005	002	001	000	000	90
95	010	004	002	001	000	000	95
100	008	003	001	000	000	000	100

When 000 appears in the table, the sum requisite is less than ten shillings. Thus, less than ten shillings improved for 85 years at 10 per cent., will produce a thousand pounds.

There are six primary results which will be necessary.

1. The *present value* of 1*l*., meaning that fraction of a pound which will, when improved at interest, produce 1*l*. in a certain number of years. Since the present value of 1000*l*. is tabulated, that of 1*l*. is virtually so, and may be found by placing the decimal point before any result of the preceding table. Thus, ·458*l*. is the fraction of a pound which will, when improved at 5 per cent. compound interest for sixteen years, produce 1*l*.; or, in technical language, the present value of 1*l*. due sixteen years hence, at 5 per cent. To find the present value for any year not in the table, multiply the present values for any years in the table the sum of which is the number of years in question. Thus, to find the present value of 1*l*. due fifty-seven years hence, at 3 per cent., multiply together ·337 and ·961, the present values for fifty-five and two years, the result of which is ·324, the fraction of 1*l*. required.

2. The present value of a *perpetuity* of 1*l*., or the sum which will, when improved at interest, pay 1*l*. at the end of every 365 days from the present time, for ever. The number of pounds required is found by dividing 100*l*. by the rate of interest; being, in fact, nothing but the sum which will at interest produce 1*l*. per annum. The results are in the following table: —

Rate	P.V. of Perp.	Rate	P.V. of Perp.	Rate	P.V. of Perp.
2	£50	4	£25	7	£14$\frac{2}{7}$
2$\frac{1}{2}$	£40	4$\frac{1}{2}$	£22$\frac{2}{9}$	8	£12$\frac{1}{2}$
3	£33$\frac{1}{3}$	5	£20	9	£11$\frac{1}{9}$
3$\frac{1}{2}$	£28$\frac{4}{7}$	6	£16$\frac{2}{3}$	10	£10

Thus 20*l*. improved at 5 per cent. yields 1*l*. a year for ever, provided the first payment be due at the interval of a year from the time of putting the money out at

interest. This is also said to be 20 years' purchase, meaning the sum which would buy at once the annual payments of twenty years.

3. The present value of an *annuity certain* of 1*l.* (so called to distinguish it from a life annuity.) By an annuity of 1*l.* is meant, a right to receive 1*l.* at the expiration of every complete year after the creation of the annuity, which is said to commence a year before any payment is made; (or a term before payment is made, whether the term be yearly, half-yearly, &c.) Thus an annuity certain of five years, commencing January 1, 1838, is paid on the first days of 1839, 1840, 1841, 1842, and 1843.

When an annuity is to be designated which makes one payment immediately, I shall call it an *annuity due;* and a perpetuity of which one payment is to be made immediately, a *perpetuity due.* Thus on January 1, 1839, the preceding annuity becomes an *annuity due* of four years, comprising an immediate payment of 1*l.* and a commencing annuity of four years. To find the present value of an annuity of 1*l.*, use the following

RULE. Multiply the perpetuity (present value of the perpetuity) by the excess of one pound over the present value of the last payment. Thus, for an annuity of five years, at 4 per cent, multiply 25 by the excess of 1 over ·822, or by ·178; the result is 4·45*l.*, nearly

4. The *amount* of 1*l.* at compound interest, meaning the sum to which 1*l.* with its interest will amount in any number of years. It is found by dividing 1 by the present value of 1*l.* due that number of years hence. Thus 1*l.* in 25 years, at 6 per cent., will amount to (1*l.*÷·233), or 4·29*l.*

5. The *amount* of an annuity at compound interest, meaning the sum of which the annuitant would be possessed immediately after receiving the last annual payment, if he had made compound interest upon every preceding payment. This amount is found by multi-

plying the number of years' purchase in the perpetuity by the interest (not the amount) of one pound improved during the existence of the annuity. Thus, at 6 per cent., and in 50 years, 1*l.* will become (1*l.* ÷ ·054), or 18·5*l.*, so that the interest is 17·5*l.* This multiplied by $16\frac{2}{3}$, or $\frac{100}{6}$, gives 292*l.* The more correct answer is 290·34*l.*, but the error is not ten shillings in a hundred pounds.

6. The present value of an annuity for a term of years, to begin after the expiration of another term; technically called a deferred annuity. To find this, multiply the present value of such an annuity created immediately by the present value of 1*l.* due at the end of the term of deferment. Thus an annuity of 10*l.* for five years, at four per cent., to commence at the end of twenty years, is thus found: were it to commence immediately, it would (see last page) be worth 4·45 × 10*l.* or 44·5*l.*, and the present value of 1*l.* to be received twenty years hence is ·456*l.* Multiply 44·5 by ·456, which gives 20·3*l.*, the present value required.

Extensive tables of the results of the preceding processes are given in all works on interest or annuities; the present treatise is not meant to contain more than enough to enable the student to exercise himself in first principles, or the proficient to obtain approximate results, when no more extensive work is at hand. I now pass to the consideration of annuities in general.

Though the term annuity be generally understood as meaning a sum of money paid yearly or half-yearly to an individual, yet it is important that the reader should consider the word as implying any sum of money paid at fixed intervals, for any term, definite or indefinite provided only that the payment can never be suspended and resumed again. Thus 1*l.* to be paid every year for ten years provided A live as long, with an additional condition that payment is to cease if C in the term should die, B being alive at the time of C's death, is an annuity within the meaning of the word: nothing can make the payment cease at all, without making it cease altogether. But 1*l.* to be paid as long as A, B, and C are alive, to cease when one dies, and to recommence when a second dies, ceasing finally with the death of the third, is not one annuity, but two annuities. And by whatever name periodical payments may be called, whether rent, salary, rent-charge, interest, &c. &c., they are considered in this subject as annuities.

An annuity may be in possession or in reversion. In the first case, a payment will become due at the end of a term after the creation of the annuity, unless, in the mean time, one of the conditions on which payment depends cease to exist. In the second, the annuity is not to begin to be paid until some circumstances happen, or cease to happen, which are named in the agreement. Sometimes the term reversionary annuity implies that there is a contingency in the time of its beginning; and an annuity not to commence till after a fixed time has elapsed, would be called a deferred annuity. There is, however, a little variation in the use of these words: reversion sometimes implies a remainder of something existing, while a deferred interest is one which does not commence till a future time. Thus if B be to take an annuity on his own life as soon as A is dead, A enjoying it during his own life, B's reversion is a certain part of an annuity on his own life, reckoned from the present time. But if no annuity at all were paid until A's death, then B's interest might be called a deferred annuity. No difference arises from these distinctions in formulæ or calculations; and they are useful in describing the circumstances of complicated problems.

A reversion, and also a deferred annuity, may be certain or contingent: and the same of a reversionary or deferred fixed sum. Thus A may have B's estate after his death, provided he survive B; or else A and his heirs or assigns may have the same: in the first case the reversion is contingent, in the second certain. A life insurance may be of one or the other kind: thus A may covenant for his executors to receive 100*l.* at his death, in any case, or else if B should be then surviving. And if, in return for such insurance, A should engage to pay a yearly premium, making the first payment (as is usually done) at the time of contracting, then the premiums altogether constitute an *annuity due* upon A's life.

All problems relating to this subject might be solved from first principles, as will be shown; but, in such case, even the most simple of them would be attended with laborious calculation. Tables are therefore constructed of such results as will most facilitate the solutions of all problems: and these tables give the values of annuities on single lives, and on two joint lives. The annuity presumed is always 1*l.* per annum, from which the value of any other annuity is found by simple multiplication.

By the value of an annuity is meant the sum which must be paid down in order to enable the grantor of the annuity to pay it as it becomes due during the term of its continuance. If money made no interest, the average duration of such sets of circumstances as are conditions for the payment of the annuity would immediately point out its value. For instance, according to the Carlisle tables, persons aged 40 live, one with another, 27·6 years. Now, remembering that half a year was added (p. 163.), as being the part of their last year which, one with another, they pass through, but which will not (in our meaning of the term annuity) entitle them to any payment, we see that 27·1 is the average number of payments which such annuitants, aged 40, will receive; that is, one hundred with another, 100 annuitants will receive 2710 payments. Consequently, money making no interest, each of them must pay 27·10*l.* for a life annuity of 1*l.*, or 27·10 years' purchase. But if each had been entitled to his fraction of the sum which would have become due had he lived to the end of the year, then 27·60 years' purchase would have been necessary.

Let us now suppose money to make interest. It has before now seemed, to more than one writer, that the value of a life annuity must be the same as that of an annuity certain during the average duration of the life. Many of my readers will not see the fallacy of supposing that, money making 4 per cent., 16·7*l.*, the value of an annuity certain for 28 years, must be but very little more than the value of an annuity on a life aged 40. Such a rule, if it be absolutely true, must be true in any extreme case, however physically impossible. Suppose, then, that of 101 persons aged 40, one lives for ever, and the rest die between the ages of 41 and 42: whence 100 of them will receive only one payment, and the remaining one will receive a perpetuity of 1*l.* The present value of an annuity of 1*l.* for one year, at 4 per cent., is ·96*l.*, and 96*l.* increased by the value of 1*l.* for ever, is 121*l.*, the present value of all payments. If, then, it be not known which is to have the perpetuity, $\frac{121}{101}$*l.* is what each should pay, or 1·21*l.* very nearly. This result is undoubtedly correct: but the average life of the 101 persons is infinitely great, since there is no number of years which is not exceeded by them all put together. Each, then, would have to pay for a perpetuity, if the preceding fallacy were admitted; or all together would pay 25 × 101*l.*, that is 2525*l.*, more than 200 times the truth. It is true that, applied to any actually existing law of life, the incorrect notion cannot produce results so grossly false as the preceding: but it is quite sufficient for us to know that a rule is incorrect in its principle, to make it necessary to apply correct reasoning before we can attempt to say how far it is wrong.

The rule for calculating the real value of an annuity is made up of a collection of individual cases, not more complicated than the following. A is to receive one pound ten years hence, if a halfpenny which is to be thrown up give a head; what is the present worth of his chance? If he were to receive the money immediately, in case of winning, its value, on the principles in chapter V., would be ·5*l.*: but the present value of 1*l.* deferred for ten years at 4 per cent. is ·709*l.*: whence that of ·5*l.* similarly deferred is ·355*l.* This simple process contains the method of valuing the sum which must be now paid down to secure the tenth payment of an annuity to a life belonging to a class of which just one half die in ten years. A repetition of a similar process for every year in which the annuity may become payable gives the present values of the other payments. The sum of all the present values of the different payments is the present value of the whole annuity. Thus if we take a very old life, and suppose that of 10 alive at the present time there will be left at the end of successive years, 7, 5, 3, 0, there are three possible payments of the annuity, and the chances of having to make them are $\frac{7}{10}$, $\frac{5}{10}$, and $\frac{3}{10}$. But the first, if made at all, is not made for a year, and the second and third are not made for two and three years. If then *a*, *b*, and *c* be the present values of 1*l.* to be received at the end of one, two, and three years, the values of the several payments are $\frac{7}{10}$ of *a*, $\frac{5}{10}$ of *b*, and $\frac{3}{10}$ of *c*, the sum of which is the value of an annuity of 1*l.* on any one life of the kind in question.

By the *status* of an annuity, I mean the state or condition of things during the continuance of which the annuity is to be paid. This status may be simple or complicated: in the latter case the method of finding the chances of its continuance or termination will also be complicated; but this does not affect the simple rule by which those chances, when found, are made, in conjunction with the rules of compound interest, to give the value of the annuity. Thus the status in question may be of a compound character, allowing of several distinct changes. For instance, A is to enjoy an annuity to the end of his life, unless B should die before C, in which case it is to cease. This annuity will be enjoyed as long as either of the following status exist.

A, B, and C all living.
A and B living, and C dead.
A living, and B and C dead, C having died first.

Tables of the values of annuities are constructed in the case of single lives and two joint lives; that is to say, the mere inspection of the tables will enable us to answer the questions — what is the value of an annuity to be paid so long as A shall live, and what is the value of an annuity to be paid so long as A and B shall both be alive, but to cease when either of them dies. The authorities upon the subject have been already mentioned; namely, Mr. Milne * for the Carlisle tables, and Dr. Price † and Mr. Morgan ‡ for the Northampton tables. Besides these we have the work of Mr. Baily §, now completely out of print; in connexion with which must be mentioned the treatises || on interest and on leases by the same author, which are also out of print and scarce. The work of Mr. Davies ¶ contains, in connexion with the Northampton tables, a very considerable amount of results, not elsewhere published. The work of Mr. Baily on insurances has lately been translated into French.** The results of the Carlisle and Northampton tables have been lately published on one large sheet, by Mr. M'Kean.†† All the preceding are valuable works, and very much surpass, in useful application, all that has been done in any other country, or in all countries put together.

I now give such a brief abstract of the results of the Carlisle and Northampton tables as is consistent with the plan of this work, in regard to annuities upon single lives, and upon two joint lives.

* A Treatise on the Valuation of Annuities and Assurances, &c. &c., by Joshua Milne. London. Longman and Co. 1815.

† Observations on Reversionary Payments, &c. &c., by Richard Price, D.D. F.R.S.; seventh edition, edited by William Morgan, F.R.S. London. Cadell and Davies, 1812.

‡ The Principles and Doctrines of Assurances, Annuities, &c., by William Morgan, F.R.S. London. Longman and Co. 1821.

§ The Doctrine of Annuities and Insurances, &c., by Francis Baily. London. Richardson, 1816.

|| The Doctrine of Interest and Annuities, &c., by Francis Baily. London. Richardson, 1808.

Tables for the purchasing and renewing of Leases, &c., by Francis Baily. London. Richardson, 1807, (second edition).

¶ Tables of Life Contingencies, &c., by Griffith Davies. London. Longman and Co., and Richardson, 1825.

** Théorie des Annuités Viagères, &c., traduit de l'Anglais par Alfred de Courcy. Paris. Bachelier, 1836.

†† Practical Life Tables, by Alex. M'Kean. London. Richardson, Butterworth, &c., 1837. The annuities on two lives are given for every difference of age; whereas, in the works of Messrs. Morgan and Milne, they are only given for multiples of five years, and the rest must be supplied by interpolation. This sheet is a beautiful specimen of typography.

Values of an Annuity of £10, or (with the last figure made a decimal) of £1, upon a single Life.

	Northampton.				Carlisle.				
Age.	3 p. c.	4 p. c.	5 p. c.	6 p. c.	3 p. c.	4 p. c.	5 p. c.	6 p. c.	Age.
0	123	103	89	76	173	143	121	104	0
5	205	172	148	130	237	196	166	143	5
10	207	175	151	133	235	196	167	144	10
15	197	168	146	129	226	190	162	141	15
20	186	160	140	124	217	184	158	138	20
25	178	154	136	121	207	176	153	135	25
30	169	148	131	117	196	169	147	130	30
35	159	140	125	112	184	160	141	126	35
40	148	132	118	107	171	151	134	120	40
45	137	123	111	101	159	141	126	114	45
50	124	113	103	94	143	129	117	106	50
55	112	102	94	87	124	113	103	95	55
60	98	90	84	78	105	97	89	83	60
65	83	78	73	68	89	83	78	73	65
70	67	64	60	57	71	67	63	60	70
75	52	50	47	45	55	52	50	48	75
80	38	36	35	34	44	42	40	39	80
85	26	25	25	24	32	31	30	29	85
90	18	18	17	17	25	24	23	23	90
95	2	2	2	2	28	27	26	25	95
100	0	0	0	0	17	17	16	16	100

According to this first table it appears, for instance, that at the Northampton rate of mortality, the value of an annuity upon a life aged 45, at 3 per cent., is 13·7 years' purchase, or 13*l.* 14*s.* 0*d.* for an annuity of 1*l.*, 137*l.* for an annuity of 10*l.*, and 1370*l.* for one of 100*l.* per annum. Or rather, since the table is correct within one twentieth of a year's purchase, we should say that the value of such an annuity is between 13·65 and 13·75 years' purchase. The same remarks apply to the table of annuities on joint lives.

Values of an annuity of £10, or (with the last figure made a decimal) of £1, upon two joint lives.

Ages.		Northampton.				Carlisle.				Ages.	
		3	4	5	6	3	4	5	6		
5	5	156	136	120	107	198	168	145	127	5	5
10	10	163	143	127	113	200	170	148	130	10	10
15	15	152	134	120	108	189	163	142	126	15	15
20	20	141	125	112	102	180	156	137	122	20	20
25	25	134	119	108	98	169	148	131	117	25	25
30	30	126	113	103	94	158	139	124	112	30	30
35	35	117	106	97	89	147	131	118	107	35	35
40	40	108	98	90	83	135	121	110	100	40	40
45	45	98	90	83	77	124	112	103	94	45	45
50	50	87	81	75	70	109	101	93	86	50	50
55	55	77	72	67	63	91	85	79	74	55	55
60	60	66	62	59	56	73	69	65	61	60	60
65	65	55	52	50	47	60	57	55	52	65	65
70	70	43	41	39	38	46	44	42	40	70	70
75	75	31	30	29	28	32	31	30	29	75	75
80	80	21	21	20	20	25	24	23	23	80	80
85	85	13	13	13	12	17	16	16	15	85	85
90	90	9	9	9	9	11	11	10	10	90	90
5	10	160	139	123	110	199	169	146	129	5	10
10	15	158	138	123	110	194	166	145	128	10	15
15	20	147	130	116	105	184	159	140	124	15	20
20	25	137	122	110	100	174	152	134	120	20	25
25	30	130	116	105	96	163	143	127	114	25	30
30	35	121	109	100	91	152	135	121	109	30	35
35	40	112	102	93	86	140	126	114	103	35	40
40	45	102	94	86	80	129	116	106	97	40	45
45	50	92	85	79	74	116	106	97	90	45	50
50	55	82	76	71	67	99	92	85	80	50	55
55	60	71	67	63	59	81	76	71	67	55	60
60	65	60	57	54	51	66	62	59	56	60	65
65	70	48	46	44	42	52	50	47	45	65	70
70	75	36	35	33	32	38	37	35	34	70	75
75	80	25	24	24	23	28	27	26	25	75	80
80	85	16	16	16	15	20	19	19	19	80	85
85	90	11	11	10	10	13	13	13	13	85	90
5	15	154	135	120	107	193	165	143	126	5	15
10	20	152	134	119	107	189	163	142	126	10	20
15	25	142	126	113	102	178	155	136	121	15	25
20	30	133	119	107	97	167	147	130	116	20	30
25	35	125	112	102	93	157	158	124	111	25	35
30	40	116	105	96	88	144	129	116	105	30	40
35	45	106	97	89	82	133	120	109	100	35	45
40	50	96	88	82	76	120	109	100	92	40	50
45	55	86	79	74	69	104	96	89	82	45	55
50	60	75	70	66	62	87	81	76	71	50	60
55	65	63	60	57	54	72	68	64	61	55	65
60	70	51	49	47	45	56	53	50	48	60	70
65	75	40	38	37	35	43	41	39	38	65	75
70	80	28	28	27	26	32	31	30	29	70	80
75	85	19	19	18	18	22	22	21	20	75	85
80	90	13	13	13	12	16	16	15	15	80	90

Ages.		Northampton.				Carlisle.				Ages.	
		3	4	5	6	3	4	5	6		
5	20	148	130	116	104	187	161	140	124	5	20
10	25	147	130	116	105	182	158	139	123	10	25
15	30	137	122	110	100	171	149	132	118	15	30
20	35	127	114	104	95	160	141	126	113	20	35
25	40	119	107	98	90	148	132	119	107	25	40
30	45	109	100	91	84	137	123	111	101	30	45
35	50	99	91	84	78	123	112	102	94	35	50
40	55	89	82	77	71	107	98	90	84	40	55
45	60	78	73	68	64	91	84	78	73	45	60
50	65	66	62	59	56	77	72	68	64	50	65
55	70	54	51	49	47	60	57	54	52	55	70
60	75	42	40	39	37	45	43	41	40	60	75
65	80	31	30	29	28	35	34	33	32	65	80
70	85	21	20	20	19	25	24	24	23	70	85
75	90	14	14	14	14	18	17	17	16	75	90
5	25	143	126	113	102	180	156	137	121	5	25
10	30	142	126	113	102	174	152	134	120	10	30
15	35	132	118	107	97	163	143	128	115	15	35
20	40	121	109	99	91	151	134	121	109	20	40
25	45	112	102	93	86	140	125	113	103	25	45
30	50	102	93	86	80	126	114	104	96	30	50
35	55	91	84	78	73	109	100	92	86	35	55
40	60	80	75	70	66	92	86	80	74	40	60
45	65	69	65	61	58	79	74	70	66	45	65
50	70	56	53	51	48	63	60	57	54	50	70
55	75	44	42	40	39	48	46	44	42	55	75
60	80	32	31	30	29	37	36	34	33	60	80
65	85	22	22	21	21	27	26	26	25	65	85
70	90	15	15	15	15	20	19	19	18	70	90
5	30	138	122	110	99	172	150	132	118	5	30
10	35	135	121	109	99	166	146	130	116	10	35
15	40	125	112	102	93	153	136	122	110	15	40
20	45	114	103	94	87	142	127	115	105	20	45
25	50	104	95	87	81	128	116	106	97	25	50
30	55	93	86	80	75	111	102	94	87	30	55
35	60	82	77	72	67	94	87	81	76	35	60
40	65	70	66	62	59	80	75	70	66	40	65
45	70	57	55	52	50	65	61	58	55	45	70
50	75	45	43	41	40	50	48	46	44	50	75
55	80	33	32	31	30	39	38	36	35	55	80
60	85	23	22	22	21	28	27	26	26	60	85
65	90	16	16	15	15	21	21	20	20	65	90
5	35	131	117	106	96	164	144	128	114	5	35
10	40	128	115	104	95	156	138	124	112	10	40
15	45	117	106	97	89	144	129	116	106	15	45
20	50	105	96	89	82	130	118	107	98	20	50
25	55	95	88	81	76	113	103	95	88	25	55
30	60	84	78	73	68	95	88	82	76	30	60
35	65	72	67	64	60	81	76	71	67	35	65
40	70	59	56	53	50	65	62	58	55	40	70
45	75	46	44	42	40	51	49	46	44	45	75
50	80	34	32	31	30	41	39	37	36	50	80
55	85	23	23	22	22	30	29	28	27	55	85
60	90	16	16	16	15	22	21	21	20	60	90

Ages.		Northampton.				Carlisle.				Ages.	
		3	4	5	6	3	4	5	6		
5	40	124	112	101	92	154	136	122	110	5	40
10	45	120	109	99	91	146	131	118	107	10	45
15	50	108	99	91	84	131	119	108	99	15	50
20	55	96	89	82	76	114	105	96	89	20	55
25	60	85	79	74	69	97	89	83	77	25	60
30	65	73	68	64	61	82	77	72	68	30	65
35	70	60	57	54	51	66	62	59	56	35	70
40	75	47	45	43	41	51	49	47	44	40	75
45	80	34	33	32	31	41	39	38	36	45	80
50	85	24	23	23	22	30	29	28	28	50	85
55	90	17	16	16	16	23	22	22	21	55	90
5	45	116	105	96	88	144	129	116	105	5	45
10	50	110	101	93	85	133	120	110	100	10	50
15	55	99	91	84	78	115	105	97	90	15	55
20	60	86	80	75	70	98	90	84	78	20	60
25	65	74	69	65	62	83	78	73	69	25	65
30	70	60	57	54	52	67	63	60	56	30	70
35	75	47	45	43	42	52	49	47	45	35	75
40	80	35	33	32	31	41	39	38	36	40	80
45	85	24	24	23	22	31	30	29	28	45	85
50	90	17	17	16	16	24	23	22	22	50	90
5	50	107	97	89	82	131	118	108	99	5	50
10	55	101	93	86	80	117	107	98	90	10	55
15	60	88	82	76	71	99	91	84	79	15	60
20	65	74	70	66	62	84	79	74	69	20	65
25	70	61	58	55	52	67	64	60	57	25	70
30	75	48	46	44	42	52	50	47	45	30	75
35	80	35	34	33	32	41	40	38	37	35	80
40	85	24	24	23	23	31	30	29	28	40	85
45	90	17	17	16	16	24	23	22	22	45	90
5	55	97	89	83	77	115	105	96	89	5	55
10	60	90	83	78	73	100	92	85	79	10	60
15	65	76	71	67	63	85	79	74	70	15	65
20	70	61	58	55	53	68	64	61	57	20	70
25	75	48	46	44	42	53	50	48	46	25	75
30	80	35	34	33	32	42	40	38	37	30	80
35	85	25	24	23	23	31	30	29	28	35	85
40	90	17	17	16	16	24	23	22	22	40	90
5	60	86	80	75	70	98	90	84	79	5	60
10	65	77	72	68	64	85	80	75	70	10	65
15	70	63	59	56	54	68	64	61	58	15	70
20	75	48	46	44	42	53	50	48	46	20	75
25	80	36	34	33	32	42	40	39	37	25	80
30	85	25	24	23	23	31	30	29	28	30	85
35	90	17	17	17	16	24	23	23	22	35	90
5	65	74	70	65	62	84	78	73	69	5	65
10	70	63	60	57	54	69	65	61	58	10	70
15	75	49	47	45	43	53	51	48	46	15	75
20	80	36	34	33	32	42	41	39	37	20	80
25	85	25	24	24	23	31	30	29	28	25	85
30	90	17	17	17	16	24	23	23	22	30	90

Ages.		Northampton.				Carlisle.				Ages.	
		3	4	5	6	3	4	5	6		
5	70	61	58	55	52	67	64	60	57	5	70
10	75	50	47	45	44	54	51	49	46	10	75
15	80	36	35	34	33	42	41	39	38	15	80
20	85	25	24	24	23	31	30	29	28	20	85
25	90	17	17	17	16	24	23	23	22	25	90
5	75	48	46	44	42	52	50	48	45	5	75
10	80	36	35	34	33	43	41	39	38	10	80
15	85	25	25	24	23	31	30	29	28	15	85
20	90	17	17	17	16	24	24	23	22	20	90
5	80					42	40	38	37	5	80
10	85					32	31	30	29	10	85
15	90					24	24	23	22	15	90
5	85					31	30	29	28	5	85
10	90					25	24	23	22	10	90

The preceding table contains the values of annuities upon two lives, for all ages which are multiples of 5. Thus, for the ages 25 and 40, look to that part of the table in which the ages differ by 15 years, and there, opposite to 25 40, will be found, under 4 per cent., 132 in the Carlisle table and 107 in the Northampton. That is, money making 4 per cent., an annuity of 10*l.*, which is to continue as long as lives of 25 and 40 are both in being, is worth something between 131*l.* 10*s.* 0*d.* and 132*l.* 10*s.* 0*d.* according to the Carlisle tables, and something between 106*l.* 10*s.* 0*d.* and 107*l.* 10*s.* 0*d.* according to the Northampton tables.

When the two required lives have ages which do not end with the figures 0 or 5, proceed as follows: — Let the value of annuity be required on joint lives of 38 and 47, (Carlisle tables at 3 per cent.). First take 35 and 45, and 40 and 45, and between the corresponding annuities insert such a mean as would represent 38 and 45 upon the supposition uniformly diminishing values. Then between 35 and 50 and 40 and 50 insert such a mean as answers to 38 and 50. Having then 38 and 45 and 38 and 50, find such a mean as answers to 38 and 47. This process, which will be intelligible to a reader who has practised similar ones before, will only be comprehended (if at all) by others from the example.

35 and 45	133	35 and 50	123	
40 and 45	129	40 and 50	120	
	4		3	
	3		3	
	5) 12		5) 9	
	2		2	
38 and 45	131	38 and 50	121	
38 and 50	121			
	10			
	2	38 and 47	127	Ans
	5) 20			
	4			

Before proceeding further, I shall describe the notation of which I intend to make use. It was not the practice of the earlier writers to invent any distinctive notation of different contingencies, the first attempt at which is found in the work of Mr. Baily. Here, however, it was not carried to the full extent, and Mr. Milne endeavoured to organise a system which should take in every case, in which he succeeded perfectly as far as distinct representation of all the cases which occur. His symbols, however, are complicated and strange, though I am clearly of opinion that they are much preferable to the attempt to dispense with notation altogether. The new principle which the notation I now propose involves, lies in the treatment of terms of years certain as lives not subject to contingencies. Thus, if AB represent an annuity on the joint lives of A and B, meaning that it is to cease when either A or B dies, then tB may represent an annuity on the joint term of t years and B's life, to cease with the first which expires; or what would be called a temporary annuity on the life of B to last t years, provided B should live t years.

1. Any simple *status* on the existence or termination of which a benefit depends, is denoted by juxtaposition of large and small letters, the large letters denoting separately the values of annuities on given lives or status, the small letters (or rather certain small letters, m, n, t, for the most part) denoting given terms of years. Thus $ABCt$ is in existence as long as t years last, provided A, B, and C (or the persons on whom annuities *now* granted have these values), remain alive all the time.

2. A compound status, or one which exists as long as either of two or more status remain, is denoted by colons placed between the symbols of the simple status; thus $A:B:t$ is in existence as long as A, or B, or t years, any or all, are in existence. The symbol is unmeaning between two certainties: thus, $n:n+t$ is $n+t$.

3. A bar placed over two status indicates that the one is to succeed the other and that the compound symbol denotes a status in being as long as the one, or the other after it shall exist: thus $\overline{A}\,\overline{P}$ denotes a status which remains in being during the life of A, and also during a life to be named at the end of the year in which A dies, having *then* the value P. If there be occasion, a thicker bar, or one with an accent, or a double bar, may be used where there are two successions involved, between which it is necessary to distinguish.

4. The symbol $|$ in all cases gives the whole symbol the meaning of the present value of a benefit to be received, and a figure attached, as in $|_4$, denotes the rate per cent. at which the value is to be calculated. This benefit is always 1*l.* at each one or more payments. The description of the status which must end before the benefit begins will be found on the left of $|$; and the description of the status during which the annual payments of the said benefit are to last will be found on the right. It is further to be understood that the first payment made

under $A|B$ will take place at the end of the year in which A drops, provided B be then in existence: thus $A|\overline{1}$ denotes * the present value † of an annuity of 1*l.*, the *first* (and only) payment of which is to be made at the end of the year in which A drops; while $A|1$ denotes the present value of 1*l.*, to be paid in a year from this time, if A be then dead.

5. The last moment of a term certain is a part of that term, unless the contrary be expressed by symbols: thus $6|\overline{1}$ refers to a pound payable at the end of seven years, and is $6|7$. But when the last moment of a term is considered as having followed the end of the term, a small hyphen (considered as an abridged negative sign) is placed after the term in question: thus, $t\text{-}|\overline{n}$ denotes an annuity of n payments, the first of which is to be made at the end of t years; and is the same with $t-1|\overline{n}$.

6. The absence of symbols on the left of | indicates that the first payment takes place in a year from the present time; but -| indicates an annuity now due. The absence of symbols on the right of | indicates a perpetuity in reversion; and | itself indicates a perpetuity, the first payment of which is to be made in a year; while -| indicates a perpetuity now due.

7. Dots between two symbols of status indicate that the joint status shall be held to exist throughout the year in which the first is determined, provided the second remain at the end of the year; and dots placed under several status denote that the succeeding benefit is not due unless all those status shall drop in the same year: thus $|A...B$ denotes an annuity on the joint lives of A and B, payable also at the end of the year in which A dies, if B be then alive; and $\underset{\cdots}{A : B}|\overline{1}$ is the value of 1*l.* at the end of the life of the longest survivor of A and B, provided they

* One bar may be omitted in very simple cases.

† Remember particularly that in $A|1$, 1 means one *year*, not one *pound*.

both die in the same year; also $|AB\ldots$ denotes the value of an annuity on the joint lives to be paid in addition at the end of the year in which the joint existence fails.

8. When the condition is that a given status shall be in existence at the moment in which another status drops (whether the first last to the end of the year or not), single dots are placed over the two: thus $\dot{A}|\overline{1}\,\dot{B}$ means the present value of 1*l.*, to be received at the end of the year in which A dies, provided B be alive at that moment; while $A|\overline{1}B$ means the same, provided B be alive when the payment is to be made.

9. When it is a condition that deaths are to happen in a specified order, it must be represented by writing small figures under the status. Thus, $\underset{1}{A} : \underset{2}{B}|C$ means an annuity on the life of C, to begin at the end of the year in which B dies, provided A have died before B; and $A : \underset{1}{(BC)}|\overline{1}$ denotes the present value of 1*l.* payable at the end of the year in which the longest of the two status A and BC drops, provided that the status BC is determined by the death of B.

10. When the joint existence of one number of lives, out of a larger number, is a condition, a figure may be annexed as follows: $(ABCD)_2$ indicates a status which exists as long as any two out of the four are alive.

11. The double sign ‖ indicates the premium which is to be paid during the continuance of the status on the left, in consideration of the deferred benefit described on the right; premium being always interpreted as an annuity due. And where a specific event, as distinguished from the duration of a status, is a condition, the premium is to be held payable as long as any status exists out of which that event may happen. Thus $\underset{1}{A} : \underset{2}{B}\|C$ denotes the premium which should be paid as long as A lives with B, or B after A, to secure an annuity to C when both are dead, provided A die first.

' now proceed to some further instances:

$|n$ means the present value of an annuity of 1*l.* to last n years.

$m\text{-}|\overline{n}$ the present value of an annuity of 1*l.*, which is to commence *payment* at the end of m years, and then to last $n-1$ years, or $n-1$ more payments.

$n|$ the present value of a perpetuity of 1*l.*, commencing at the end of n years, or first paid at the end of $n+1$ years.

$n\text{-}|$ the present value of a perpetuity, first paid at the end of n years.

$|A$ the present value of an annuity on the life of A.

$|AB$ the present value of an annuity on the joint lives of A and B, to cease with either.

$t\,|A$ the present value of an annuity on the life of A, to begin in t years; that is, the first payment to be made at the end of $t+1$ years, if A should then be alive.

$|At$ the present value of an annuity on A's life, or t years, whichever drops first.

$A|$ the present value of 1*l.* for ever, to be first received at the end of the year in which A dies.

$AB|$ the present value of 1*l.* for ever, to be first received at the end of the first year in which A or B dies.

$A|\overline{t}$ the value of an annuity for t years, payment to begin at the end of the year in which A dies.

$AB|C$ the present value of an annuity which begins payment at the end of the year in which either A or B dies (the first), provided C be then alive, and which continues during the life of C.

$|A:B$ signifies the present value of an annuity which is to be paid as long as either A or B is alive.

$AB:C|D.E$ the present value of an annuity which is not to be paid as long as A and B are both alive, nor as long as C is alive, but which begins when the joint existence of A and B, and that of C, are both terminated; and continues as long as either D or E are alive.

Brackets, as distinguished from colons, will serve the same purpose as in algebra, namely, to give compound terms the meaning of single ones. Thus,

(AB) C|D : E denotes the last-mentioned annuity, on the supposition that payment is to begin when either of two events happens, the failure of the joint existence of A and B, or that of C.

A B|A : B the present value of an annuity to begin when one of the two, A and B, dies, and to continue during the life of the survivor. There is in this particular case the expression of an event which cannot happen; for if B die first, it is only A who can receive the annuity. Thus B|B is an expression for nothing; for the present value of an annuity on the life of B, to begin at the death of B, is nothing. In the expression A B|A : B, part is nothing, and the rest has a value.

A|$\overline{1}$ is the present value of 1*l.* to be received at the end of the year in which A dies; *t*-|$\overline{1}$ is the present value of 1*l.* due *t* years hence; A B|1$\overline{\text{C}}$ is the present value of an annuity which, commencing with the first death out of the two, A and B, lasts till either one payment, or the life of C drops: that is to say, the value of 1*l.* to be received at the end of the year in which the joint existence of A and B fails, provided C be then alive.

Let a colon placed after the final letter denote that a perpetuity is one of the status during which the annuity is to last. Thus,

A|(C :) denotes the present value of an annuity to last for ever, after the death of A. The symbol denotes that C being alive at the time of the first payment is a necessary condition. This being satisfied, the longest of the two, C, or a perpetuity, is of course a perpetuity.

The presence of the colon always indicates the longest of the two status, and when the colon is a final symbol, one of the status is an infinite number of years, or a perpetuity.

A_1 : B_2|C denotes the present value of an annuity which is to be paid during the life of C, after the deaths of A and B, if A die before B.

$A : \underset{2\,3}{B} : C_{|}\overline{1}\ E$ is the present value of 1*l.* to be paid at the end of the year in which the last of the lives, A, B, C, drops, on condition that B shall have died second or third, and that E shall be alive.

$\overline{A}|\overline{P}$ denotes the present value of an annuity which is to begin payment at the end of the year in which A dies, and to last during the life which shall then have the value P. If there be several conditions, put a symbol over the status which ends and before the one which begins. Thus,

$|\overline{ABC} : \overline{P}$ denotes the value of an annuity which is to be paid during the joint existence of A, B, and C, and further during that of a life which is to be nominated (then having the value P) at the end of the year in which either of the three drops: while $\overline{ABC}|\overline{P}$ denotes that part of the annuity which is paid during the life of P. An author's interest in his works, which is now denoted by $|A : 28$ was proposed, in a bill lately before the House of Commons, to be changed into $\overline{A} : \overline{60}$.

The preceding notation will admit of almost any degree of extension, and will be found perfectly capable of expressing any case which now occurs in practice. It must receive some generalisation before it can be applied to an indefinite number of lives, in the manner of Mr. Milne. But since it very rarely happens that more than three lives occur in a practical question, I shall leave farther extension to those who may find the want of it. I shall merely now add, that any case must admit of expression by means of a notation which provides for the conditions under which the benefit begins, the number of years which it is to last, and the conditions of discontinuance: and also that in problems of any degree of complexity, the invention of the notation will be a useful preliminary to the actual solution of the problem. Thus, suppose it required to represent the value of an annuity which is to continue as long as any two of the three, A, B, and C are alive. This must be done thus:

$$|\, AB : AC : BC$$

This might be abbreviated into $|\,(ABC)_2$, or any such symbol; which, however, I should recommend to no one who is not very familiar with the developed form.

A method of making the notation of chances analogous to that of annuities was devised by Mr. Milne, which is much too ingenious and efficient to allow of its being dispensed with in any future system. To adapt the principle of this connexion to the system which I have proposed, let $n \dagger a$ express the chance that A shall be alive in n years, in which the small letters answering to the capitals which denote lives refer to the chances of those lives, and certain letters m, n, t (or more if necessary), are reserved to signify terms of years. Then all which precedes the sign † (or any other which may be preferred), refers to lives or terms expired, and those which follow the sign, to lives or terms which are to be then in existence. Where no given term of years is included, the chance must be understood to refer to the whole continuance of all the status mentioned. Thus,

$a \dagger b$ is the chance that A shall die before B.

$a\, n \dagger b$ is the chance that one of the two, A or n years, shall fail before B.

$a : n \dagger b$ is the chance that both A and n years shall fail before B; that is, that B shall outlive A, and also live more than n years.

$a \dagger n$ the chance that A shall not survive n years.

$\underset{1}{a} : \underset{2}{n} \dagger b$ the chance that A shall die in n years, and that B shall outlive that term.

$n \dagger ab$ the chance that the joint existence of A and B (or both A and B) shall outlast n years.

$\underset{1}{n} : \underset{2}{a} \dagger (n+1)$ the chance that A shall die in the $(n+1)$th year from this time.

The tables, of which an abstract has been given, will enable us to find $|A$ and $|AB$, the values of an annuity on one or two lives, by simple inspection or easy interpolation, but not $|ABC$, the value of an annuity on three joint lives. A sufficient approximation to $|ABC$ is found by finding the single life Z, whose annuity is equal, or as nearly equal as the tables will give, to the annuity on the joint lives of the two elder of A, B, and C. This new life must be placed instead of those of B and C, which reduces the three lives to two. That is $|Z$ being equal to $|BC$ (A being the youngest of the three) $|ABC$ is $|AZ$ very nearly. Thus, the ages of A, B, and C being 45, 50, and 65, we have $|BC$ (Carlisle table, 5 per cent.) $= 6{\cdot}8$, and in the table of single lives 7·8 and 6·3 are the values of a single life at 65 and 70 years, giving a diminution of 1·5 in five years or ·3 per year of age. Hence 7·8 will become 6·8 in about three years, or 68 is the age of Z. Again, writing the ages at the feet of the letters signifying lives, $|A_{45}\ Z_{65}$ is 7·0 and $|A_{45}\ Z_{70}$ is 5·8, giving a diminution of ·24 for every year of *Z*'s age, so that $|A_{45}\ Z_{68}$ will be $7{\cdot}0 - {\cdot}7$ or 6·3, which is near the value of $|A_{45}\ B_{50}\ C_{65}$.

The values of annuities upon single or joint lives being thus found, it is easy to solve many other problems of annuities. Such annuities may either be required for a temporary purpose, or as a provision for future years, or for some parties after the death of others. I write the result required in symbols at the beginning of each question, and leave the reader to refresh his memory of them by observing the demonstration.

PROBLEM. An annuity of 1*l.* on the life of A, aged n, is to be bought, to begin payment at the end of t years, if A should live so long: required its value.

The question is proposed in the most usual form, but a little change will facilitate its solution. The right of A is evidently an annuity to begin in $t-1$ years, should he live so long. And the rule is expressed thus,

$$t-1|A_n=(t-1\dagger a_n)\times(t-2|\bar{1})\times A_{n+t-1}$$

At the end of $t-1$ years, if A be then alive (of which find the chance), he enters upon an annuity of 1*l.*, being then $n+t-1$ years of age. The value of an annuity at that age, multiplied by the chance of attaining that age, and by the present value of 1*l.*, to be received $t-1$ years hence, is, on the principles explained in p. 189., the present value of the annuity.

Example $n=45$, $t=11$ (Carlisle tables 5 per cent.)

$10\dagger a_{45}=\frac{4073}{4727}=\cdot 862 \quad 9|\bar{1} = \cdot 614 \quad |A_{55}=10\cdot 3$
$\cdot 862\times\cdot 614\times 10\cdot 3=5\cdot 45$ £5·45 *Answer.*

Problem. An annuity of 1*l.* is granted to A, aged n, for t years, provided he live so long: what is its value?

$$|A_n\, t = A_n - t|A_n$$

The last equation is obvious; for the whole annuity on A's life is made up of a contingent annuity for t years, and a contingent annuity to *commence payment* in $t+1$ years. Consequently, from the whole value of an annuity for A's life subtract that of an annuity to commence payment in $t+1$ years, if he should be then alive, and the remainder is the value of an annuity for t years, if he should live so long.

Example $n=45$, $t=10$

By last problem $10|A_{45}=5\cdot 5$, $|A_{45}=12\cdot 6$.
Therefore $|(A\ 10)=|A_{45}-10|A_{45}=7\cdot 1$ £7·1 *Answer.*

Problem. To find $A_m\,|B_n$, the value of an annuity on the life of B, aged n, the first payment of which is to be made at the end of the year in which the life of A, aged m, fails. This is called a survivorship annuity, since it can never be paid unless B survive A. To give this annuity is evidently to give a complete annuity to B, on condition that he shall restore it as long as A is alive; that is,

$$A|B=|B-|AB;$$

or, from the value of an annuity on B's life subtract that of an annuity on the joint lives. Thus (Carlisle tables 4 per cent.), the value of an annuity on the life of B, aged 30, to commence after the death of A aged 35, is $16\cdot9-13\cdot5$, or $3\cdot4$ years' purchase.

Problem. To find $AB|A:B$, the value of an annuity on the life of the survivor, whichever it may be, after the death of the other.

$$AB|A:B=|A+|B-2(|AB)$$

This is the same thing as giving an annuity to both, on condition that both restore it as long as both are alive. From the sum of the annuities, therefore, on the lives of A and of B, subtract twice the value of an annuity on the joint lives.

Problem. To find $ABC|AB:BC:CA$, or $ABC|(ABC)_2$, the present value of an annuity to begin payment at the end of the year in which one of the three dies, and to continue as long as both of the other two are alive. Give each pair an annuity on their joint existence, and withdraw all three annuities as long as all three are alive.

$$ABC|(ABC)_2=|AB+|BC+|CA-3(|ABC)$$

If the annuity be to commence immediately, without waiting for the death of one, this is an additional grant of $|ABC$ or

$$|AB:BC:CA=|AB+|BC+|CA-2\,(|ABC)$$

Problem. To determine $|A:B$, the present value of an annuity to be continued as long as either A or B shall be alive. Give both an annuity, but withdraw it from one as long as both shall be alive.

$$|A:B=|A+|B-|AB$$

Problem. To determine $|A:B:C$, the present value of an annuity to be continued so long as any one of the three shall be alive. Give each an annuity, and withdraw one of the annuities from any pair as long as that pair shall be both alive: but as this would take away the annuity during the joint continuance of the three lives, grant an additional annuity of 1*l.* on that joint continuance. Thus,

$$|A:B:C=|A+|B+|C-|AB-|BC-|CA+|ABC;$$

in the process of finding which it becomes evident that

$$ABC|A:B:C=|A+|B+|C-|AB-|BC-|CA$$

It may be worth while to point out how, in every possible case, the preceding grants and withdrawals produce the required effect; namely, one annuity to be paid as long as any one life lasts. The following are the payments and withdrawals: —

Living.	Dead.	Pay.	Withdraw.	Balance.
ABC	- -	1+1+1+1	1+1+1	1
AB	C	1+1	1	1
BC	A	1+1	1	1
CA	B	1+1	1	1
A	BC	1	0	1
B	AC	1	0	1
C	BA	1	0	1

Problem. Required C|AB, the value of an annuity on the joint lives of A and B as long as they shall survive C. Grant a complete annuity on the joint lives, and withdraw it while all three are alive.

$$C|AB=|AB-|ABC$$

Problem. Required AB|C, the value of an annuity on the life of C, to commence after the failure of the joint existence of A and B. Grant an annuity on the life of C, and withdraw it as long as all three are alive. Thus,

$$AB|C=|C-|ABC$$

Problem. Required the value of an annuity B : C|A, to be paid to A after B and C are both dead. Grant to A an annuity on his own life, withdrawing it as long as either B or C survives with A; that is, withdrawing

$$|AB : AC \text{ or } |AB + |AC - |ABC. \quad \text{Hence,}$$

$$B : C|A = |A - |AB - |AC + |ABC$$

PROBLEM. Required C|A : B, the value of an annuity to be paid as long as either A or B shall survive C. Grant one annuity to be paid as long as A or B is alive (|A : B), and withdraw it as long as A or B lives with C; that is, withdraw |AC : BC. Hence,

$$C|A : B = |A : B - |AC : BC$$

$$= |A + |B - |AB - (|AC + |BC - |ABC)$$

$$= |A + |B - (|AB + |AC + |BC) + |ABC$$

The same case also amounts to granting |A : B : C and withdrawing |C.

PROBLEM. An annuity C|A : B, payable as long as either A or B shall survive C, is to be divided equally between them, while they both live, and is then to go to the survivor. What is the value of the interest of each?

The interest of A is an annuity of half a pound while both A and B survive C, and of a whole pound as long as he shall survive both C and B: or $\frac{1}{2}$ (C|AB) + C : B|A. But,

$$\tfrac{1}{2}(C|AB) = \tfrac{1}{2}(|AB) - \tfrac{1}{2}(|ABC)$$

$$C : B|A = |A - |AB - |AC + |ABC$$

the sum of which is

$$|A - \tfrac{1}{2}(|AB) - |AC + \tfrac{1}{2}(|ABC)$$

and similarly, the value of B's interest is

$$|B - \tfrac{1}{2}(|AB) - |BC + \tfrac{1}{2}(|ABC)$$

the sum of which makes up, as it should do, the value of C|A B , given above.

Problem. An annuity on the longest survivor of A and B, or $|A:B$, is to be equally divided between them during their joint lives, and afterwards to go to the survivor. What is the value of the interest of each? That of A is evidently $\frac{1}{2}(|AB) + B|A$, which is

$$\tfrac{1}{2}(|AB) + |A - |AB \text{ or } |A - \tfrac{1}{2}(|AB)$$

$$\text{Similarly that of B is } |B - \tfrac{1}{2}(|AB)$$

which results are obtainable by a yet more evident process, since the interest of each is evidently an annuity on his own life, with half of it withdrawn as long as both are alive.

Problem. To determine $(\underset{1}{B}C)|A$, the value of an annuity on the life of A, to commence with the failure of the joint existence of B and C, provided it be B who dies first.

There are no tables for the accurate solution of this problem; but the following reasoning leads to a result which cannot be far wrong, unless some of the lives be very old, and which will always be near enough for the species of application contemplated in this work. The interest of A may be divided into two annuities, one of which is $B|AC$, and the other a portion of $B:C|A$. For A is certain of an annuity during C's life after the death of B, and of another after both are dead, provided B die first. Suppose A's interest in the latter annuity to be worth one half of it, which is strictly true if B and C be of the same age, and not much beside the truth for considerable differences of age, particularly when A is the oldest of the three. On this supposition A's interest is

$$B|AC + \tfrac{1}{2}(B:C|A)$$

$$\text{or } |AC - |ACB + \tfrac{1}{2}\left(|A - |AB - |AC + |ABC\right)$$

$$\text{or } \tfrac{1}{2}\left(|A - |AB + |AC - |ABC\right) = (\underset{1}{B}C)|A$$

which is obtained by supposing

$$\tfrac{1}{2}\left(A - |AB - |AC + |ABC\right) = \underset{1}{B}:C|A$$

The facility with which the preceding rules may be applied, in every part of the process except that of finding the annuities on three lives, makes it unnecessary to present examples. In order that examples may be readily obtained in cases involving three lives, some partial tables are given both by Mr. Morgan and Mr. Milne, from which the following selection is made: —

Values of an Annuity of £10 (or, with the last figure made a decimal, of £1,) on the joint continuance of three lives of equal ages, from the Northampton and Carlisle tables; in the former at 4 per cent., and in the latter at 5 per cent.

Common Age.	Northampton. 4 per cent.	Carlisle. 5 per cent.	Common Age.	Northampton. 4 per cent.	Carlisle. 5 per cent.
5	112	129	45	71	88
10	122	134	50	63	79
15	113	127	55	56	65
20	103	122	60	48	51
25	98	115	65	39	42
30	92	108	70	30	32
35	86	102	75	21	21
40	79	94	80	14	16

The following is from the Northampton table, at 4 per cent., the annuity being £10, and the ages as specified.

Ages.	Annuity.	Ages.	Annuity.
5, 15, 25	107	45, 55, 65	51
10, 20, 30	104	50, 60, 70	42
15, 25, 35	97	55, 65, 75	33
20, 30, 40	90	60, 70, 80	24
25, 35, 45	83	65, 75, 85	16
30, 40, 50	76	70, 80, 90	11
35, 45, 55	68	75, 85, 95	2
40, 50, 60	60		

The following is from the Carlisle table, at 5 per cent: —

Ages.	Annuity.	Ages.	Annuity.
5, 30, 35	111	45, 70, 75	33
10, 35, 40	106	50, 75, 80	25
15, 40, 45	99	55, 80, 85	18
20, 45, 50	91	60, 85, 90	12
25, 50, 55	80	65, 90, 95	11
30, 55, 60	66	70, 95, 100	9
35, 60, 65	55		
40, 65, 70	44		

I shall proceed in the next chapter to consider the methods of finding the value of reversionary interests, including life insurances.

CHAPTER X.

ON THE VALUE OF REVERSIONS AND INSURANCES.

The distinction between the problems of this chapter and the last, lies more in names and in the circumstances under which they occur for solution, than in difference of methods, principles, or (according to the scheme which I have suggested), even of notation. Every interest, the symbol of which has any thing preceding the |, is properly a reversion, being something of which the benefit is not to begin until the happening of some event, or the determination of some existing status.

It may be proper here again to remark, that all the rules in the preceding chapter, though the status mentioned are technically called *lives*, are equally true for any species of circumstances, temporary or permanent, certain or contingent. Thus an annuity for t years, to begin after n years, signified by $n|n+t$, or by $n|\overline{t}$, is determined by the same formula as an annuity on the life of B, to begin after the death of A, signified by A|B.

$$n|n+t = |n+t - |(n\ n+t) \qquad A|B = |B - |AB$$

but in the first I write $|n$ instead of $|(n\ n+t)$ since the joint continuance of n and $n+t$ years must be n years.

The only difficulty of this notation is the necessity of remembering the distinction by which it appears whether $n|n+t$ means that a payment is to be made at the end of the nth year or of the $(n+1)$th. In the case of A|B the first payment is to be made at the end of the year in which A dies, which is always intelligible, since it is considered as an infinitely small probability that A should die at the moment which divides two years. But since the term of n years does expire at such a moment, the analogy which connects the symbols of terms certain and contingencies points out no rule. Let $n|$ stand for a perpetuity of 1*l.*, the first payment to be due at the end of $n+1$ years, then $n-1|$ stands for a similar perpetuity due at the end of n years. But since the introduction of a new figure may turn the attention off n, the datum of the problem, and since analogy does not require us to strike off a whole year from the term, let n-| signify a perpetuity deferred for a term (no matter how little) short of n years, that is, payable in n years. And by analogy -|A will signify an annuity due on the life of A. The symbol n-$|n+t$ must be treated as if it were

$$n-1|n+t \text{ or } n-1|\overline{t+1}$$

The symbol of a perpetuity of 1*l.* to commence from the present time (that is, payment at the end of a year), is simply |. This is found as in p. 184., and by subtracting |A, the value of an annuity on the life of A, we obtain | − |A, the value of the reversion of a perpetuity, of which payment is made at the end of the year in which A dies. At the end of that year, the holder of the reversion will have in possession and expectation, an equivalent to -| or 1+|, or the value of a perpetuity due. Consequently, since the present value of 1+| to be received at the death of A is |−|A, that of 1*l.* will be found by dividing the latter by the former: or,

The present value of 1*l.* to be received at the end of the year in which A dies, is found by subtracting the present value of an annuity on the life of A from that of a perpetuity, and dividing the remainder by the present value of a perpetuity due, or one year's purchase more than the present value of a perpetuity.

Example. (Northampton tables, 3 per cent.) What is the value of 1*l.*, to be received at the end of the year in which a life of 30 shall fail?

Perpetuity of £1 at 3 per cent.	£33·3
Value of annuity on life of 30	£16·9

34·3) 16·4(·478

In the preceding rule any status may be substituted for a single life, and the value of the annuity which is to be paid as long as the status lasts is connected with the present value of 1*l.* to be received at the end of she year in which the status fails, by the preceding timple rule.

The premium which should be paid (first down, and afterwards at the end of each year), is an annuity due upon the life or status, and is therefore worth -| A or 1+|A year's purchase. Consequently the premium which should be paid for the 1*l.* above described is the preceding present value divided by one year's purchase more than the annuity is worth. In the example, divide ·478 by 1+16·9 or 17·9, which gives ·0267, so that 2*l.* 13*s.* 6*d.* is the premium for insuring 100*l.* at the end of the year in which a life of 30 fails.

The following rule is somewhat shorter, in the case in which the premium only is required, and not the present value.

Question. To find the premium which should be paid (first down, &c.), during the continuance of a status, to insure 1*l.* at the end of the year in which that status drops.

Rule I. From the quotient of a perpetuity divided by a perpetuity due, subtract that of an annuity on the status divided by an annuity due.

34·3) 33·3 (·9708   17·9) 16·9 (·9441
·9441

·0267 Answer, as before.

Rule II. Divide 1 by the value of an annuity due, and by that of a perpetuity due, the difference of the quotients is the premium required.

34·3)1(·0292   17·9)1(·0559
·0292

·0267 Answer, as before.

A perpetuity divided by a perpetuity due, is the present value of 1*l.* to be received a year hence, and may be taken from the following table: —

p. c.		p. c.		p. c.	
2	·9804	4½	·9569	9	·9174
2½	·9756	5	·9524	10	·9091
3	·9709	6	·9434		
3½	·9662	7	·9436		
4	·9615	8	·9259		

From the preceding rule an illustration of the reason of it may be derived, which I give professedly as an exercise of ingenuity to those who may be beginners in the subject. Let there be two persons, one of whom holds a perpetuity and the other a life annuity, each of 1*l.* Both the perpetuitant * and the annuitant desire

* If the holder of an annuity be an annuitant, the extension of language is justifiable, by which the holder of a perpetuity may be called a perpetuitant.

to commute their interests for interests due: that is, the perpetuitant, instead of 1*l.* a year hence and so on, desires to receive a fraction of a pound *now*, and the same fraction at the end of every year; and the same for the annuitant. Say the value of money is four per cent., then the perpetuitant desires to change an interest which is worth twenty-five years' purchase into an equivalent interest worth twenty-six years' purchase (or income); consequently his year's income (now due, &c.) must be only $\frac{25}{26}l.$ Say that the annuity is worth ten years' purchase; then by the same reasoning the yearly income of the annuitant (now due, &c.) must be only $\frac{10}{11}l.$ The second is less than the first; whereas the original incomes were the same, both 1*l.* But there must be some consideration which the commutation gives to the annuitant, and for which this greater diminution of his income is the payment; and it is as follows:—Since the commutation forestalls each successive payment, giving it (or the substitute for it) a year before it becomes due, the annuitant would receive, if his income were made equal to that of the perpetuitant, the 1*l.* which, had he lived, would have become due at the end of his last year, but which his death hinders from becoming due. This difference of income $(\frac{25}{26}-\frac{10}{11})l.$ is therefore equivalent to preventing his receiving 1*l.* at the end of the year in which he dies, and it is taken from him now and in every succeeding year of his life. Consequently it is the premium which such an annuitant should pay to receive 1*l.* at the end of the year in which he dies; and it is also the result of the first preceding rule.

The second rule may receive an explanation of a similar kind. I now reverse the problem, and ask the following

QUESTION. If an office charge the premium p for insuring 1*l.* at the end of the year in which a life (or other terminable status) drops, what should we infer that they suppose to be the greatest possible value of an annuity to continue during the remainder of that life or status; that is, what is the value of an annuity on that status, which is such that the office must be ruined if the truth falls below it?

Rule. Take the rate of interest which money really makes *, and subtract the premium for 1*l.* from the present value of 1*l.* to be received at the end of a year (see last table): divide the remainder by the excess of unity over the remainder, and the quotient is the number of years' purchase in the present value of the annuity.

Example. A society professes to insure lives of 35 at a premium of 3 per cent. on the nominal sum insured; what is the lowest value of the annuity on such lives at which this can be done?

At 4 per cent. |1	= ·9615	1·0000
(page 200.), A||1	= ·0300	·9315
	·0685)·9315(13·6	·0685

Answer. Such an office cannot permanently stand (as far as this one species of bargain is concerned), unless the value of an annuity on lives of 35 (at 4 per cent.) be more than 13·6 years' purchase.

Generally speaking, contracts of insurance are not made for the end of the year in which the party dies, but for payment at a given number of months after the parties' death is proved and the claim made. If this agreement were always made for six months after the real death, the office would, one party with another, neither gain nor lose, while for every month less than six, the office gives that month's interest to the parties' executor, while for every month more than six by which payment is deferred, the office takes a month's interest. I believe no office defers its payments more than six months after the claim is made; and the difference is rendered immaterial by the probable errors of the tables, which require too large a covering profit to make it worth while to take such a circumstance into account.

* This is an essential element, but cannot be very accurately determined: something above the truth should be assumed.

The premium demanded by an office is that charged by their tables at the age which the party will attain at his next birthday; thus if a person desire to insure his life the day after he attains 31 years complete, he will be required to pay the same as if he had deferred completing the insurance till the day before his thirty-second birthday. This is, one party with another, a gain of half a year to the office. Thus, the Northampton table at 3 per cent. giving 16·7 and 16·5 as the values of annuities at the above-mentioned ages, all parties who have passed 31 years at their last birth day are considered as having lives worth 16·5, whereas they are worth, one with another, 16·6. The tables are not sufficiently accurate to make the effect worth caring for.

A party having made an insurance, and paid one or more premiums, the instrument by which the right to receive the stipulated sum at death on payment of a stipulated premium is conveyed, is called a *policy of insurance.* The value of this policy is then easily determined; at least what we may call its *office value,* supposing the tables of the office to be perfectly correct. A person aged thirty insures for 100*l.*, for which he pays, say 3*l.*; he continues to pay this premium until the age of fifty, at which time, if he had began to insure, the annual premium would have been, say 5*l.* Suppose that the holder of the policy wishes to sell his interest just before he would otherwise have had to pay another premium, it is plain that he then offers for an insurance on the life of 50, a better bargain than the office would offer, since the buyer of the policy (who pays all future premiums) will acquire, in consideration of an annuity due of 3*l.* upon the life of A, that which the office would not sell for less than an annuity due of 5*l.* upon the same life. The difference, or an annuity due of 2*l.* upon the same life, is the value of the policy.

Rule. To find the present value of a policy of insurance, at the moment before a premium becomes due, subtract the premium which is to be paid from the premium which would be paid if the same party made the same insurance at the present time. Find the present value of an annuity on the life of the party insured, of the same yearly amount as the preceding difference, and this value, increased by one year's purchase, is the present value of the policy.

To find the value of the policy immediately after a premium is paid, add the premium just paid to the result of the preceding rule. It would not be worth while, in the present work, to give a rule for any intermediate value. (See Milne, p. 283.)

But in finding the real value of a policy, there are one or two circumstances to be considered, of which no mention is made in the preceding rule. The buyer of the policy, being uncommitted by any previous act of his own, is not bound to consider the premium of any one office as a standard. Suppose that in the preceding example, another office of equal solvency can be found, which will insure a life of 50 at 4 per cent. instead of 5: the buyer, therefore, may consider that the seller offers him for *3l.* a year during his life a benefit which he might buy elsewhere for *4l.*, and that he should therefore pay only the value of an annuity due of *1l.* instead of *2l.* But since the two offices cannot be together parties to any transfer of policy, the preceding case will only serve to show that it may be more prudent for a person who has money to invest, to lay it out at once in insuring lives in a cheap office, than in buying existing policies in a dear one. It is to be remembered that the lower premium in the preceding rule is to be paid, by bargain already existing, while the higher one is hypothetical, depending on the buyer's opinion of tables of mortality. That the office which demanded and obtained the *3l.* would demand the *5l.* for an insurance now to commence, must be no consideration for a person who is merely thinking how to lay out his money to the best advantage; it may be by buying the policy which is offered to him, or by insuring his own life, or that of some one else, in the same or another office. It is his business to consider what he is likely to have to pay, in the shape of future premiums, and not what an office, which must be on the safe side, has thought fit to suppose it will have to receive. Putting out of view the state of health * of the party insured, I should think it most advisable to calculate the value of policies by finding the present value of the sum insured, and also that of the premiums to be paid, from the tables which best represent healthy life, and using the rate of interest which money will really obtain, rather above than below; that is, I should use the Carlisle tables at 4 per cent. The profits guaranteed by the office, if any, should be duly considered. Thus suppose a person at the age of 30 had insured for 1000*l.* in an office which demands 25*l.* premium for that insurance, and returns no profits, and suppose that twenty years have elapsed, so that the life insured is now at the age of 50, what is the real value of his policy? The value of 1*l.* to be received at the death of a person aged fifty, by the Carlisle tables at 4 per cent., is (p. 214.), $25-12{\cdot}9$ divided by $25+1$ or $\cdot 465$: that of 1000*l.* is therefore 465*l.* If a premium be just becoming due, the present value of all the premiums is therefore $1+12{\cdot}9$ or $13{\cdot}9$ years' purchase; and $13{\cdot}9 \times 25l.$ is $347{\cdot}5l.$ Consequently $465-347\frac{1}{2}$, or $117\frac{1}{2}l.$, is what I should consider to be the value of that policy. But if I took the tables of the same office, which require a premium of 47*l.* at the age of fifty, and which, with some variation, are derived from the Northampton tables at 3 per cent., I should find by the rule in p. 218., $(1+12{\cdot}4)\times(47-25)$, or 295*l.* nearly. So great is the difference between policies valued by the nearest approximation which exists to the actual truth, and then valued by the tables which offices adopt for their own security.

The office itself, which takes an advantage of the buyer when the policy is first created, may reasonably

* Of course the policy of a person whose health has very much declined since he effected the insurance, is of higher value on that account: but this cannot be made the subject of calculation.

allow that advantage to the insured, if he afterwards desire to sell his policy *to the office itself.* I am not aware of the exact rule which is followed by the offices in this respect, except in one or two cases, in which the plan is, or was, to follow their own tables, with a certain deduction from the result, and to give the difference to an insured party who desired to sell his policy. This is well enough in the case of offices which return profits; but if such a rule be followed by those which do not, it may amount to a contradiction of their profession in the case of the sale of policies; and may become in effect an allowance of that share in the profits to those who desire to leave the office, which they refuse to grant to those who continue. To prevent such a result, I believe the offices who would be liable to it, make a large deduction from the value of policies, as indicated by their tables.

All the preceding rules apply to any given status as well as to a given life. Thus, to effect an insurance on the survivor of two lives, the present value and the premium (payable as long as either is alive) are to be found by using $|A+|B-|AB$ for the value of the annuity, instead of $|A$. I now proceed to some simple cases of insurance, where the payment on one party's death is made conditional upon another party being alive to receive it.

The symbol $A|(\overline{1}B)$ denotes the value of an annuity upon the joint continuance of one year and the life of B, payment being made at the end of the year in which A dies. It is therefore necessary that B should be alive at the end of the year in which A dies. But in the usual conditions of contingent insurances, it is sufficient that B should be alive at the moment in which A dies. Let this be expressed by $\dot{A}|\overline{1}\dot{B}$; it is then evident that $\dot{A}|\overline{1}\dot{B}$ is greater than $A|\overline{1}B$. The following preliminary considerations will be necessary.

Problem.—Required the value of an annuity on the joint lives of A and B, to be paid at every end of a year at which B shall be alive, provided A were alive at the *beginning* of the year. This may be denoted by | (A...B).

The condition that a life shall be alive at the beginning of the year must be, in tables of averages, the same as that of a life a year younger being alive at the end of the year. For example, suppose that of 500 persons of the age of $n-1$, 493 attain to that of n, then 500 annuities granted on the lives of A_{n-1} will be equivalent to 493 granted to A_n, if the latter be payable at the end of any year in which A shall have been alive at the beginning; and the same for any joint lives combined with both. Thus 500 annuities granted on the joint lives of A_{n-1} and B, payable as usual, are equivalent to 493 on the joint lives of A_n and B, payable as long as B is alive at the end, and A was alive at the beginning, of a year. Hence | (A...B) is $\frac{500}{493}$ of |A,B, where A, stands for a life a year younger than A. Hence the following Rule. To solve the preceding question, multiply the value of a joint annuity on B and one year younger than A by the number alive at that younger age in the table, and divide by the number alive at A's age: the result is the present value required. Or more concisely, divide | (A,B) by the chance which A, has of living a year.

Now let us ask, by how much does |(A...B) as above described exceed |AB. The only possible case in which a payment will ever be made upon the first annuity, and not upon the second, is when A dies before B, for both are determined by the death of B. When A dies before B, the annuity will be paid at the end of the year upon |(A....B), if B be alive, but not upon |AB. Consequently the excess of the first over the second is the value of £1 to be received at the end of the year in which A dies, provided B be then alive.

Or,

$$A|\overline{1}B = |(A\ldots B) - |AB, \qquad B|\overline{1}A = |(B\ldots A) - |AB$$

Also the present value of an annuity to be paid at the end of any year in which both A and B were alive at the beginning, or $|AB\ldots$, is evidently $|AB$ increased by $AB|\overline{1}$, the present value of an insurance of one pound on the joint lives.

Problem. Required the present value of £1 to be paid at the end of any year, provided that both A and B die in that year; which has been signified by $\underset{\cdots\cdot}{A:B}|\overline{1}$.

Grant the following annuities; —

$|AB$, on the joint lives of A and B.

$|(AB\ldots)$ the same as $|AB$, but to be also paid at the end of the year in which the joint existence fails.

Take in exchange the following annuities: —

$|(A\ldots B)$ and $|(B\ldots.A)$ two annuities to be paid during the joint lives, and also at the end of the year in which the joint existence fails, provided B in the first, and A in the second, be alive at the end of the year.

The balance of this transaction will be £1 to be paid at the end of any year, provided B and A both die in the year. For as long as both are alive, two annuities are payable each way; if A die and B remain alive till the end of the year, $|AB$ has ceased, $|(AB\ldots)$ is payable, but $|(B\ldots A)$ has ceased, and $(A\ldots B)$ is payable; similarly in the case of B dying and A remaining alive. If both die in one year $|AB$ has ceased, but $|(AB\ldots)$ is payable, while $|B\ldots A$ and $|A\ldots B$ have both ceased. Consequently, the only possible payment which the grantor has to make, over and above those which he receives, is the £1 in the question proposed; or

$$\underset{\cdots\cdot\cdot}{A:B}|\overline{1} = |AB + |AB\ldots - |A\ldots B - |B\ldots A$$

We are now in a condition to solve the final Problem. Required the value of £1 to be paid at the end of the year in which A dies, if B should have been alive at the moment of A's death. This is denoted by $\dot{A}|\overline{1}\dot{B}$.

When A dies before B, either B survives till the end of the year, or dies in the intermediate time. The insurance on the first risk is worth $A|\overline{1}B$ determined in p. 222.; on the second it is worth half the result of the last problem, if it be considered that the chances of

A dying before and after B, in any one given year, are equal. We have therefore

$$|A\ldots B - |AB + \tfrac{1}{2}\{|AB + |AB\ldots - |A\ldots B - |B\ldots A\}$$

Or,

$$\tfrac{1}{2}\{|AB\ldots - |AB + |A\ldots B - |B\ldots A\}$$

But, (p. 223.)

$$|AB\ldots - |AB \text{ is } AB|\overline{1}$$

Whence the final result is,

$$\tfrac{1}{2}\{AB|\overline{1} + |A\ldots B - |B\ldots A\}$$

Rule. For determining the value of 1*l.* payable at the end of the year of the death of A, provided B be alive at the moment of A's death.

(1.) From the value of a perpetuity subtract the value of the joint annuity, and divide by that of a perpetuity due.

(2.) Multiply the value of the joint lives of B and a year younger than A by the number alive in the table at the younger age, and divide by the number alive at the age of A.

(3.) Repeat the preceding process, substituting B for A, and A for B.

(4.) From the sum of the results of (1) and (2) subtract that of (3), and half the difference is the present value required. The premium payable during the joint lives is found by dividing the result by the value of an annuity due on the joint lives.

Though I have deduced the preceding rule at length as an instance of the very improving exercise of deducing the more complicated results of this subject by what we may call the *balance of annuities*, yet for the rough purposes of this work, if not for others still more exact, the more simple process implied in finding $|A\ldots B - |AB$ is sufficient. It must be obvious that the fraction of the whole value of a survivorship insurance, which depends on the risk of both parties dying in the same year, is a small one; so that it is nearly sufficient (and certainly within the probable errors of tables, &c.) to consider this question independently of the double risk, and as if it were certain that both parties would not die in the same year. Instead of $\dot{A}|\bar{1}\dot{B}$, we then employ $A|\bar{1}\,B$ (page 222). Mr. Milne (in his page 352. &c.) has given three examples, which I here repeat by the latter formula, taking the data from the work cited:

1. A is aged 17, B is aged 57: Carlisle Tab., 4 per cent.

$$\begin{array}{r} |A\ldots B = 10{\cdot}009 \\ |AB = 9{\cdot}923 \\ \hline 0{\cdot}086 \end{array} \qquad A|\bar{1}\,B = {\cdot}086, \text{ which in work cited [is } {\cdot}08656.$$

2. A is aged 45, B is aged 35; do. do. 5 per cent.

$$\begin{array}{r} |A\ldots B = 11{\cdot}172 \\ |AB = 10{\cdot}912 \\ \hline 0{\cdot}260 \end{array} \qquad A|\bar{1}\,B = {\cdot}260, \text{ which in work cited [is } {\cdot}266331.$$

3. A is aged 64, B is aged 19; do. do. 5 per cent.

$$\begin{array}{r} |A\ldots B = 8{\cdot}115 \\ |AB = 7{\cdot}593 \\ \hline 0{\cdot}522 \end{array} \qquad A|\bar{1}\,B = {\cdot}522, \text{ which in work cited [is } {\cdot}52556.$$

The present value of a pound, to be received at the end of the year in which A dies, provided he survives B, or $\underset{1}{B}:\underset{2}{A}|\bar{1}$, is readily found from the preceding: for since A must either die before or after B, the sum of the two must be the present value of 1*l.* to be received at the death of A, independently of B. That is,

$$\underset{1}{B}:\underset{2}{A}|\bar{1} = A|\bar{1} - \dot{A}|\bar{1}\dot{B} = A|\bar{1} - A|\bar{1}\,B \text{ very nearly.}$$

The present value of an insurance is also that of the reversion of a fixed sum; since it is the same thing whether 1*l.* is to be received from an office, or conditionally under a will, or in any other way. The reversion of a perpetuity should be treated as that of the value of a perpetuity due at the end of the year in which the life drops.

Problem. Required the present value of 1*l.* to be received at the end of the year in which A dies, provided that event take place before the expiration of *t* years from the present time: signified by $A|\overline{1}\ t$.

Suppose a person to have the certain reversion of a perpetuity due at the end of *t* years, or sooner, if A die before *t* years are expired: the reversion of the perpetuity after the failure of the joint existence of A and *t* years, is $|-|\ A\ t$, which can be found from page 206. But this is more than that fraction of a perpetuity due at the end of the year in which A dies, which will pay for the chance of entering on it before *t* years are expired: for part of it expresses the value of a perpetuity due, which, though A should be alive, is to be entered on by the failure of *t* years. If $t \dagger a$ be the chance that A is alive at the end of *t* years, then $t \dagger a \times t|$ must be deducted, as being expressed twice in the preceding: consequently

$$|-|A\ t - t \dagger a \times t|$$

is the present value of a perpetuity due, or $-|$, to be entered upon at the end of the year in which A dies, if before *t* years. The preceding then divided by $-|$ gives the present value of 1*l.* to be received under the conditions of the question. But a perpetuity created at the end of *t* years, or $t|$, divided by a perpetuity now due, gives the present value of 1*l.* to be received at the end of $t+1$ years; which gives the following

Rule. From the value of a perpetuity subtract that of an annuity on the given life for *t* years, and divide by the value of a perpetuity due. From the quotient subtract the present value of one pound to be received at the end of $t+1$ years, if the life be in being at the end of *t* years: the difference is the present value of 1*l.* to be received at the end of the year in which the life drops, if before *t* years have expired. The present value just found, divided by that of an annuity due on the given life for *t* years, gives the requisite premium.

And since the present value of such an insurance as the preceding, together with the present value of 1*l.* to be received at the end of the year in which A dies, if after *t* years, make up the present value of 1*l.* to be received in any case at the death of A, the third diminished by the first will give the second. But it will be preferable to make an independent investigation of this case.

Problem. Required $\underset{1}{t} : \underset{2}{\mathrm{A}}\big|\overline{1}$ the present value of 1*l.* to be received at the end of the year in which A dies, provided *t* years shall have previously expired.

If from the present value of a perpetuity deferred for *t* years, we deduct that of an annuity on the life of A deferred for *t* years, we have the value of a deferred perpetuity, further suspended during the term by which A outlasts *t* years, and to commence at the end of the year in which A dies: or not to be suspended at all if A should die in less than *t* years. Take away the value of a perpetuity beginning from the end of *t* years, if A should have died in the interval, and we have remaining the present value of a perpetuity due at the end of the year in which A dies, if that be deferred beyond *t* years. This last is therefore

$$t| - t|\mathrm{A} - (1 - t \dagger a) \times t|$$

where $t \dagger a$ is the chance of A living *t* years. This can be reduced to

$$t \dagger a \times t| - t|\mathrm{A}$$

Divide this by the value of a perpetuity due, and we have the present value of 1*l.* receivable on the same conditions. But $t|$ divided by $-|$ gives $t\,|\,\overline{1}$, as before; whence the following

Rule. Multiply the present value of 1*l.* receivable at the end of $t+1$ years by the chance which A has of living *t* years; and from the product subtract the quotient of a deferred annuity on A's life, divided by a perpetuity due: the remainder is the present value of 1*l.* to be received, if A outlive t years, at the end of the year in which he dies.

The preceding pages contain all those cases which most usually occur in practice; but it is to be noticed that various modifying circumstances will present themselves in different problems, for which no general rule can be given. I now proceed to problems in which successions occur; that is, in which a benefit depends upon the continuance of a status which does not begin until another status is finished.

In the case of an annuity for a certain term of years t, to begin payment at the end of the year in which A dies, we have obviously to consider a benefit which, at the end of the said year, will amount to an annuity due, of t payments; or 1*l.* augmented by the present (as it will be then) value of $t-1$ future payments. This *then* value can be found as in page 185., and, being multiplied by the *present* value of 1*l.* receivable at the end of the year in which A dies (found in page 214.), gives the present value of this contingently deferred annuity.

In the case of A|B, an annuity on the life of B after the death of A, we certainly have a succession, but it is one which may never exist. To make a problem which may come under the present division of our subject, we must imagine that, at the end of the year in which A dies, a new life may be nominated at pleasure, which is then to be of a given age. If P be the value of an annuity upon such a life, then, according as the benefit is an annuity, or an annuity due at the end of A's year of death, we find the present value of P or $1+P$ to be received at the end of that year. The result is the present value of the succession. This problem includes that of finding the value of the next presentation to a living. The patron of a living of 500*l.* a year may consider that he gives the clergyman whom he presents 100*l.* a year (or whatever may be called liberal remuneration for a curate) for work and labour, and the remaining 400*l.* as a free gift. If he sell the next presentation he must therefore consider that he sells 400*l.* a year (not 500*l.*, since that would be to allow the clergyman no salary* for his labour), to be paid yearly during the continuance of a life to be named by the buyer, at the decease of the present incumbent. And, since the right to name new incumbents of 24 years of age is part of the bargain, the patron will require a sum corresponding to the value of an annuity upon a life of that age. Deducting a sum for first fruits, probability of expenses from dilapidations, &c., which must be determined by the circumstances of each case, the remainder is the net present value of the living. It would probably be most fair to value the interest of the purchaser as if the new incumbent would come into half a year's revenue at the end of the year in which the present incumbent dies.

Question. What is the present value of the next presentation to a living of which the average annual income is £s, the salary of a curate † being £v, and f the estimated expenses at entry. Let A be the value of the incumbent's life, and P that of a life of 24 years of age. Find the present value of $P+\frac{1}{2}$, to be received at the end of the year in which A dies (p. 214.), and multiply the result by the excess of s over v; from this deduct the present value of f, to be received at the end of the year in which A dies, and the remainder is the net present value of the next presentation.

The perpetual value of an advowson (that is, of the right to nominate the incumbent in all time to come, after the decease of the present one) is generally valued as the reversion of the net income after the death of the present incumbent. But the expenses of entry, first fruits, &c., should be considered as a fine levied on the property at the death of every tenant, in diminution of

* The right of selling livings is therefore a *bonâ fide* right to alienate all the church property which is in private hands, with the exception only of that minimum which will obtain a curate.

† If the living be one on which a curate must be kept by the incumbent, the salary of two curates should be deducted from the yearly revenue in the valuation.

the total value. To the problems connected with this subject I now pass.

A great many interests are held in this country on the consideration of rents, fines, or whatever they may be called, which are not paid at any fixed time, but at the deaths of successive lives which are named, each life being nominated, and the rent or fine paid, at the death of the preceding nominee. Leases held under ecclesiastical and other corporations, copyholds, &c., are instances. By a statute of Henry VIII., corporations are permitted to lease lands for *three lives*, or *twenty-one years;* so that it may be suspected the legislature imagined the average term of the duration of three lives to be 21 years; or that, any three mature lives being named in one set, and a large number of such sets being taken, and each set being considered as a status to last as long as any one of its lives was in being, the average duration of such a status was 21 years. If this were the opinion, and grounded upon any thing like experience, the value of life in that day must have been incredibly below what it is at present: but it must be remembered that in that day of insecurity few people would venture on the life of a child or a woman; and that in all probability the lives contemplated were those of men of middle age. However this may be, since that time the tenure of lease upon lives has become extremely common, it being understood that the lives which drop are renewable upon the payment of a fine, either fixed or at the discretion of the lessor.

It is, of course, the interest of the lessor that the lives should be as bad, and of the lessee that they should be as good, as it is possible: but the lessee, having the nomination of the lives, will choose the best the tables afford. The rate of interest being settled, the highest life annuity in the tables gives the age which the life nominated ought to have. The best age in the Northampton tables is 8 years, and in the Carlisle 7 years: for which ages I subjoin the values of annuities at various rates of interest, adding also the age of 24, which will be useful in the calculation of the values of advowsons, with the correction above proposed.

Northampton.					Carlisle.				
Ages.	3 p. c	4 p. c.	5 p. c	6 p. c.	3 p.c.	4 p. c.	5 p. c.	6 p. c.	Ages.
8	20·9	17·7	15·2	13·3	23·9	19·8	16·8	14·5	7
24	18·0	15·6	13·7	12·1	20·9	17·8	15·4	13·5	24

QUESTION. At the end of the year in which A dies, a fine of 1*l.* is to be paid, and a new life nominated, of which the value will then be P: at the end of the year in which P dies, another fine of 1*l.* is to be paid, and a new life P nominated, and so on for ever: what is the present value of all the fines, or what present money must a person be considered as paying who receives an estate charged with the preceding liabilities?

This problem, as will be more fully explained in the second appendix, was incorrectly solved by every writer on the subject, down to the time of Mr. Milne, whose solution, though perfectly correct, is in a difficult form. The coincidence of the rule I now give with that of Mr. Milne will be shown in the appendix cited.

Let us suppose a fine of 1*l.* *per annum*, first payable at the end of the year in which A dies. If, then, a receiver P were appointed for his life, his interest in the fines, at the end of the year in which A dies, would be 1 + |P; and if at his death a second receiver were appointed, of the same age at which the first was when his term began, the interest of this second receiver at his entrance would also be 1 + |P, and so on. But if the tenant compounded with each receiver on his entrance, for the rents payable during the life of that receiver, it would evidently be equivalent to paying a fine of 1 + |P at the end of the year in which each dies, and also at the end of the year in which A dies. But the present value of all the rents is a perpetuity diminished by the value of an annuity on A's life, or | — |A. And if this be the value of a fine of 1 + |P, then | — |A, divided by

$1+|P$, gives the value of a fine of 1*l.* in the same circumstances. Hence the following

RULE. From the value of a perpetuity subtract that of an annuity on A's life, and divide the remainder by the value of an annuity due on the renewal life at the time of renewal.

If there be several lives in the lease, apply the preceding rule to each life, and add the results: for the several contingencies do not interfere with or depend upon each other, nor will the case of more lives than one in one lease differ from that of several leases each on one life. The most convenient method is as follows:—

RULE (for several lives). Multiply the value of a perpetuity by the number of the lives, and subtract the sum of the values of the annuities on the different lives: divide the result by the value of an annuity due on the renewal life at the time of renewal.

QUESTION. An estate of the clear annual value of £*a* per annum is to be leased on *n* lives, A, B, C, &c., with liberty to renew at the end of each year in which a life drops, the best life in the tables being P: what fine should be paid, on the supposition that the purchaser is to have a given rate of interest for his money?

RULE. Find the value of the perpetuity of £*a* per annum; multiply it by the value of an annuity due on the renewal life at the time of renewal, and divide by the excess of *n* times the value of a perpetuity of 1*l.* over the sum of the values of annuities on the lives of A, B, C, &c.: the quotient is the value of each fine required. But if a sum £*s* be paid down, and the rest of the value of the estate is to be paid in fines, then subtract *s* from the perpetuity of £*a* per annum, before using it in the preceding rule.

EXAMPLE 1. The lives in possession, A, B, and C, are 35, 48, and 60 years of age, and the fine paid on renewal is 300*l.* What is the present value of all the fines, using the Carlisle tables, and interest at 4 per cent.? *

* I have taken Mr. Milne's example, in order to show the accordance of

$\rvert=25$	$\lvert A=16{\cdot}041$	$\lvert P=19{\cdot}792$
3	$\lvert B=13{\cdot}419$	1
—	$\lvert C=\ 9{\cdot}663$	———
75	———	20·792)35·877(1·7255
39·123	39·123	300
———		———
35·877		517·65
		In Milne 517·6296

The present value of every single pound of the fine is 1·7255*l.*, which, multiplied by 300, gives 517·65*l.*

Example 2. All things remaining as above, the preceding lease, worth 120*l.* per annum, is purchased for 2500*l.* and a contract for fines on renewal. What should the fine be?

$\rvert=25$	$1+\lvert P=20{\cdot}792$
120	500
———	———
3000	35·877)10396(289·77
2500	289·782 in Milne.
———	
500	

The answer is 289·77*l.*

Question. What yearly rental should the fines be considered as amounting to; and what should be paid by the lessee annually to an insurance office which would undertake to pay all the fines as they become due?

These two questions are the same, and the answer to both is,—the yearly interest upon the present value of the fines. Thus, in the first preceding example, the lessor's interest, at 4 per cent., is worth 20·7*l.* per annum for ever; which the lessee might either pay to his landlord, as a commutation of fines, or to an insurance office, which should take them upon itself.

Question. What is the present value of the next fine upon the renewal of the first life which drops of the three, A, B, C?

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the rules. The slight difference arises from Mr. Milne's rule requiring an interpolation, which he has very properly thought it not worth while to make. I have taken more decimal places than those previously given, in order to show the accordance more clearly. (Milne, page 365.)

This is evidently the present value of 1*l*. to be received upon the failure of the joint existence of A, B, and C, and is to be found (page 214.) by subtracting |ABC, the value of an annuity on the joint lives, from that of a perpetuity, and dividing by the present value of a perpetuity due.

Question. If the tenant wish to exchange one life for another and a better, how much should he pay to be allowed to do so?

Rule. If the fine be 1*l*., subtract the value of the inferior life from that of the better one, and divide the difference by the value of a perpetuity due on the renewal life at the time of renewal. To exchange two lives, or three lives, use the sums of the values of the better lives, diminished by that of the inferior ones.

Question.* If the lessor have only a life interest in the estate on which he grants leases for lives, what is the value of his interest?

In strict justice to future holders, it *ought* not to be worth more than the rental calculated in the last question but three, continued for his life. But it is the nature of this species of property, that the life interest of the holder is subject to considerable fluctuations of value, the preceding annuity being at one time less and at another greater. A lessor, for instance, who enters when the lives in possession are very old, himself being very young, has nearly a certainty of one fine on account of each, and not much less than an even chance of a second, while his prospect of a third may be worth calculating. But a lessor who enters at an advanced age, against lives which are very young, has a present interest in each coming fine, which may be determined by finding the present value of 1*l*., to be paid when the life drops, on condition the lessor survives (page 223). Indeed, whatever may be the lives in the lease, provided the lessor enters at an advanced age, his interest is deter-

* This problem, properly treated, would be of extreme complication, and I do not remember having seen it proposed. The method in the text is an approximation.

mined with sufficient accuracy by finding the values of insurances on the lives in possession, on condition of the lessor surviving. But when the lessor is young, I am not aware of any rule to which I would trust, as making as good an approximation to the value of his life interest as can be made in other cases. Each case must be determined by its own details; and it will always be safe to begin by calculating what we may call the mean value, namely, the annuity first mentioned; which may, for any thing I see to the contrary, be a perfectly correct mode of proceeding in all cases.

Question. If the lessor should refuse to renew, and if it be pretty certain that his successor will adopt the same course, what is the present value of the tenant's interest?

Evidently the clear annual value of the estate, considered as an annuity upon the longest of the three lives, the value of which is determined in page 208.

This awkward contrivance for limiting the rights of corporators over property is prejudicial in its effects, both upon the tenants and the lessors. The former, holding an interest of a comparatively precarious character, have not the same inducement to improve their property which is felt by leaseholders for fixed terms of years. On the other hand, the lessor, in all cases in which he has a personal interest in the proceeds of the estate, has two distinct periods of temptation to an act of equivocal morality. If he be young, he may, as it is called, run his life against those in possession; that is, refuse all renewals, upon the prospect of a large ultimate gain from the falling in of the old leases: if he be old, he may induce the tenants, by offering easy terms, to change their old lives for young ones, thus impoverishing the successor, by leaving him nothing but long leases, or leases on young lives.

It must very often be a question for the lessee, whether it would not be his wisest plan to refuse all renewal, and to insure a certain sum upon the last survivor of the three lives by which he now holds. The prudence of such a step must depend upon the fine demanded for a renewal. In the case of church leases, I believe it could not often be desirable: and certainly not if they are let so much below their value as has been asserted. But if the fine demanded should be exorbitant, it would then become cheaper to insure the longest of the lives in possession than to pay the demand. The premium for such insurance would be found as in page 214, the value of an annuity on the longest of the lives having been previously found in page 208.

QUESTION. The average value* of a living is £s per annum, and the proper allowance to the incumbent for the performance of the duties is £v; the unavoidable expenses at entrance are £f for each new incumbent; and the value of the life of the present incumbent is |A, while that of the new incumbent will be |P. What is the value of the perpetual advowson of such living?

RULE. The value of a reversion of 1*l. per annum* after the death of A being found (by subtracting |A from the value of a perpetuity), and multiplied by the excess of s over v, will give the value of the perpetual advowson, as it would be but for the expenses at entry. For these, deduct the present value of a fine f, payable at the end of the year in which each incumbent dies, the value of each pound of which determined by dividing the reversion aforesaid by one year's purchase more than |P. The difference is the net present value of the advowson required. According to a frequent practice of valuing advowsons, in which the expenses at entrance are neglected, the buyer pays them twice over.

* This should include all real profits: for instance, the value of the parsonage house as a residence, considered as taken on a strict repairing lease.

CHAPTER XI.

ON THE NATURE OF THE CONTRACT OF INSURANCE, AND ON THE RISKS OF INSURANCE OFFICES IN GENERAL.

In laying down the following considerations, I think it right to state most explicitly, that I intend no direct reflection upon any office now in existence, or whose establishment is contemplated. In a set of societies so numerous and varied as those in question, there must be details in one and another, of which any individual, who turns his attention to the subject, must disapprove; but the studied exclusion of the name of every office whatsoever will, I hope, be taken as earnest of my desire to confine myself to the enabling other persons to discover the grounds of censure, without directing their attention to the quarter in which they are to be found. I have not much fear that any part of this chapter could have been misconstrued into allusion; and perhaps even the present disclaimer may have no other effect than to make some imagine that there must be more meaning somewhere than is openly expressed. Leaving such lovers of mystery to their search, I proceed to the subject before me.

The avowed levellers in politics, a rare and scanty sect among educated persons, would have an argument of some force, from considerations of general expediency, if it could not be shown that any attempt to equalise property would be attended with a vast diminution of the fund itself, so that the great majority *

* The only great alteration of property which is likely to be agitated, is the question of the national debt, the entire abolition of which is not without its advocates. This enormous sum, as it appears, is really little more than one year's income of the country, and perhaps not so much, if all the colonies be considered. The honesty of a sponge not being considered, there would still remain this question :—Would the ultimate loss occasioned by the subversion of such a debt, amount to a year's income of the country? If so, there would be no gain arising from the abolition of the debt.

would really have even less than they now possess, and also less facilities of increasing their stock. The differences of talent and of life would still remain, constantly working towards a restoration of the ancient inequalities, in which they would almost instantaneously show their power. A division of property, to be permanent. must be accompanied by a division of intellect, a division of manual skill, and a division of life; nor would the sum of the parts make up the whole in any one of the four, *except the last.* A law which should tax the property of all who live beyond a certain time of life, to provide an addition to the maintenance of the widows and children of those who die before it, would not be so utterly impracticable, nor so pernicious, as an attempt at equalization of fortune, intellect, or skill. Such a law would, however, fail in its operation, by the mere difficulty of arranging its enormous details, the frauds to which it would give rise, and the temptation to idleness which it would hold out to the young. A small community, consisting of members of known honesty, living under a government in which they reposed entire confidence, and possessing sufficient inducements not to relax in their exertions, by the certainty of a provision for their families, might live under such a law: and such communities actually do exist, under the name of insurance companies.

If a large number of persons, all of the same income and prospects, and all certain of the same duration of life, were to choose a common bank in which to deposit their savings, each laying by a given proportion of his income, it is obvious that each would receive the same sum as the rest at his decease; but, if the lives were of unequal and uncertain duration, this result would no longer be produced. It might, however, be attained by a covenant, that all sums paid in should remain till all were dead, and then be equally divided among the executors of the parties. Such a bank might be called an equalization office, and it would present the first approximation towards an insurance office such as those which at present exist.

As yet we have not mentioned the interest of money. Suppose the equalization office to pay no interest; and suppose all the lives to be 20 years of age, such as are described in the Carlisle tables, the average duration of which is 41½ years. If, then, every person pay 1*l.* per annum, each will ultimately receive 41½*l.*, which is the mere compensation of the inequality of life. Such persons would enter into a mutual covenant, by which those who live beyond the average term would divide the surplus of their savings among those who fall short of it.

Probably, if the following question were put to all those whose lives are now insured, What is the *advantage* which you derive from investing your surplus income in an insurance office? more than half would reply, The *certainty* of my executors receiving a sum at my death, were that to take place to-morrow. This is but half an answer; for not only does the office undertake the equalization of life, as above described, but also the *return of the sums invested, with compound interest.*

No one can form an accurate idea of such an establishment, who does not consider it as a savings bank, yielding interest, and interest upon interest. This is the reason why an office which charges for its insurance more than it is worth, as an insurance, may nevertheless put its contributors in a better position than they could have held if there had been no such institution. To make this clear, let us consider the working of a simple investment office. A large number of individuals subscribe a sum, which they intrust to an individual or a company to employ, yielding them the return at some fixed, but distant, period. Let each share be 100*l.* The best thing which an individual could do with such a small sum, so as to have perfect security for its return, would be to invest it in the funds, at 3½ per cent. He might also invest the interest, and thus obtain compound interest: but it is not easy for an individual to do this. Unless he provide an agent to draw the dividends immediately on their becoming due, various circumstances will happen to prevent the immediate investment of the interest. It is not at all an unfair calculation to suppose that, upon each half yearly dividend a month will be lost, so that nominal compound interest for 42 years will only be really for 35 years. A single pound, therefore, laid up by a man of 20 and improved for the average term of his life, at $3\frac{1}{2}$ per cent., would only become $3\frac{1}{3}$*l.*; while, in the hands of a person who lost no time, it would become $4\frac{1}{4}$*l.*, or nearly a pound more. On the other hand, a company, or a skilful individual who can command large sums of money, can always make the best interest which the market will afford. The funds, from the security of their tenure, and the conveniences which they offer, will always, in ordinary times, represent the lowest rate of interest which money will yield. Other investments, which offer better interest, are generally only accessible to those who can command considerable sums, and are frequently attended with risk; so that it requires knowledge to distinguish between the sound and the unsound. A company, employing the whole time of a person or persons skilled in money matters, and having continual large investments to make, can realise not only more interest, but so much more, that there shall remain a surplus worth considering, after the skill employed has been paid for. It is not assuming too much to say that, all expenses paid, they can command $3\frac{1}{2}$ per cent. compound interest. More than this, they can obtain such interest without any delay in investing the interest. The process is extremely simple: it is not difficult to ascertain what sum should lie permanently at the banker's, in order to meet current expenses, so that the banker has general directions to buy stock as soon as the balance in his hands exceeds that sum; and all cash received is paid into the bank at the close of each day. Suppose it should happen that ten individuals paid 100*l.* into the office on account of life insurance premiums, in the same hour in which the executors of a deceased contributor received a claim of 100*l.* The hundred pounds, which, in the theory of the process, should be sold out, or otherwise set free, to meet the claim, is in its practice supplied by the new premiums, so that the premiums of those contributors are making interest from the hour in which they are paid. But there is always an unemployed sum lying at the banker's. This is true; but the interest of that sum is the salary of an officer of the institution, namely, the banker himself. All such expenses paid, I believe it may be stated, with correctness, that an investment office can net $3\frac{1}{2}$ per cent. compound interest. Hence 1*l.*, improved during the average life of an individual aged 20 years, would become $4\frac{1}{4}$*l.*

The institution we have hitherto described is simply an office for the investment of premiums and the equalization of results: it becomes an insurance office when it undertakes to pay a fixed sum for a fixed premium, at the end of a given time after the decease of the party. It then begins to incur a risk of a twofold character: in the first place, the lives which it undertakes to insure may not die, one with another, in or near the same manner as those from which the tables were constructed; in the second place, the rate of interest, upon which it calculates the premiums, may be higher than it is afterwards able to obtain. According to the Carlisle table, the premium which should now be paid to insure 100*l.* upon the life of an individual aged 20, is one pound seven shillings, or 1·32*l.*, at *four* per cent. According to the Northampton table, and at *three* per cent., the same premium should be 2·2*l.* Taking the first premium, and assuming its table, the office will not be sure of avoiding loss, until the party has lived 35 years; by which time the premiums, with their accumulated interest, will have passed 100*l.* It is a little more than 2 to 1 that a life of such an age shall live beyond 32 years after the contract. Taking the premium of the Northampton table, the party must live 28 years before the office can gain by him; and it is about 10 to

7 that he will outlive this term. We have now to ask, What are the principles which should guide the office in the determination of its premiums, it being remembered that there is an absolute security required, and that the remote chance of bankruptcy, which is almost essential to the ordinary run of commercial affairs, is not to be encountered?

The basis of the tables is the observation of the lives of a comparatively small number of individuals; it being well known that the value of life varies considerably in passing from one class of society to another. Now, we have seen (page 91.) that we cannot depend upon a law of probability, derived from a limited number of instances, with the same degree of confidence, as upon one which we know to exist *à priori*. If we were sure beforehand that the great average of life in England was according to the Northampton, or any other table, we might rely upon such a document as being extremely likely to exhibit, with small fluctuations, the future course of the lives of the two thousand or ten thousand persons insured in any given office. Let such a table be assumed, and let the premiums be so calculated, that it shall be a thousand to one against any ruinous amount of fluctuation, taking the law of the tables as that which will certainly prevail in the long run. Then return from the hypothesis to the truth, and, taking the number of lives from which the table was actually formed, say 5000, suppose another 5000 persons to have commenced an insurance office. The degree of fluctuation within which it was 1000 to 1 that the future results should be contained, is now larger than before, in the proportion of the square root of 2 to 1, or in that of 14 to 10, nearly. Larger premiums would then be required to make ruinous fluctuation as unlikely as upon the preceding supposition. These considerations, which may easily be reduced to calculation by the rules in chapters IV. and V., will serve to show that there may be danger in the assumption of any table formed from experience: and they ought to operate powerfully as a caution against lightly admitting a change of premiums, on the authority of any small number of facts. But more particularly should they be attended to in the formation of new varieties of contingency offices, the chances of which have not yet stood the test of experience.

But there are reasons why the premiums of an insurance office need not be so high as the very limited number of data in their tables might seem to require. If the fluctuations from the average, which are within the most cautious definition of reasonable probability, were all to be encountered at once, or might be encountered at once, it is difficult to say what premiums should be considered as too high. But this cannot be the case, unless, indeed, a pestilence should single out the members of an insurance office, or an earthquake should, by one extraordinary event, swallow them all up in the place where, by a most remarkable coincidence, they were all assembled together. Such extreme cases are not worth consideration; and we may take the chances of life and death as distributed over a large number of years. In the meanwhile the surplus fund increases at compound interest; and the problem is, not whether a given number of lives will, on the whole, drop so much before the predicted time that a given fund will be destroyed, but whether this can happen so fast, that it will outrun the increase of the fund at compound interest. If, indeed, there were compound mortality to set against compound interest; that is, if the number of deaths must become larger from year to year, or if the rate of mortality were increasing, the fear of such a result might be entertained; but all experience is on the other side, and tends to show that the value of life is increasing, instead of decreasing.

The tables of an insurance office must be considered as collections of limited data, the premiums deduced from which are increased by a percentage, to meet the possible fluctuations of mortality. As soon as these tables are formed, and the directors have published their proposals, an *insurance* office is created, with all these fundamental characters already described, and which are but ill represented by the term. The word *insurance* or *assurance* has given rise to some wrong notions, and it will be worth while to examine the nature of the contract.

A and Co. engage with B that, in consideration of 1*l*. a year, paid by him during his life, they will pay 20*l*. to his representatives as soon as he shall be dead. Both parties run a risk; A and Co. that of having to pay B more than they receive; B, that of paying more than will at his death produce 20*l*. But the risk of the office is of immediate loss, and that of B, of deferred loss: that of the former is also continually lessening, and that of the latter increasing; until, should B live long enough, both risks become certainties. If the insurance be only for a term of years, B runs the risk of losing his premiums altogether.

The office does not inquire what reason B may have for insuring his own life or that of another person, nor do any possible contingencies, except those of life, affect the office calculations. We cannot, therefore, be too much surprised at the ignorance shown by that judge* who declared that life insurance † was of its own nature a contract of indemnity; that is to say, if, by any lucky chance, B can be proved to have accomplished the object for which he insured by other means, he has no claim upon the office. The circumstances are as follows; and the absurd conclusion is law, and would be practice, if the insurance offices had not refused to acknowledge the decision, or protect themselves by the precedent. A and Co. covenanted with B to pay 500*l*., if C should die within the term of seven years next ensuing, in consideration of the usual premium. C did die within the term; and A and Co., in

* Godsall *v*. Boldero. See the report of the case in Mr. Babbage's "Comparitive View of Institutions for the Assurance of Lives."

† He might have said that the law would refuse to consider an assurance in any other light; but he was palpably wrong in asserting that the contract, as understood by the parties, was merely one of indemnity.

answer to a claim of 500*l.*, replied, that the intention of B in insuring the life of C, was to obtain security for the payment of a debt of 500*l.*, due by C to B, which debt had been already paid by C's executors: consequently they owed nothing to B. An action was brought by B, and defended by A and Co. on the above plea; and a special case being made, the point was decided by the court of King's Bench against the plaintiffs; thereby establishing the principle, that life insurance is a thing similar to fire or ship insurance; namely, a contract of indemnity, to be fulfilled with allowance for salvage.

The defendants' case rested upon the asserted nature of the contract, and the statute 14 Geo. III. c. 48., which enacts, that "no greater sum shall be recovered from the insurers than the amount or value of the interest of the insured in such life." The act does not state at what time this interest is to be reckoned, but the plaintiffs contended that the time of death was the meaning of the statute; the defendants averred, and the court decided, that the time of bringing the action was to be understood. The plaintiffs contended that the debt was not the object of insurance, but the life of the insured; the court decided, that "This action is, in point of law, founded upon a supposed damnification of the plaintiffs, occasioned by the death, existing and continuing to exist at the time of the action brought; and, being so founded, it follows, of course, that if, before the action was brought, the damage which was at first supposed likely to result to the creditor were wholly obviated and prevented by the payment of his debt, the foundation of any action on his part, on the ground of such insurance, fails." This sentence contains nothing but very good sense, and, no doubt, very good law: but the application of it was accompanied by a mistake as to the nature of the damnification which the plaintiffs had sustained. The counsel on both sides, the court, the insurance office, and the plaintiffs themselves, showed a very partial knowledge of the nature of the contract; and I make no doubt, that almost every person who heard it agreed with the court, however much they might impugn the decision on other grounds, that the damage* to the creditor "was wholly obviated and prevented by the payment of his debt."

In order to show that such was not the case, we must suppose that an exactly similar transaction had taken place before any insurance office existed. How this could have been may not be apparent, if we take the notion which the law formerly entertained of such an office; namely, that it is a species of gambling house: but if we prefer to consider it as a savings bank, with an equalization system (page 238.), which is unquestionably the correct notion, we may return to the circumstances which the case would have presented had there been no insurance. C, a person whose credit has become doubtful, is indebted to B to an amount which B could not afford to lose; consequently, B, knowing that his chance of payment is precarious, resolves to diminish his expenses, hoping by economy to restore to his family the sum which he may have lost by his engagements with C. He collects, accordingly, a small fund, which he places with his banker, avowing the purpose of its collection. In the mean time C dies, and some friends pay off his debts, and that due to B among the rest. The latter having now no further occasion for such economy, draws upon his banker for the amount, and is answered, that, since the purpose of the saving was fulfilled by the payment of C's debt, he, B, has no further claim upon his own money. An action is brought, and the courts decide that the banker is right, and that B, having really attained his object in one way, has no right of property in the proceeds of another attempt to serve the same purpose.

* The defendants paid into court a sum somewhat less than the amount of the premiums they had received from the plaintiffs, doubtless as a precaution, in case the court or jury should think the premiums ought to have been returned.

The only distinction between the case just put and that which actually occurred is, that the banker was a person who gained his profits by receiving such savings during a contingent term, and guaranteeing a fixed sum; standing the loss, if there were any, and paying himself for it out of the gain which would accrue in another instance: the premium having been calculated so as to insure a moral certainty of profit upon the average of similar cases. It is not pretended, on either side, that the chance of indemnification at the hands of C's executors was made to lessen the consideration paid by B for the guarantee; and the legal iniquity of the decision may, I think, be made clear, as follows: —

It will hardly be disputed, firstly, that the legislature is the judge of what shall constitute valuable consideration; and, secondly, that a consideration which is expressly allowed to be good in a statute, should be admitted as such in the decisions of the courts. Now, the contract of insurance, be it gambling, or be it not, rests entirely upon the permission given by the law to consider a high chance of a small sum as good consideration for a low chance of a large sum. If I now pay *2l.* of premium for *100l.*, in case I should die in a year, and if my executors can maintain an action for *100l.*, it must be because the law sanctions the notion that *2l.*, nearly certain, may, with consent of parties, be considered as an actual equivalent for a distant chance of *100l.*, as much so as one weight of silver for another of bread, or food, clothing, and wages for personal service. It is true that the same law, fearing certain reputed immoral practices, to which the power of making a particular bargain offers temptations, may limit the circumstances under which it will permit such bargains to be made; but this is equally true in regard to the other sort of contracts mentioned: indeed, there is no sort of bargain which is not under regulation. The law, then, allows risks, and permits unequal chances to be compensated by giving odds; the courts declare that, after the cast shall have been made, and one of the parties shall have stood *his* risk, which turns out in his favour, the other party shall receive an *ex post facto* release from the conditions of his bargain, because circumstances afterwards arise, which, had they existed* at the time of making the bargain, would have made it illegal. The several principles on which the decision was founded, well carried out, as they say in parliament, would require that the previous contracts of a man who becomes insane should be null and void; that the meat which a man buys for his dinner should be returnable to the butcher under the cost†, if a friend should invite him in the mean time; and, in the case before us, supposing that C should have outlived the term, and his debt were paid, as before, then B might have brought his action against the office, for the return of the premiums; alleging that, as it turned out, the office would have been indemnified, and, therefore, should be considered as having run no risk.

But, said the judge, the damage was "wholly" obviated and prevented by the payment of the debt. To try this point, let us make a debtor and creditor account of the whole transaction. The following is the way in which it will stand.

Cr.	*Dr*
£500 worth of goods furnished to C.	£500 paid by C's executors.
Certain small premiums paid to an insurance office, with imminent risk of their entire loss; such premiums, multiplied by the risk of loss, as in chapter V., being good legal consideration for a remote chance of gaining £500, and so considered by both parties.	Those same premiums returned by the office, instead of £500.

* This is admitting more than is absolutely necessary; for, unless there were mathematical certainty that a third party would step in and pay C's debts, it is difficult to see how B's insurable interest would cease.

† The sum paid into court by the insurance office, was less than the amount of the premiums: but the plaintiffs waived that point.

The advantage of the moral security which a contract of insurance gives is obvious in the transaction which led to this decision; namely, the insurance of the life of a creditor by a debtor at his own expense. Commercially speaking, such a transaction is literally this: C owes *500l.* to B, who, doubting his chance of payment if the debtor should die, buys *500l.* from a third person, and makes believe that it is the *500l.* which C owes him. Morally speaking, it is the determination of B to retrench his own expenditure, as soon as he finds that a part of his property consists in bad debts. This the office enables him to do in a manner which will make the retrenchment proportional to the necessity for it. In the mean time, it is much to be wished that the law of life insurance were settled upon a fixed basis, which should proceed upon such a definition of the contract as has been here explained, and not on the notions which have been drawn from a supposed analogy between it and the insurance of a ship or a house. The effect of the present state of the law is, that the offices have no law except that of honour, which, though it more than suffices for the protection of the insured, yet may at any time involve the offices in the necessity of paying really questionable policies, without having the means of submitting to open examination the point on which they wish to resist. Policies of insurance are sold daily to persons who have no interest in the lives of the insured parties, on the faith of the good conduct of the offices. If an office were to resist the payment of a policy so transferred, say on the ground of fraudulent representation, the parties so resisted might give out that the opposition of the office arose out of an intention to cover themselves by the present letter of the law. Neither could such a case be carried into court without proof that the plaintiffs possessed that insurable interest in the life of the deceased which the law requires.

The nature of the contract, both in law and usage, having been laid down, we must next ask what are the means which the offices employ to reduce the risk so as to render themselves safe against fluctuation. The state of opinion upon this matter is somewhat unsettled; one party advocating the practice of approaching near to the line which separates security from insecurity; another insisting upon what appears to the first a most superfluous degree of caution. Without expressing an opinion, I will describe the various risks, and the method of avoiding them which has usually prevailed.

1. The insecurity of data, that is, of existing tables of mortality. This divides itself into two parts; that relating to the young and middle aged, and that relating to old lives. With regard to the first, the data might probably be obtained in sufficient numbers to justify a considerable degree of confidence in them as to the chances of a single life, or even of a considerable number; but when the number of lives is to be as great as the number of persons who may choose to offer themselves, the considerations in Chapter IV., again adverted to in page 242, present themselves in force. I am not aware that any writer on the subject in this country has formally taken into consideration the uncertainty of tables, arising from their limited numbers, except Mr. Lubbock, who has made use of (Cambridge *Phil. Trans.*, and Treatise on Probability *Lib. Usef. Know.*) the correction which the probability of living a given number of years should receive on that account. But, considering the probable errors of the data, this correction is small, and the question how far an office proceeding upon such data can deal with the public to any amount is yet in its infancy, though the necessity for its consideration is approaching, and it is one of vital importance to the interests of the middle and lower classes.

The constructor of tables of mortality draws a number of balls from an urn which contains an infinite number and, having sorted them into red, blue, black, &c. presents them to the world as a necessary representation (or very nearly so) of the proportions in which those colours are scattered throughout the whole urn. He commits an error which is in all probability very small, and which has hitherto been carefully guarded in the deduction of office results. But there is a much more important question behind. Suppose the calculator had undoubtedly succeeded in exhibiting the real law of mortality, and that it were quite certain the next hundred million inhabitants of Great Britain would die in the manner pointed out in a table. In such a case, many will say, the office may charge the real premiums deduced from the table, with a very slight addition for expenses of management. They may leave the fluctuations to take care of themselves, and trust in the long run. This assertion I now proceed to discuss.

If the banker of a gaming-table were to follow the same plan, that is, if he were to stake against all comers with only just enough of advantage to cover expenses, he would infallibly be ruined at last. It might not be in this year nor the next, nor in this century nor the next; but ruined at some time or other he must be (see page 110, and also Appendix I.). If the case of the office managers were precisely analogous to that of the bankers of the gaming-table, I would repeat with as much confidence of the former what I have said of the latter. But, in the first place, the fluctuations of mortality are not, by very much, so great as those which take place in the assortment of cards, nor even so great as those which take place in harvests, in the price of provisions, &c. This is much in favour of the insurance office; but who can say *how much?*

In the second place, the fluctuations of mortality have of themselves a tendency to create opposite fluctuations. Thus, a very sickly season carries off the weak, and deprives the succeeding years of those who were most likely to have died; causing, therefore, a season of remarkable health. This is a very important item in the theory of the fluctuations of mortality, and there is nothing similar to it in the case of the gaming house. It reduces annual fluctuation itself to a species of regularity, and is, perhaps, the sufficient reason for the slightness of the total fluctuations.

In the third place, with a merchant or a banker, the liability to a demand and the demand itself come so nearly upon one another, that real insolvency and bankruptcy are never far asunder. When credit cannot be sustained by monthly, and even daily, proofs of substance. it takes its departure altogether: but it is not necessarily so with an insurance office, of whose existence it is the essence to be always receiving consideration for bills which, one with another, have a long time to run. Such an establishment, as will presently more distinctly appear, may be in reality *insolvent* many years before the symptoms of *bankruptcy* come on. As no large concern of the kind has hitherto failed, it is difficult to say how they would finally come on: but this much is certain, that an insurance office which could really pay only ten shillings in the pound might, by introducing a better system, or by mere force of circumstances, not only recover its ground, but ultimately become exceedingly profitable. But I throw this part of the argument (though it shows a strong principle of vitality inherent in the constitution of such offices) out of the question; for, surely, no sane and honest person would trifle with important matters so far as to assert that the possibility of temporary insolvency, to be redeemed by the chapter of accidents, or prudence, when it was wanted, should enter into the deliberate calculations on which men should be invited to stake the subsistence of their children.

If the last contingency be rejected, that is, if it be held absolutely necessary to calculate on permanent solvency, both real and apparent, then I assert that there is not sufficient ground to gainsay the conclusion, that any insurance office charging only *real** premiums (increased for expenses of management) must inevitably have its phases of solvency and insolvency, at the very best. Begin by considering the office as identical in principles with the gaming house, and beat down the

* By real premiums I mean those which only cover the risks of life.

certainty of ruin which is thus known to exist, if they play upon equality of chance, by allowing for the first two of the three preceding considerations. There must still remain more risk than it is safe to face of insolvency, either temporary or permanent. And though, in consequence of the smallness of the portion which the office risks upon one hazard, a very small mathematical advantage might be sufficient, yet, so long as the necessity for such an advantage exists, and its absolute amount is unknown, so long must an office guard itself by requiring, in the first instance, a sensible addition to the real premiums.

With regard to the old lives, there is an additional ground of insecurity. Not only are the probable errors of the tables exceedingly large with respect to them, but, from the smallness of the number which will enter an office, there will be a liability to great fluctuations in the results of transactions with them. The first circumstance would prevent the second from becoming ruinous, but at a risk of loss to the capital invested by younger lives: it is usual, therefore, to exclude all lives above a certain age from entering the office, upon the principle that no risks are to be taken of which the numerical amount is not well understood, and of which the number is not large enough to secure an average. But, since the tables of old lives are only a very unsatisfactory approximation, and since the premiums payable by young lives depend in part on the chances of those lives becoming very old, how does it happen that the insecurity of the latter part of the tables does not affect the premiums throughout? It *does* affect them, but not sensibly, for the following reason. If, assuming the Northampton table, we suppose a person aged 40 to insure his life, we see that the portion of the present value of his insurance which depends upon his dying in his 85th year is very small, on two accounts: firstly, because the chance of his living to the age of 84 is very small; and, secondly, because the present value of a sum to be received

45 years hence is small, compared with that of a sum now due, or receivable soon. This last consideration works as follows:—When a percentage comes to be added to the whole present value or to the premium deducible from it, for the security of the office, that percentage being made upon a much larger sum than the present value just mentioned, a very trifling deduction from the whole additional sum will cover a very serious mistake in the mortality of the older years. For example, in the Northampton tables, the chance of 40 living to 85 is about $\frac{1}{20}$, and the present value of 1*l.* due in 45 years is about 5*s.* at 3 per cent. From this it follows that 100*l.*, to be paid if a person aged 40 dies after 85, cannot be worth so much as 1·25*l.* But the present value of the whole insurance is 53·8*l.*; and if this be the real value, and 10 per cent. be added for security, then 5·38*l.* is added; so that if 1·25*l.* were considered as added solely for the chances after 85 years, it follows that we might consider ourselves as having allowed for not being able to calculate the chances on old lives within one half, and as having added 8 per cent. to the whole present value besides. Thus, it appears that our comparatively little knowledge of old life, though not unimportant, yet can be made to be of less importance than might have been expected by one who has not considered the matter. Of course, the preceding reasoning must be considered only as addressed to a person to whom, for any thing he sees to the contrary, it is of as much consequence to know the *entire* law of mortality in the insurance of young lives as of old ones.

There is one use of the table of old lives, by which an insurance office might make its existence very problematical, to use a gentle term; namely, by inverting the order of security, and selling *guaranteed** benefits, which are to increase with the age of the party, and to be accumulated solely out of his premiums. To take

* This, of course, does not apply to divisions of profit *gained*, but to contracts for sums to be accumulated after the date of the engagement.

an extreme case, suppose an office should name a premium for which it would undertake to pay 100*l.*, if the party dies in the subsequent year; 200*l.* if he dies in the second subsequent year; 300*l.* if he dies in the third; and so on. In this case, every fluctuation which bears the appearance of lengthened life, were it only to amount to deferring one death for a single year, would be a new claim of 100*l.* upon the office. The fluctuations which are observable in the very old lives, would become matters of extreme importance; and though, assuming a given table fairly to represent the average, premiums might be calculated which should be sufficient in the long run, yet there is no possibility of saying what capital might become necessary to meet the fluctuations of half a century. Such an attempt as the preceding can be compared to nothing but gambling, and its stability to nothing but that of a ship running before the wind, with all the heavy cargo lashed to the topgallant mast. Other cases might be mentioned, which should partake of the same species of danger in a less degree; but every attempt to *guarantee* increased benefits with increasing life should be looked at with caution, as being of its own nature the addition of risks in which the errors of the unsafe part of the tables are, or may be, multiplied into importance. There is an opposite plan, which I am not aware has been tried, but which I should strongly recommend to any new insurance office, as being of a safe character, and also meeting the views under which many insure their lives. It is that of insuring *decreasing* sums, upon either fixed or decreasing premiums. Many persons are so situated, that they will be able to provide for their families if they live a few years. To provide for the hazardous period, they are under the necessity either of insuring for their whole lives, that is, of buying more insurance than they want; or of insuring for a fixed term of years, which does not meet several contingencies; or of making complicated survivorship insurances. But, if a person so circumstanced found, by estimation of his income, that he should want 5000*l.* if he died in one year, 4800*l.* if he died in the second, and so on, it would be desirable that he should be able to insure for these several sums, contingently upon his dying in any of the several years to which they are made to belong. Various modifications of this scheme might be proposed, all having this difference from the usual plans, in that the latter enable a person to make a provision for his family, while the former would only supply the deficiencies which his death would leave in the proceeds derived from other sources. In an appendix (on the value of increasing annuities) will be found the method of calculating the present values of such insurances.

2. The possible fluctuations of the rate of interest. These may be either general and national fluctuations, or alterations in the value of the property held by the office. The former cannot be guarded against or predicted; and, as the rate of interest has been slowly falling for centuries, there is some reason to suppose that this depreciation of money may continue. But this gradual sinking of the rate of interest may be only partly dependent upon the fall of profits, and part may be due to the increase of security. I question whether the political economist has found the historical materials for determining this most important element; namely, the extremes of interest at which loans were contracted in the different periods of our history. The legal maximum of interest, at the beginning of the reign of James I., was 10 per cent., and at the end of the century, 6 per cent. But, at the beginning of the century, land was commonly bought at 20 years' purchase, and never at less than 16 years' purchase; while at the end of the century it was still at 20 years' purchase. No method of proving such a point is better than the examination of the works on interest which appeared during the century. If, then, we suppose, with Adam Smith, and I believe with most others, that the changes in the legal maximum of interest followed, and did not

precede, those of the market, there is good ground for imagining that the diminution of the rate of interest between borrower and lender (from 10 to 4 per cent.) has arisen more from the increase of security than from any other cause. If such be the case, there is strong presumption that the fall is near its end. But, if the preceding surmise should not be well founded, and if (as was the case in Holland during a part of the last century) the rate of interest should fall until government can borrow at 2 per cent., and others at 3 per cent., the change may happen in a manner which will seriously affect the insurance offices, unless it should come about so gradually that they will have *time* to introduce new premiums for incomers, and *surplus* to meet the claims of those to whom they are already engaged. It is, in the meanwhile, a question well worth the attention of those connected with them, what the causes have been which have determined the rate of interest, and the rapidity and amount of its variations.

The offices depend for the existence of their present system upon the national debt; and they are differently situated from the government which owes the debt, in that the engagements of the latter are all maxima, while theirs are minima. If the rate of interest should really fall, the government will have the means of reducing the interest of the debt, never to rise again; while the offices have, in fact, guaranteed to their existing customers a rate per cent., which is never to fall during their lives. The rate assumed by the offices should, therefore, never be above that at which the government can borrow.

With respect to the second reason for a variation in the rate of interest, as experienced by the office, namely, a depreciation in its own property, such an establishment, not being allowed to run the usual risks of mercantile life, should not deal in any but the most secure investments, and those which depend on the personal security of others should be altogether avoided. The only point which it is incumbent to mention, in addition to general cautions, is a mistake to which such offices are subject in the valuation of their property; namely, the estimation of different items by their reputed worth, or by the price which was given for them, instead of the actual income which they produce. We shall see the effect of such a mistake in considering the proper method of inquiring into the state of their affairs.

The precedent are the contingent risks to which an office is subject: its certain expenses are the ordinary charges of management, including rent, salaries, interest of sums lying at the banker's, &c., advertisements, and the *commission,* as it is called, which most of the offices pay to those who bring them business.

Commission, in general, means either a per centage paid to a factor for the transaction of business, or a voluntary relinquishment in favour of the person who brings business of a part of the profit which the said person, being honourably free to choose between one competitor and another, has brought to the trader who, therefore, allows the commission. It answers to the profit which the retail dealer is allowed by the wholesale merchant from whom he buys. But, when an insurance office announces to the solicitor, attorney, or agent of a party desiring to insure, that they will allow him a liberal *commission,* the term has a different meaning. As between one office and another, the attorney is in a judicial capacity; and, as regards his client, *he is already the paid protector of the interests of another person.* He has, therefore, no liberty of choice between one office and another, but is already bound to choose that which he judges best for his client. All who have written on the subject of late years have attacked this *bribe,* for such it is; but they have directed all their censures against the offices, as if they were the only parties to blame. If, indeed, the bribe had been offered to the needy and ignorant only, this partial distribution of blame might have been allowed; but when the parties who receive the bribe are men of education, and moving in those professions which bring the successful to affluence, I do not see the justice of allowing them to escape. I have little doubt that an increasing sense of right and wrong will banish this unworthy practice, either by failure of givers or receivers. A barrister cannot offer an attorney commission on the briefs which he brings, nor can a physician pay an apothecary for his recommendation; a jury never receives a hint that the plaintiff will give commission on the damages which they award; and the time will come when the offer of money to a person whose unbiassed opinion is already the property of another, will be deemed to be what it really is, namely, *bribery and corruption*. It is one among many proofs how low is the standard of collective morality; and how easy it is for honourable individuals to do in concert that from which they would separately shrink.

It appears, then, from all which precedes, that the ordinary risks of an insurance office are alterations of, and mistakes in determining, the rate of mortality, and reduction of the rate of interest: which are guarded against by assuming a rate of mortality beyond all question greater than exists, and a rate of interest below that which the funds will yield. At the peace of 1815, every insurance office used the Northampton table at 3 per cent. This was at a time when the real rate of interest was higher than at present, and the offices must have made considerable profit. It was well known that they did so; and, accordingly, new offices were formed, and have continued to be formed up to the present time, some upon lower premiums than others, and most of them returning all or part of the profits to the insured. At the same time, an opinion has become very prevalent, that it is possible for such offices to maintain their ground at *much* lower rates of premium than those in use; a notion which I proceed to examine.

Mr. Finlaison, whose experience in such matters is well known to the public, and for whose opinion I entertain a high respect, stands foremost among those who contend for low rates of premium, having published a table, which he certifies to be "abundantly safe and practicable," and "so high as to insure, beyond all doubt, a surplus of profit;" which table charges premiums at the ages in which most insurances are made, falling short of those actually in use about 15 per cent. These premiums are supposed not chargeable with the management of the office, and at a rate of interest of $3\frac{1}{2}$ per cent. I take Mr. Finlaison's proposition as a modified one, for there are some which go beyond it.

On the other hand, the late Mr. Morgan could never be persuaded that it was safe to abandon the Northampton table; and considered that the superior vitality of the members of the Equitable was altogether a consequence of their being select lives. He seems to have thought that, whatever run of success an office might have, it should always be on the look out for reverses; and that even the enormous accumulations of his office were no more than the seven good harvests, a provision for other seven of a different character.

In holding an opinion which comes between that of these two authorities, I form it on a ground on which neither would have rested the truth or falsehood of his own. I consider the fluctuations of mortality as very little to be feared, compared with those of the rate of interest. It has long been matter of observation, that the phenomena of the natural state of man vary but little compared with those of his social condition. The price of provisions swings to and fro like a pendulum; the variations of mortality which follow its changes, though sensible, bear no proportion to the magnitude of their cause. The rate of interest has been halved within the memory of man, and a heavy war might double it again. That same war, with all its casualties, direct and indirect, included, would not alter the mortality of the country by any serious amount. I consider it, then, as next to certain, that the insurance offices have more to look for, whether as matter of hope or fear, from the fluctuations of the rate of interest, than from those of mortality. If the interest of money could be made as stable as the duration of human life, I could then see no objection to an immediate and considerable reduction of the premiums charged, to an amount at least equal to that proposed by Mr. Finlaison. But here lies the difficulty; that these tables, at $3\frac{1}{2}$ per cent., already involve a rate of interest which the office cannot much exceed, if at all; so that the security which the precautions, nominally made against mortality, really afforded against fluctuations of interest, is partially or wholly destroyed, while no safeguard is introduced to supply its place.

An office raises its premiums either because its previous notions of existing mortality were wrong, or because it finds that it had calculated upon too high a rate of interest. A mistake on either of these points might be compensated by a contrary mistake as to the other. Now, though the offices which existed during the war have demonstrated that the mortality and rate of interest together yielded a large profit, it by no means follows that one of those causes of profit may be fully corrected, while the other has been correcting itself. To make both perfectly accurate, would bring the office to the very line which divides security from insecurity; a position which it would not be safe to endeavour to maintain. We are already in a very different position as to the rate of interest, which has been gradually falling since the war. The opinion as to the extent to which tables of mortality may be safely corrected, is formed upon arguments which dwell on the favourable rate of mortality, without sufficiently considering the counterpoise (for, as far as it goes, it is a counterpoise) existing in the alteration of the value of money.

Assuming the necessity of calculating upon a rate of interest something less than that which can actually be attained, I should think that no office would be justified in supposing more than 3 per cent., *with tables which are sufficiently high to come any ways near to the actual experience of mortality.* With regard to one point, and that of fundamental importance, namely, the possibility of a still further fall in the rate of interest, it may even be doubted whether, *with such tables*, a still lower rate of interest should not be allowed. But I am not here advocating one result or another, but only the necessity of taking into consideration all the possible sources of danger. To those who would use tables of greater vitality, I concede that, so far as mortality alone is concerned, the alteration is admissible; and for this reason, that experience shows human life to be of a higher value than formerly; but the concession is accompanied by the requisition of a lower rate of interest, and for the selfsame reason, that experience shows the value of money to be less than it was.

The preceding conclusion is reinforced by the consideration, that the worst is to be made of every circumstance in our previous calculations. When mortality is diminishing, the whole diminution is not to be allowed; but when it is increasing, a larger increase is to be contemplated. A person who would walk dry-shod on the sea shore, must not advance so fast as the ebb, and must retreat faster than the flow. Upon this consideration, the necessity of providing for a further fall in the interest of money is increased; or, which amounts to the same thing, the amount by which the favourable alteration in the rate of mortality may be allowed to affect the premiums is less than it would be if it were certain that the value of money would remain unaltered.

A very common security or guarantee to the public is the announcement of a large subscribed capital, either paid up in whole or part, or liable to be called for. This is equivalent to the personal security of a number of shareholders, collectively making themselves answerable for the engagements of the office up to a certain amount. Such a provision in itself is an obvious good; but, it being remembered that this security must be paid for, it becomes a question how much it is worth, and whether it may not be bought at too high a price. It is easily understood that the consideration which tempts men to lend names or money to an insurance office, is the offer of payment for the risk, or of higher than market interest for the money. If the capital be paid up, the office makes common interest upon it, which is returned, with an augmentation, to the proprietors: if the capital be only paid in part, or merely nominal, still the office has to pay something more than it receives.

Now, I take it for granted that an office charging premiums* such as are commonly demanded, managed with prudence and economy, and successful in obtaining business, will not ultimately need any capital at all: firstly, because the premiums are such as must, in the long run, realise a profit after paying the expenses of management; so that the only use of the capital would be as a provision against extraordinary temporary fluctuation: secondly, because a sufficient supply of business renders the probability of ruinous fluctuation extremely small, and altogether beneath consideration.† Now, since it is well known that the premiums are sufficient, it follows that the only need which a commencing insurance has of capital is for safeguard against the early expenses of management, and against failure of business; as follows.

The expenses of carrying on an insurance office, though they vary somewhat with the amount of business, yet do not by any means increase as fast. In the first year of its existence, it would not be surprising if all the premiums paid were swallowed up by house-rent, salaries, &c.; while, in process of time, increase of business might reduce such expenditure to 2 per cent. upon the yearly premiums. Some capital, therefore, is necessary at the commencement; for, if there be none,

* If the premiums were really too low, capital would be an injury, and not a benefit; for, since this capital is really paid for, in whole or in part, out of premiums, it would not preserve the office from insolvency, but would rather accelerate its progress towards bankruptcy.

† The most probable cause of ruin to the insurance offices, or rather the least improbable, is a national bankruptcy. Any contingency, then, which is much less likely than a national bankruptcy, need not be considered.

those who first insure their lives are entirely dependent upon the future success of the office. But this capital need not be large: in the present state of things, an engaged capital of one hundred thousand pounds is certainly above the mark, even for an office which is entirely without connection, and starts without one single life insured. If, as very often happens, a tolerably large number of customers has been obtained before the prospectus of the office is announced, then a capital, the interest of which will cover the expenses of management, is sufficient. But here it must be observed that the proprietors of this capital run some risk of losing a portion of their principal, and a still greater one of losing the interest for a limited time. This risk is the greater the smaller the original subscription, and it must be paid for accordingly. At the same time, it must be remembered that the mere existence of the capital diminishes the risk, by making it the interest of every proprietor to procure business for the office. The connection thus created is the secret of the successful start which has frequently been made; and it may be considered as very unlikely that an office will fail, from want of business, which is so well supported in the first instance as is implied when a capital of the preceding amount is announced.

There is, however, one case in which a larger capital is desirable, and even requisite; that is, where an office is established which is to insure some new and yet untried risk. Whatever pains may be taken in such a case to procure facts and deduce proper tables, there is always a risk that the experience of the office may be at variance with the facts of the tables. When, for instance, the general conclusions drawn from the mortality of towns were first applied to the insurance of life, it was a risk of unknown amount as to whether the lives of those who would come to insure would be of the same class as those from which the tables were made. They might turn out better, or worse. This risk has been tried, and found to be in favour of the offices; but in another speculation, of another kind, the same species of risk might give a contrary result.

Among the sources from which the insurance offices have drawn profit, we must reckon lapsed policies. It has frequently happened that an individual insuring his life has continued to pay the premiums for a few years, and then, either through incapacity to continue the payment, or because the object of his insurance was otherwise attained, has allowed his policy to lapse to the office by non-payment. The office, of course, is benefited, but not, as might be supposed, by the total amount of his premiums. What they have received does not all become profit by the lapse of the policy, but only that portion by which the premium for the whole life exceeds the premium for a temporary insurance. Every premium which is paid by an insurer contains the consideration given for the chance of his dying in each and every subsequent year. If, then, he remain a member of the office, and stand the risk of death during a certain number of years, all such part of his premiums as was consideration for the risks of those years became due to the office, and was taken by the office, as compensation for those risks, and cannot therefore be said to fall to them as profit upon the lapse of the policy. Two individuals, A and B, go to the office on the same day, and insure their lives for the same sum, A upon his whole life, and B for seven years. A pays, say 10*l.* of premium, and B 7*l.* At the end of seven years, A allows his policy to lapse, just at the time when B's policy expires by its own construction. What does the office gain by the lapse? Evidently the temporary annuity of 3*l.*, by which the two premiums differ. The 7*l.* paid by A out of 10*l.* is not more than sufficient to pay his share of the claims which arose during the years which he continued in the office: the remaining 3*l.* was a reserve for future years, which becomes profit to the office on his declining to stand the risks of those years.

Perhaps no part of the subject is less understood than this. Persons having insured for their whole lives, and being afterwards desirous to discontinue, are surprised to find that they cannot get for their policies even as much as the amount of their premiums, to say nothing of interest. Each of them reasons thus: — Since I did not die, the office lost nothing by me, and, as it has turned out, ran no risk: why, then, should they not restore me the premiums which I have paid? To which it should be answered: Because the risk, which turned out favourably in your case, did not produce the same result in another case; and it is the very essence of an insurance office, that those who live pay for those who die. If you can induce the executors of those who have died during your tenure of your policy to refund what they have received from the office, with compound interest, when the office will repay you your premiums, also, with compound interest. The above-mentioned reasoning of the insured party is much on a par with that of the judge in Godsal's case.

A respectable weekly newspaper has lately allowed the following doctrine to be promulgated in its columns; namely, that it is an undeniable fact, demonstrable by the books of an insurance office, that very much the larger portion of their profits has always arisen from lapsed policies! Till I saw that article, I could hardly have believed that even a newspaper would have admitted so palpable a mistake. On the supposition (no matter how false) that all the back premiums of a lapsed policy are, as they say in book-keeping, to be carried to profit and loss, how could such an assertion be made, in the face of the well-known fact, that premiums are deduced from a table of much higher mortality than that actually experienced? Those persons, who, one with another, were expected to live twenty years, have lived twenty-four years. A small proportion of them have allowed their policies to lapse, enough to give, perhaps, a perceptible profit to the office, but not enough materially to increase its funds; for it must be remembered that, though the number of policies allowed to lapse bears a proportion to the whole which might give some colour to the preceding assertion, yet the value of these policies is generally small. It is seldom that a policy is abandoned which involves a large sum, or on which many premiums have been paid. If, instead of comparing the number abandoned with the whole number of policies, we were to calculate the value of those policies, and compare them with the value of all the liabilities of the office, the former would be found a very small portion of the latter. It is well that it has been so, for this source of profit is diminishing as the subject becomes better understood. It is known that a policy of a very few years' standing is worth *something*, and had better be sold at any price than abandoned.

All that precedes has reference to the relation in which the office stands to the public, and to the collective body of the insured. All dangers, and all remedies, have been considered merely with reference to the general security of the establishment, and without inquiring into the effect produced on the relative interests of the insured. Since it is the first principle that no interest of one or the other class of insurers must be consulted to the detriment of the whole, the order of discussion which I have followed is necessary to the subject. It now remains to treat of the internal management of an office, and to this subject I proceed in the next chapter.

CHAPTER XII.

ON THE ADJUSTMENT OF THE INTERESTS OF THE DIFFERENT MEMBERS IN AN INSURANCE OFFICE.

There is not a circumstance against which it is necessary to guard in the general management of an office, but what is accompanied by this inconvenience, that the measures adopted, whether of precaution or remedy, may be made to press unequally upon the different classes of insurers. If we take, for instance, a fixed rate of interest, sufficiently below that which can really be obtained, we find that many of those insured must pay their premiums at a time when interest is comparatively higher, and *vice versâ*. With regard to the tables of mortality, most probably (it has always so happened) a table which is generally too high will be unequally too high; so that some classes of insurers will contribute more largely to the safety fund than others. And even in the distribution of the profits, however good the will may be to apportion them duly, there are yet practical difficulties in selecting an equitable method out of those which do not require calculations of insupportable minuteness.

It will only here be necessary to dwell upon two points, the distribution of the premiums, and the method of appropriating the profits.

In the last chapter, in speaking of the use of too high a table of mortality, as a safeguard, I was merely considering the collective security of the office. There are two different ways of answering the same end: either by using a table of mortality confessedly too high, or constructing premiums from a true table of mortality, and increasing these by such a percentage as will produce the same receipts to the office. For general security, these two plans are equally good; but they may produce very different consequences upon the relative state of the members. For instance, the Northampton table, which is the basis of most of those now in use, is certainly, as already noticed, too favourable to the older lives. Mr. Morgan gives the following table*, exhibiting the number who did die, and those who should have died, if the Northampton table had been correct, all in the twelve years preceding 1828.

* View of the Rise and Progress of the Equitable Society, London, 1828, page 42.

Age.	Number.	Of whom did die.	Of whom should have died.
20 — 30	4,720	29	68
30 — 40	15,951	106	243
40 — 50	27,072	201	506
50 — 60	23,307	339	545
60 — 70	14,705	426	502
70 — 80	5,056	289	290
80 — 95	701	99	94

From this comparison, Mr. Morgan concluded that the superior vitality of the young and middle ages was the effect of selection, which wore out, so to speak, after the age at which no new members were admitted; thereby proving, in his opinion, at once the effect of selection, and the excellence of the Northampton table. Now, it obviously cannot prove both of these things: granting the latter, it would certainly go a great way to prove the former; and granting the former, it does not impugn the latter: which is all that can be said. But, if it should happen that the mortality of the Northampton table is near the truth at the older ages, and very much above it at the younger, the sort of result shown in the preceding comparison would follow of course; and this circumstance, demonstrated as it is by other and independent tables, is, no doubt, the true explanation.

If such be the case, where is the fairness of using a table which demands premiums very much larger than the real risks from the young, while it admits older lives on more easy terms? Ought the older lives to enjoy any privilege in this respect? Quite the reverse; for, (page 253.) belonging to a class which is less known, and entering also in smaller numbers, with results therefore more subject to fluctuation, the percentage, added to the premiums deduced from a true table, ought rather to be larger in the case of old lives than in that of young ones. The best customers, both in number and quality, ought not to come worst off.

The proposed table of Mr. Finlaison (page 259) affords a striking illustration of this point. It is accompanied by a table representing the average premiums of all the offices. At the age of thirty, Mr. Finlaison proposes to demand 17 per cent. less than the average of what is now asked by the offices; at the age of 60, this same able and strenuous advocate of reduction would only reduce the average premium of the offices by $3\frac{1}{2}$ per cent. I now put down the present value of 100*l*., payable at the end of the year in which a life drops, from the Northampton and Carlisle tables, at 3 per cent., and for different ages, together with the percentage which must be taken from the former to reduce it to the latter.

Age.	Northampton.	Carlisle.	Percentage of difference.
20	£ 42·8	£ 33·9	20·8
30	47·8	40·0	16·1
40	53·8	47·1	12·5
45	57·2	50·8	11·2
50	60·9	55·4	9·0
55	64·6	60·9	5·7
60	68·6	66·5	3·1
65	72·9	71·1	2·5

In offices, then, which continue to use the Northampton table throughout, the *safety rate* is levied upon those who enter at the age of 20, to the amount of 21 per cent. out of the total sum they pay; while on those aged 65 it only amounts to $2\frac{1}{2}$ per cent. The Carlisle table represents the experience of the Equitable Society very nearly.

Again, the Amicable Society now charges premiums deduced from its own experience, and in which the fundamental inequality of the Northampton table is corrected. It will be worth while to compare the average of all the offices given by Mr. Finlaison, with the present premiums charged by the Amicable. The supposition is for 100*l.* insured.

Age.	Average.	Amicable.	Mr. F.'s proposed Premiums.
20	£2·02	£2·03	£1·76
30	2·50	2·53	2·07
40	3·26	3·25	2·78
50	4·47	4·83	4·06
55	5·38	5·90	5·00
60	6·58	7·33	6·25

From such comparisons as the preceding, I have long been of opinion that, safe as the offices are, each considered as a whole, the proportions of the premiums demanded at different ages are, in the first instance, inequitable. To a certain extent, the young are made to work for the old; that is to say, the person who insures early in life, the more prudent of the two, is made to pay a part of the premium of the one who does not begin till he is old.

The evil is not so great as it might at first sight appear, for two reasons: firstly, because those who enter at the older ages are few in number compared with those who begin between 30 and 50 years of age; secondly, because many offices make compensation to the younger members in the division of the profits. Still, however, the inequality is of a sufficient magnitude to demand alteration, which will be brought about in an obvious way; namely, by the younger insurers giving the preference to those offices in which, premiums and returns considered together, the inequality is the least.

There is another point, though not of so much consequence, in which an inequality falls more heavily upon the young than upon the old; namely, the method of paying the expenses of management. The yearly contribution of every member to this fund ought to be the same. Suppose, then, that from every premium a given sum is subtracted, to answer this end, the inequality of the remainders is increased; it being obvious that any disproportion which exists between two numbers is made larger by taking away the same from both.

The way to correct the inequality, without altering the actual receipts of the office, is as follows. The proportions in which the different ages exist in the office at any one time can be pretty nearly found. Let the office table of premiums be taken, and from it let an average premium be formed, by taking into account as well the several premiums, as the numbers who pay them. Suppose, for instance, that A persons pay the premium *a*, B pay *b*, &c. &c.; then the average premium is found by dividing the sum of the products of A and *a*, B and *b*, &c., by the sum of A, B, &c. Let the actual average premium be called P; and let the average premium, formed in the same manner from a true table of mortality (in which *a*, *b*, &c. are different, but A, B, &c. the same as before), be Q. Let P exceed Q by *k* per cent. of Q; then the premiums given by the true table, increased by *k* per cent., are those which should be substituted for the existing premiums, in order that all inequalities may be corrected, without diminishing the receipts of the office. It matters nothing, in the preceding rule, whether the premiums of what has been called the true table are correct or not, so long as their proportions are correct; and one office might, by this rule, adopt the proportions of another, without altering its own receipts.

If such a process as the preceding were performed, deducting from the receipts required by the office the whole expense of management, and afterwards adding the last-mentioned item in equal shares to all the *policies*, the distribution of the premiums would be theoretically perfect. It remains to consider the more difficult part of the question,—the method of dividing the profits.

Hitherto, I have had no occasion to speak of a most important difference of system which distinguishes one office from another; the distinction of *mutual* and *proprietary*. The former have no capital, except what arises from their own accumulations, and each member is a guarantee to the rest for the fulfilment of all engagements. If the office possess a charter, this guarantee operates no further than to pledge the premiums already paid by any member for the discharge of all claims which arise before his own, since a corporation is considered in law as an individual. If, on the other hand, there be no charter, the whole fortune of every member is pledged for the discharge of all claims. The risk, however, at the commencement is not great in character, and small in amount; and the quantity of risk diminishes so much faster than the amount increases, that it may safely be said there is nothing in the commercial world which approaches, even remotely, to the security of a well established and prudently managed insurance office.

A proprietary insurance office has a capital, the proprietors of which may or may not be insured in the office, and for which such a bonus is paid, in addition to the market rate of interest, as is mentioned in p. 263. It would perhaps be difficult, at the present time, to establish a new proprietary office with a very large capital. The public now begins to see that much capital is not necessary, and that nearly all the bonus which is paid for its use is so much taken away from the savings of the insured, without any adequate benefit received in return. One by one, the proprietary offices must (as some have done) admit the insured to a share in the profits: the necessity for which will be taught by the decline of business, if not previously learnt.

The question as to how profits should be divided, is of the same nature in both species of offices; the difference being, that the offices which are partly proprietary have less to distribute among the insured than those which are mutual. The first inquiry must be, What is the profit of an insurance office; and how is the amount to be ascertained? Firstly, as to the profit which an insurance office may be expected to realise, judging by the premiums they receive, and the mortality they have hitherto experienced. Certain limits may be obtained which may sometimes serve as a useful check.

Perhaps the average age of admission to an insurance office is about 40, as many entering younger as older. The average premium charged by the offices at that age is 3·26*l.* per cent. Now the most extreme supposition which can be imagined in favour of the insured is, that the Carlisle table should be taken as the law of mortality, and 4 per cent. as the interest of money. Upon these suppositions, the accumulations of the office would amount, upon a premium of 3·26*l.*, at the death of parties aged 40, one with another, to 137*l.* But this pushes every favourable supposition to its extreme, and moreover allows nothing for expenses of management. I am inclined to think, however, that the usual premiums will, as long as the rate of mortality continues at its present amount, yield about 125*l.* for 100*l.* nominally insured, and perhaps something more.

It must not be left out of sight, that the offices consider every person as having the age which he will attain at his *next* birth-day. If, for instance, a person who attains 40 years of age on the 12th of March were to insure his life on the 13th, he would be said to be 41 years of age, and would have to pay accordingly. The effect of this very proper* regulation is, that, one party with another, all are half a year younger than their *office age*. Again, all the tables are computed on the supposition that interest is made yearly, whereas in fact it is made quarterly. Circumstances of this sort, trivial as they appear, do nevertheless produce a sensible effect in a large number of years. To the above we must add the profits arising from the purchase of policies, which is always done by the offices on terms very favourable to themselves; fines for non-payment of premiums; the profits of lapsed policies; and so on.

Leaving all speculation as to the probable profits, I now proceed to show how to ascertain, from the

* Proper as long as there is no subdivision of a year. I think the offices might very rationally divide the year into quarters.

actual statistics of an office, what its real condition is. And here I must observe, that though in the construction of premiums, a table of more than the real mortality must be used, yet no such thing is absolutely necessary in the valuation of its liabilities and assets. Here truth, and not security, is the object; and if by any means a true table can be obtained, its results should be calculated; though I do not say that in the declaration of profit, such results should be admitted to their full extent. The most simple theoretical way of conducting the process, is to ascertain the value of every policy, as in page 218.; that is, to ascertain how much should be given to the holder of each policy to renounce his claim, the office also abandoning the future premiums. When this is done, it is obvious that the office is not solvent, unless the assets arising from the accumulations of former years be sufficient to pay the values of all the policies, and thus to buy them all up. Supposing the office able to do this, with a capital remaining larger than would be necessary to create a permanent fund for the expenses of management, the surplus of that capital is profit. Otherwise, calculate the present value of all premiums due to the office, and also the present value of all claims to which it is liable. To the former add the sum total of the assets of the office, and to the latter add the present value of a perpetuity equal to the expenses of management. Thus, let

P = present value of all premiums.
C = present value of all claims.
A = total assets of the office.
M = present value of all expenses of management.

If then P and A together exceed C and M together, the office is solvent, and the excess is profit.

On each of these items a few remarks may be made.

(P.) All the parties who are of the same office age, may have their several policies considered as one collective policy, in respect of which the sum of the premiums is paid as one premium, and the sum of the possible claims is one claim. But as these premiums are payable at all periods of the year, they may be considered as, one with another, due at six months after the valuation, at which time the present office age of the parties may be considered to be their real age.

(C.) All bonuses which have actually been added to policies (if any) must be included in the claims; and the value of each claim must be carefully found, with reference to the time after death at which it is paid. (See Appendix the Second.)

(A.) The principal of the assets must be deduced entirely by means of the income it yields, and must be ascertained from the income by means of the rate of interest assumed. On this subject, which contains a difficulty of a peculiar character, see the Sixth Appendix.

(M.) Against the expenses of management may be set, as far as they go, the incidental profits, when they can be tolerably well ascertained.

The profit being thus found, and that share of it which belongs to the insured (if the office be not mutual), it remains to inquire, What principle of division should be adopted? And, firstly, it may be doubted whether the whole of the profit is immediately divisible, consistently with prudence. To use an astronomical phrase, the increase of the surplus is partly secular, and partly periodic; that is to say, instead of a steady and uniform increase, there is a fluctuating rate of augmentation, compounded of that permanent rate which the largeness of the premiums necessarily gives, and the alternate accelerations and retardations occasioned by the departures of the incidents of the several years from the average. The only way of obtaining the permanent part of the surplus is by estimating it on the average of a considerable number of past years, regard being had to the relative, not the absolute, surplus. Let us suppose, for instance, that the present value of all claims is ascertained to be one million, and the present value of all premiums 700,000*l.*, the office possessing besides (clear of charges of management) 500,000*l.*:

there is then a surplus of 200,000*l.*; which having been accumulated out of premiums, and profits having been regularly paid up to the present time, it may be presumed that the premiums themselves are capable of maintaining this rate of surplus. The office must then be presumed able to pay 120*l.* for every 100*l.* insured.

But it is important to note, that the present rate of profit must not always be assumed as that which can be permanently maintained. Suppose, for instance, an office which begins for the first time to divide profits: its accumulations are therefore, in part, the reserves of profit which should have been added to former claims, had any division of surplus previously existed. The same remark may be necessary when any change is made in the way of dividing profits, since the surplus existing at the moment of the change is the result of a former state of things. Thus, an office which has proceeded injudiciously, in making too large divisions, may possibly, when it adopts a more prudent system, be justified in forming a system which would require a larger surplus than the one which it actually possesses at the time of discovering the error; for the then existing surplus has been unduly weakened, and is not to be considered as representing the permanent effect of the improved mode of proceeding.

I have stated, that the percentage which can be added to each 100*l.* insured should be determined by the average of a number of years. If this number be too great, the incidental fluctuations of mortality may be compensated; but at the same time the real and secular changes of mortality may be prevented from producing their proper effect. As long as the value of life is increasing, too long an average is a defect on the safe side: but if it were diminishing, it might happen that the mean of a number of preceding years would present a higher result than would be consistent with security. As yet the offices have had nothing to encounter except the diminution of mortality, and its consequences; but in constructing rules for their own guidance, they should be careful not to fall into such errors on the safe side as become errors on the wrong side when circumstances change. I hold an opinion which I think, from his writings, was also that of the late Mr. Morgan; namely, that an insurance office must consider the last half century as having been a period of circumstances singularly favourable for the formation and growth of such institutions, more so than it would be wise to expect for the future. Perhaps from five to ten years is the length of time for which the preceding average should be computed.

The valuations should, if possible, be made *yearly*. No check which can be devised is so likely to be useful as yearly valuation; and it is absolutely necessary to any system which gives the real amount of their premiums to the insured. In a mutual insurance office, starting without much capital, it would be madness to rest upon any tables and to neglect valuations; unless, as before remarked, the returns made to the insured are meant to be very much below their payments. And in conjunction with yearly valuations should come yearly divisions of profits, or something equivalent. There is, I believe, a prejudice against frequent divisions in the minds of many who have derived their ideas on the subject from the former practice of offices. But surely, provided that the proper amount of profit be divided yearly, and no more, it matters nothing whether the apportionment be made seven times in seven years, or once only, as far as security is concerned. For it is to be remembered, that yearly division of profits does not imply an annual expenditure, but only an annual distribution of future expenditure. In septennial divisions, one of two things always takes place: either the profits are made contingent upon a party surviving one or more periods of division, which creates great inequalities between the lot of different persons (the very thing an insurance office was intended to avoid); or it declares beforehand, what the profits shall be during periods of seven years. In the latter case the annual division is unquestionably the more safe; since it is easier to predict the capabilities of one year than of seven.

In writing upon any point connected with insurance, the practice of the Equitable Society naturally suggests itself. Nevertheless, I always consider that society as a distinct and anomalous establishment, existing at this moment under circumstances of an unique character. It is the result of an experiment which it was most important to try; but which having been tried, need not be repeated. Its history is briefly this:—The Amicable Society, which, in the year 1760, was the only one existing, was originally founded rather on principles of mutual benevolence, than of mutual insurance, as now understood. A certain number of persons (the only restriction being that their ages should be between twelve and forty-five), each paying the same sum yearly, the whole fund of each year (or the greater part) was divided among the representatives of those who died within the year.* The Equitable Society was founded upon the principle of apportioning the payments to the risk of life. The tables were constructed by Dodson, who, as Mr. Morgan remarks, "for greater security assumed the probabilities of life in London, during a period of twenty years; which, including the year 1740, when the mortality was almost equal to that of a plague, rendered such premiums much higher than they ought to have been, even according to the ordinary probabilities of life in London itself." The truth of this remark will sufficiently appear, from comparing the average of the present office premiums with the original Equitable premiums, as given in the following table. And even these premiums were increased on the most frivolous pretexts. Thus *female* life and *young* life were considered as more than usually hazardous, and paid for accordingly.

* The Amicable Society now retains only one of its original characters; namely, that all members, whatever may be their age at death, or the term of their continuance in the society, participate equally in the profits.

Age.	Equitable Premium, 1771.	Equitable Premium, 1779.	Average present Premium.
	£ s. d.	£ s. d.	£ s. d.
14	2 17 0	2 5 5	.. ..
20	3 9 4	2 12 10	2 0 0
25	3 14 0	3 0 6	2 4 0
30	3 18 7	3 8 11	2 10 0
40	4 17 9	4 7 11	3 5 0
49	6 2 5	5 10 2	4 6 0

Mr. Morgan says, "that for the first twenty years, the society possessed such an excess of income, that being suffered to accumulate without interruption, it contributed, in a great measure, to form the basis of its future opulence." This circumstance, with the great number of policies which were abandoned* in the early stages of its career, and the increase of interest during the war, are quite sufficient to explain the wealth which the Equitable Society has accumulated: to these must be added the parsimony with which, at first, additions were made to the policies. The whole was an experiment, on a graduated scale of premiums, made with a caution, which, though it turned out to be superfluous, could not be known to be such, except by the result. It was at the same time a venture, and by many considered as a hazardous one; for instance, the law officers of the Crown refused a charter, on account of the lowness of the premiums. The hazard having been run, and having turned out profitably, the proceeds belong to those who ran it, and to those who, by their own free consent, became their lineal successors. Nor is it the least remarkable circumstance connected with this society, that the immense funds at its disposal have been always opened, though under restrictions, to the public. Though this has been done in a way which renders the participation of the new insurer in the

* Perhaps Mr. Morgan's statement on this point may have led to the statement alluded to in page 266.

previous accumulations a remote contingency, still it *is* done, and by a body who might without any bar, legal or moral, immediately close their doors, and divide the whole among themselves.

I have made the preceding remarks, in order that it may be clear how little the history or practice of the Equitable Society should have any direct authoritative bearing on the spirit in which the management of a more modern office should be carried on. The general lesson taught by it is,— be cautious ; but, among other things, be cautious of carrying caution so far as to leave a part of your own property for the benefit of those who are in no way related to you. If there be a Charybdis in an insurance office, there is also a Scylla : the mutual insurer, who is too much afraid of dispensing the profits to those who die *before* him, will have to leave his own share for those who die *after* him. Reversing the fable of Spenser, we should write upon the door of every mutual office but one, *be wary* ; but upon that one should be written, *be not too wary,* and over it, "Equitable Society."

An insurance office has no existence separate from that of its insurers ; and no public duty to fulfil, except to collect, improve, and *equalize* their premiums (p. 238.) : therefore, their most important object, next to the fulfilment of their guaranteed engagements, is the distribution of their profits in such manner that every one may obtain his due share. The question now becomes, What is the due share of each party ? This is, in some measure, a question of previous contract, though there are those who consider that there must be a right and a wrong way. For instance, Mr. M'Kean, the compiler of the tables alluded to in page 191., and of a useful work * which accompanies them, says, "Our conclusion, and a most important one, lies conspicuous on the very surface. It is impossible that ALL the

* "Exposition of the practical Life Tables, &c. London : Butterworth, Richardson, &c. 1837." This work is, I believe, sold separately.

offices above mentioned can be correct or just in their aws for dividing the surplus. If the plan of the *Equitable* is right, then most unquestionably the plan of the *Atlas* is wrong, and great injustice is done to the younger members, and so *vice versâ*. But, is this a state of things in which so important a system as that of life insurance, based, as that system is, on mathematical science, ought or can continue to exist? Certainly not."

On this I observe, that though life insurance be an application of (not based upon) mathematical science, yet that the entrance of exact numerical reasoning is subsequent to the admission of certain principles, and the experimental acquisition of certain facts. It is not by mathematics we learn that life is uncertain in individual cases, but nearly certain in the mass — that it is the duty of every one to provide for his family — and that this can be done without contingency, if those who survive the average term agree to surrender a part of their substance to those who do not. Calculation will point out the amount which, upon any given principle of division, belongs to one or another of the insured; but before we can come to this point, it must be settled with what intention the surplus was paid; which may be different in different offices. The following considerations might be addressed to any person who intends to insure his life: — You are aware that the premium demanded of you is, avowedly, more than has hitherto been found sufficient for the purpose, the reason being, that it is impossible to settle the exact amount, on account of our not knowing whether the future and the past will coincide in giving the same law of mortality, and the same interest of money. The surplus arising from this overcharge, for the future existence of which it is hundreds to one, is now at your own disposal, and you must choose between one office and another, according to your intentions with regard to its ultimate destination. Firstly, if you doubt the general security of the plan of insurance, and are desirous of an absolute guarantee, independently of accumulations from premiums, there are offices which will, in consideration of the surplus aforesaid, pledge their proprietary capitals for the satisfaction of your ultimate demand upon them. Secondly, if, being of the opinion aforesaid, you think the whole surplus too much to pay for the guarantee, there are proprietary offices which retain a part of the profit in consideration of the risk of their capital, and return the remainder. Thirdly, if you wish the surplus premium, as fast as it is proved to be such, to be applied in obviating the necessity of any further overcharges, there are offices which divide the profits during the life of the insured, by means of a reduction of premium. Fourthly, if you wish the surplus to accumulate, and, feeling confidence in your own life, are willing to risk losing it (the *surplus*, remember) entirely if you die young, on condition of having it proportionally increased if you live to be old, there are offices which divide all or most of the profits among old members. Fifthly, if you would prefer a certainty of profit, die when you may, there are offices which at once admit new members who die early to a full participation in all advantages. The choice between these several modes must be made by yourself, according to your own inclinations, views of fairness, or particular circumstances.

There are three modes of division which deserve particular notice; namely, periodical additions to the policies, periodical diminutions of premium, and addition to the policy at death to an amount depending upon the assets of the office, without reference to the time during which the insured has paid premiums. I may, perhaps, be thought to treat this subject with prolixity; notwithstanding, knowing that this part of the subject has created more discussion of late years than any other, I think an attempt to compare the principles of different plans not out of place.

The considerations which follow will apply to all offices which divide any profits whatever: the inquiry being, not how much surplus should be divided, but in what proportions a given sum should be divided among the insured.

Let us return to the original constitution of an insurance office (page 238.), derived from the statement of its main object; namely, that it is a savings' bank with a power of equalizing those results in which the different durations of life would cause differences. Suppose that such an office sets out with premiums imagined to be no more than sufficient, but which are afterwards found to be more than sufficient, leaving an admitted amount of surplus in hand. The first thought would be of *restitution*; namely, rendering back to each individual the amount which he had *bonâ fide* contributed towards the surplus. To do this properly, it must first be settled whether the insurance office is one or many. Does each age insure itself, or do the separate ages insure both themselves and each other? If the premiums were properly proportioned, there would be no occasion to ask this question: but if the incomers of one age pay unduly as compared with those of another, then it is but fair that they should receive in proportion. In the distribution of premiums, which I have described in p. 270., it is equitable that a remedy should be provided, by virtue of which those who enter the office young should receive more than the rest. And it is, for this reason, desirable that the proportions of the division should be regulated by a true table of mortality.

Let P be the real premium, and $P+p$ the office premium; and let the death of an individual take place after he has been n years insured, and just before the $(n+1)$th premium is paid. If the office had been a compound interest savings' bank, the deceased would, at his death, have been entitled to the following amount.

$P+p$ improved at compound interest for n years
$P+p$ . . . . $n-1$
. . . .
. . . .
$P+p$ . . . . 1 year

But under the conditions of insurance, the part P, with its accumulations, is the consideration for the sum insured ; the remaining part p, with its accumulations, is due under the name of profit or restitution, in a strictly mutual office.

The application of the preceding method would require that a calculation should be made once in every year of the quantity p and its accumulations, for every individual insured. This having been done, and the surplus $A+P-C$ having been calculated from a true table of mortality, it is then known in what proportion any two individuals insured are claimants upon this fund. Suppose that p and its accumulations amount, in the case of the persons X and Y, to 100*l.* and 150*l.* Suppose that $A+P-C$ is 100,000*l.*, and that the sum of all the excesses of premium with their accumulations, of which the 100*l.* and 150*l.* just mentioned are items, is 120,000*l.* It matters nothing that the last sum is greater than 100,000*l.*, since we are not speaking of a fund on which there are definite claims, but of one the nature of which it is to be of uncertain amount. The use of the items 100*l.* and 150*l.*, and of the sum total of 120,000*l.*, is to enable us to divide the real fund of 100,000*l.* among those who raised it, in the proportions in which they contributed towards it. Thus if X and Y were to die in the year of the valuation, it would be fair that they should receive such proportions of the 100,000*l.* as 100*l.* and 150*l.* are of 120,000*l.*; that is, five-sixths of 100*l.* and 150*l.* This method proceeds upon the principle that all the excess of premium is taken in trust as a guarantee for the main fund, and is to be returned if not wanted, or such proportion of it as is not wanted. It confines the insurance, or provision against the uncertainty of life, entirely to a stipulated sum, and regards all that part of the premium which is not really wanted to provide this sum, for one man with another, as paid into a common savings' bank, in which no equalization is supposed.

The labour of making the calculations would, I imagine, prevent any office from adopting the preceding plan, so as to carry it into execution yearly. With a good system, however, the difficulty of managing the details of such a scheme would not be so great as at first sight might be supposed. Upon its principle hang the two first plans of division mentioned; namely, periodical additions to the policies, and periodical diminutions of premuim. In both of these, the advantage of the insured is increased by the length of his life; that is to say, the excesses of his premiums are placed to his credit in the first, and considered as having been prospective payments of his future premiums in the second. But nevertheless there runs through the offices which adopt these plans more or less of a practice which prevents the surplus from being divided among the insured in equitable proportions. Suppose that there is a septennial *bonus*, as it is called, which was declared in the year 1830. Immediately after the award, two persons, A and B, aged 30 and 60, enter the office each upon a policy of 100*l.*, and were both alive when the bonus of 1837 was declared. This bonus is generally a percentage, not upon the amount of premiums paid, but upon the sum insured, and both would have the same addition made to the 100*l.* for which they have insured. But have both contributed to the accumulations of the office in the proportion which would render this mode of division equitable? To consider this point, remember that a promise to pay, say 5*l.*, at the death of a person aged 67, is of much more value than the same at the death of a person aged 37. The older life therefore receives much more than the younger life. But he has paid much more. That is true; but at the same time he has occasioned a greater risk to the office, and it is the excess of his premium above the risk (and not the whole premium) which the office acknowledges in declaring the bonus. From page 270. it sufficiently appears that the premiums of the older ages are already too small in comparison with those of the younger: this mode of dividing the surplus, therefore, only tends to increase the existing injustice. The only remedy is, to make use of the process laid down in the preceding page; and having ascertained the amount of what each person has paid over and above what was necessary, to consider each person as entitled to the sum which his overplus would purchase at his death, if the bonus be made by addition to his policy; or to a diminution of premium answering to the annuity on his life, which the overplus would buy, if the bonus be made by diminution of premium.

The knowledge, therefore, of the *real* premium is necessary for an equitable distribution of the surplus, upon the supposition that the said distribution is made on the principle of dividing the surplus fund among the contributors in proportion to their contributions. Every plan which, ceteris paribus, makes equal additions to the policies of different ages, is inequitable. I repeat again, that in the preceding cases, the principle of division ought to be considered as arising from the combination of an insurance office and a savings' bank; the portion of premium which covers the risk of life being paid to the former, and the remainder to the latter.

The third method of division supposes the establishment to be entirely an insurance office, and not at all a savings' bank. Its object is to make the returns to the different members both equal and equitable. Considering that the real risk of life is not perfectly ascertained, and that if it were it would not be safe to reduce the premiums to the lowest theoretical safety-point, such an office, instead of demanding a premium avowedly too high for the sum insured, and engaging to return all or part of the surplus, considers the sum insured as indefinite, except only in so far as a minimum is named, below which it is not to fall. Thus such an office, receiving, say *3l.* of premium, from a person aged 34, for what is called, in compliance with custom, a policy of 100*l.*, does in fact make the following bargain:—The office engages to return, at the death of the party, let that take place when it may, such a sum as will represent the average accumulation of an annuity of *3l.* continued during the life of a person aged 34, be that sum more or less; with this additional limitation, that the office undertakes that the said accumulation shall not be less than 100*l.* This last guarantee, though necessary for the satisfaction of the public, is in truth so certain, from the amount of the premium demanded, that a person acquainted with the subject looks upon the possibility of the funds of the society suffering from it as an extremely remote chance.

In order, however, to make the proceedings of such an office equitable, the proportions of the premiums paid by parties of different ages must be fairly regulated. On the supposition that the inequality pointed out in page 270. is allowed to exist, the preceding methods of division may be (I do not say are) adjusted so that every interest shall be consulted. But in the present plan, it is impracticable to remedy any such defect of proportion, at least without dividing the establishment into as many different offices as there are ages, which would not be easy, and perhaps not very safe. The simple rule for determining the relative premiums is to make them proportional to the real premiums, with the exception of a given addition to each (not premium, but) policy, for expenses of management. In a large office, however, the expenses of management may be made a part of the percentage addition to the premiums.

The method of division in such an office is extremely simple, and has been already described in page 276. Subtracting the present value of all the claims, that is, of all the minimum claims, reckoned as 100*l.* for each tabular premium paid, from the sum of the present values of all premiums, and of the assets of the office, the proportion which this remainder is of the present value of all the claims expresses the fraction of 100*l.* which may be added to each 100*l.* insured.

Let A, the assets* of the office, be 500,000*l.*; P, the present value of all premiums, 600,000*l.*; and C, the present value of all claims, 850.000*l.*: then $A + P - C$,

* Diminished for the expenses of management, as in page 275.

the surplus, is 250,000*l.*, which being 25 parts out of 85 of the whole claims, or $29\frac{7}{17}$ per cent., will afford $129\frac{7}{17}$*l.* for every 100*l.*, which is guaranteed. Those who die in the year of this valuation, may therefore receive that sum.

The principle on which the preceding division is made, is, that if the same state of things continue, every one will in turn receive the same dividend. But, can such a prediction be made? Undoubtedly not, for the fluctuations, both of those who come into the office, and those who go out, will tend to produce variations. It is very unlikely that any office should maintain itself for a long series of years nearly in the same position; and, since the idea of allowing any permanent diminution of the surplus must not be admitted, there is no alternative except an arrangement for a gradual increase, which it is the object of this mode of division to make as slow as is consistent with the certainty of having it. But in this case, it may seem as if the old system were revived, and a fund instituted by the present insurers, for the sole benefit of those who come after them. There is, however, an important difference between never paying more than the guaranteed minimum, so that all the surplus goes towards that fund, and drawing upon the surplus nearly to the full amount which safety would allow, leaving only such a trifle to augment the fund as is requisite to avoid too large an out-going. The old principle, then, which formerly prevented any bonus whatsoever, is here merely applied to such an extent as to keep the bonus within proper limits.

If the tables of mortality by which the profits are divided, be actual representatives of existing mortality, and if the number of members remain nearly the same, the indications of these tables, implicitly followed, would soon reduce the surplus of the office to that which is barely necessary for the extreme payment which the premiums will admit. To take a case: suppose that the premiums will in the long run pay 125*l.*

for every 100*l.* guaranteed; the present value of all the claims is 1,000,000*l.*, that of all the premiums 700,000*l.*, and the value of the assets of the office 600,000*l.* The surplus is therefore 300,000*l.*, and, going upon real tables, the office begins to pay 130*l.* for every 100*l.* guaranteed; and this it would be able to do in favour of all who are insured at the time of the preceding valuation. But part of this dividend does not, and, by hypothesis, cannot, arise from the premiums: it is therefore paid entirely out of surplus, and will gradually disappear. The dividend will be reduced to 125*l.*, about which it will fluctuate, being sometimes a little less and sometimes a little more. An increase of business in such an office would make the surplus disappear more rapidly, since each new comer brings in an equivalent to 125*l.* and those of the new comers who die receive 130*l.* A diminution of business would produce a contrary effect; and a total cessation of new comers would allow the dividend to remain at 130*l.* As far as any danger from fluctuations of mortality is concerned, I do not see any objection to such a division as the preceding: but when it is remembered that the possible diminution of the rate of interest must also be provided for, I think it would be prudent to reserve a small proportion of the surplus for accumulation.

There are two ways in which this reserve may be made; firstly, by employing a table of less than the real mortality in the valuation of the claims and premiums; secondly, by calculating the surplus from a real table, and dividing as upon the supposition that a given fraction of this surplus, say one eighth or one tenth, should be expunged in the calculation. The latter plan is the best of the two, in every respect but one, as follows. The mutual insurance office must be a republic, and many of its members have very little information upon the questions which are, from time to time, submitted to them. They are easily dazzled by the appearance of surplus, and are quick to believe that a larger division might be made in their favour. Add to this that the older members carry with them in the discussion of questions, that influence which age naturally and properly gives in the management of important affairs; and as to which the conduct of an insurance office only forms an exception, because questions arise in which the interests of the old and young clash * with each other, which is nowhere else the case. Under such circumstances the disposition to break in upon the surplus is the fault to which the body has a tendency, and it is not a bad thing to place some small difficulties in the way of doing this. Now if a fraction of the surplus be withdrawn from the calculation of the dividend, it is very easy to change one fraction into another. A vote of the general meeting, and a few strokes of the actuary's pen, and the thing is done. But when the requisite fraction of the surplus is deducted by the supposition of a lower rate of vitality (or of interest) than actually prevails, no change can be made without the entrance of a large number of important considerations, the discussion of which occupies some time, and places a useful check in the way of the restless.

But is it then proposed that every office shall be provided with a fund, which, though slowly, is yet indefinitely, to increase? Not necessarily; for the reserved portion of one year is not put aside, and considered as inalienable, but enters into the surplus of the next year. There may be, then, a limit to the increase of the surplus, as follows. Suppose the office to be in a stationary state, having arrived at the point where the influx of the new members compensates the efflux occasioned by death or surrender. The receipts of the office consist entirely in premiums and produce of capital, the expenditure in management and payment of claims. As long as the surplus increases, the sum of the first pair

* The members of a mutual insurance office are not properly represented in their list of directors, unless the individuals composing it are of very different ages.

of items will exceed that of the second; and, whatever may be laid by in each year, it produces a larger surplus, and larger payments on account of claims, in the next year. If, then, the surplus could increase without limit, so would the dividends; but if the surplus have a limit, the dividends also have a limit: and it is plain that the limit arrives, when the yearly outgoings from claims and management are equal to the receipts from premiums and interest of capital. A mathematical investigation of the conditions necessary in order that the fund may increase, but not without limit, gives the following result:

Suppose an insurance office, constructed upon the preceding principles, to have arrived at its stationary state, with respect to influx and efflux of members, and make the following suppositions:

A The assets of the office, for precision, say January 1, 1838.

P The *real* present value of all premiums from members then in existence.

C The *real* present value of all claims (not including additions) to which the office is then liable.

m The expenses of management till January 1, 1839.

p The amount which will accrue from premiums and interest of premiums by January 1, 1839.

c The amount of claims (not including additions from the surplus fund), which will be paid before January 1, 1839.

r The interest of one pound for one year.

t The fraction which is taken of the tabular surplus fund in the computation of the dividend.

We suppose (as must be the case in an old office), C greater than P, and (as must be the case in a solvent office) A and P together greater than C.

1. In order that there may be a surplus fund increasing, but not without limit, find the fraction which a year's interest on C is of *c*. Then *t*, or the fraction of the surplus fund (or of $A+P-C$), which enters into the formation of the dividend, must exceed that

fraction which a year's interest on C is of c, otherwise the fund would increase without limit.

2. Neither can there be such a fund unless the sum of m and c should fall short of the sum of p, and of a year's interest on the excess of C over P. But, when this is the case, the limiting *surplus* capital is found by dividing the excess of the second total just mentioned over the first, by a divisor obtained as follows:— multiply together t and c, divide the product by C, and subtract r from the quotient. To this surplus capital, add the excess of C over P, and the limiting capital is obtained.

3. If it should happen that the limiting surplus capital is less than the actually existing surplus, it is a sign that the action of the preceding plan would diminish the surplus towards that limit instead of increasing it. In such a case, the surplus is already too large for the value of t to increase it; and if t be not diminished, that is, if less of the tabular surplus be not taken into the computation of the dividend, the fund will diminish.

It is not to be supposed that any office will ever reach a stationary state; but the approach may be near enough to make the preceding process of some use in the determination of the dividends due to the insured. If, following the plan which the preceding problem supposes, we were to inquire what value should be given to the fraction t, the answer to the question must depend on the reduction of interest which is supposed within the bounds of probability. Suppose the present rate of interest to be $3\frac{1}{2}$ per cent. and that the extreme limit is supposed to be $2\frac{1}{2}$ per cent, in such a case the value of P and C must be calculated at $2\frac{1}{2}$ per cent., and such a limiting surplus must be fixed upon as will, at that rate of interest, enable the office to pay at least its guaranteed claims. But it is impossible to lay down an entire system of rules for the regulation of a species of undertaking which depends on the fluctuations of the state of society. Whatever maxims may be collected, and however sound they may be, skill and judgment will always be requisite to apply them to the cases which arise. In this respect the offices resemble the individual problems which arise in life contingencies. Many as are the cases which have been described in books upon the subject, almost every application of them requires attention to some circumstance peculiar to the instance in question.

CHAPTER XIII.

MISCELLANEOUS SUBJECTS CONNECTED WITH INSURANCE, ETC.

The limits of this treatise will only afford a few words on several points of interest, which I will therefore condense into one chapter, taking the subjects as they arise.

The management of annuity offices is somewhat more easy than that of insurance establishments; and the maxims of security in the former are, of course, the direct reverse of those in the latter, so far as any considerations of mortality are concerned. Tables must be assumed of higher than the real vitality, and a rate of interest somewhat below, or at least not above, that which can actually be obtained.

Those who wish to buy annuities on the firmest possible basis, may deal with the government. The commissioners for the reduction of the national debt are empowered to grant annuities in lieu of stock, on terms calculated from the *government tables* (page 168). The rates are high; and though a private office may really be as solvent as the nation, yet confidence springs from opinion, and the security of the national debt must always be thought the very best. The patriotic annuitant, too, may reflect that the profit derived from him goes to the reduction of the national debt.

The distinction of male and female life becomes of importance in the granting of annuities. The insurance offices have not as yet, except, I believe, in one or two instances, begun to recognise the distinction, which is of the less consequence, since, with respect to the office, it is keeping on the safe side, and, with respect to the public, very few female lives are insured. But the exact reverse takes place with regard to annuities; it would be insecure to grant them to females on the same terms as to males; and a very large proportion of the whole number of annuitants is of the former sex.

Annuities might be granted by an office which should undertake a return of profits, in the form of a payment to the executors at the death of the party; and an association of mutual annuitants would not be of difficult formation. The principal objection would be, the smallness of the number of persons who buy annuities, compared with those who insure their lives. If, however, such an office were to grant reversionary annuities, their field would be very much widened. Several of the insurance offices grant annuities, but none, I believe, in which the annuitants are sharers in the profits.

The details of a *Friendly Society* comprise every possible species of life contingency. They grant weekly payments during sickness, annuities in old age, and sums payable at death, in consideration of weekly premiums. These institutions, combined with Savings' Banks, and aided by the removal of the abuses of the Poor Law, will, in time, raise the labouring classes of this country to a degree of independence which they have never known. But, as might have been expected, the management of these important institutions has, in many instances, been wanting in prudence; and I am afraid it is hopeless to expect that the unity of system, which must prevail before a thorough knowledge of the advantages they offer can get abroad, can be attained while their several administrations are unconnected, and at liberty to pursue all possible variety of plans, subject only to the certificate of an actuary that each proposition is not unsafe. But something more than safety is required: an equitable distribution of benefits, and a certainty of the most careful management, are as necessary to the universal formation of these societies as an opinion of their safety. The government, which has within these few years been compelled, by the most decided necessity, to apply a very severe and searching remedy to an abuse of long standing, owes the labouring classes a strong expression of sympathy with the numerous cases of hardship which such a measure must create, and with the excellent conduct and temper under its operation which has pervaded the classes most immediately affected by it. It is to be regretted that the change itself was not accompanied by acts of parliament for the *encouragement* and *aid* of societies such as those of which I am now speaking, in addition to those which already existed for their *regulation.* The most determined opponent of the protective principle would hardly dispute the policy of giving effective help to the efforts of self-support, at the moment when the aid of the parish, which had been the resource against poverty, became only the last security against starvation. If the nation had been obliged to abandon a distant colony, in circumstances of danger and distress, there is no doubt that the settlers would have been furnished with arms, arsenals, ships, money, and all that could enable them to do whatever might be done for their own defence and support. Has similar help in similar circumstances been given at home? Is the labouring man, thus suddenly thrown into a position where the power and the habit of depending on himself are necessary to a degree of which his training never implied the existence, one bit nearer to the acquisition of the power or the formation of the habit, by any aid of the legislature? Have even the opponents of the measure, with their professions of benevolence, ever pressed, or even suggested, the duty of showing the labouring man, not only that by combination his class can provide for itself, but that the community which found it necessary to make a change involving him in years of uncertainty and possible hardship, was desirous that he should have that knowledge, and willing to aid him in attaining its full benefits?

It is not too late to take the necessary steps; and any one who imagines a legislature able to feel, or to think, will see the means of addressing himself to the first faculty by such considerations as the preceding, and to the second by urging the policy of giving every class a share in the artificial system of property on which the country now depends. At present, the *property* of a labouring man is all tangible, and immediately at hand; it would not be a great wonder if he were found to have no clear opinion of the rights of a landlord, a fund-holder, a mortgagee, or an annuitant. But if he himself were in possession of any of those claims which, by means of law, can be created, enforced, or transferred by virtue of the possession of a bit of paper—still more, if the support of his old age and of his sick bed were connected with this purely legal tenure of his past savings, he would then be interested in the preservation of the existing system by the share of it which belongs to himself.

The friendly societies, numerous as they are, are by no means universally distributed; and if they were, the smallness of their several amounts of investment must occasion the expenses of management to bear a larger proportion to the whole than would be the case if all were united. Besides which, it happens every now and then that the affairs of such a society fall into disorder from want of skill or care. The government has lent considerable assistance by allowing their investments a larger rate of interest than could elsewhere be obtained; but this aid, independently of its being but little known by the class whom it most concerns, does not guarantee the proper use of the funds so invested. If one large office were to be established in London, having the general management of the money raised, and the regulation of its distribution, it would not be difficult to find persons of station * and character throughout the country who would consent to act as agents, receiving the contributions and certifying the claims. The expense of management might be borne for a few years by the public purse, and this burden might be gradually thrown on the establishment itself. No very great difficulties could arise in the formation of such an institution, and certainly none the expense of conquering which would not be trifling in comparison of the greatness of the object gained. The act which should establish this universal Friendly Society would, in two generations, become the real poor law.

The subjects of fire and of marine insurance are founded on principles of great simplicity, though it is not easy to procure exact data for the computation of risks. As there exist no offices which are managed on the republican method of a mutual Life Insurance Company, no publication of the results of experience has been made. If every loss by fire or sea were a total loss, it would only be necessary to ask what proportion of all the houses or ships now existing is burnt or wrecked in a year or on a voyage, and the premium for insuring a house for one year, or a ship for one voyage, would immediately follow. Thus, if of all the ships which sail to the West Indies, one in a hundred is lost, the lowest premium at which an insurance could take place is one per cent., and all demanded above that proportion would be profit. It would not, perhaps, be very easy to ascertain this proportion with exactness, and the difficulty is increased if ships or houses be divided into different classes as to security, since the risks of each class must be ascertained separately. But

* Many of the Friendly Societies now established depend almost entirely upon the superintendence of the clergy or local gentry.

the greatest obstacle to a satisfactory adjustment of risks, lies in the necessity of taking into account the chances of only partial loss, which would make the tables (if they could be procured) nearly as complicated as life tables.

On the subject of marine insurance, nothing is known to the public, as to the experience of the underwriters; and, as it is not directly interested in the subject, it would be difficult to create any disposition to inquiry. The mercantile world, however, and the underwriters themselves, have a direct interest in the dissemination of such information, for reasons which it is no pleasant task to state, both on account of their invidious character, and their obvious want of connexion with the general objects of this treatise. But the latter circumstance may, perhaps, not be disadvantageous, since the statement of the existence of an imputation, coming from a quarter in which there is no interest whatever, either in the continuance or discontinuance of any present condition of things, need not excite any disposition, except that of calmly weighing whether it is necessary or not to produce a refutation.

Some years ago, I heard the following opinion stated in a mixed company, in reference to a then proposed attempt to render ships incapable of actually sinking, however much they and their cargo might be damaged; namely, that the mercantile world would not be inclined to patronise an invention which would make the seaman safer than the ship. Some time afterwards, I saw an article in a periodical journal, distinctly written for the purpose of making its readers believe that, in consequence of insurance, unsafe ships are allowed to be used, to an extent which has caused much more loss of life and property than could have been experienced if no such institution had existed. Other allusions, more or less direct, in various publications, have convinced me that one of two things must be true, either such an impression has a party who acknowledge it, or authentic information upon the subject is so difficult to be obtained, that the one, two, or ten, who believe it, or profess to believe it, feel that no answer can be made to the assertion.

That there are men in the *carrying* world (if mercantile world be too wide a phrase for the subject) who would, from a pitiful economy, expose the seaman to risks which a little outlay might prevent, is very possible; there are men of such a spirit in every world: that there are others who would consider such conduct as little short of murder, a like analogy would equally justify us in asserting. Which class has predominated can only be absolutely known to the public by results, without which there is but general opinion upon character to aid any individual in forming his conclusion. It is in human nature that the insured should not be so careful as one who stands risk; and it is, unfortunately, the general experience of men acting in bodies, that they are not found to be swayed by the principles which would be acknowledged and acted upon by them severally. Putting these things together, it is not wonderful that, in any case where suspicion might attach to a body of men, there should be quarters in which it does attach. It would not be wonderful, either, if the suspicion were found to be perfectly groundless; but correct feeling would point out the desirableness of forestalling such suspicion, if possible, by the publication of all necessary information. In the present instance, it would be well that the proportion of loss, among insured vessels, should be known; it would not be necessary to state the values of the several vessels, since the simple account of the number insured, and the number on which a claim has been paid, in various years, would be sufficient. The onus of proving that the loss on uninsured vessels, or on vessels which sailed before insurance was known, is or was greater than that on insured vessels, would lie upon those who make the charge. All persons, in the case of any body of men, must hold every thing short of absolute proof against them to count for nothing, when they show themselves ready to communicate those materials out of which a misdemeanor, if there be one, might be substantiated.

The offices for the insurance of fire have not given any account of the proportion of insured houses upon which claims have arisen. Their usual annual charge is, I believe, about one part in a thousand of the sum insured, upon premises of ordinary risk, such as a dwelling-house in London. There are higher rates for more hazardous insurances, constructed, I should imagine, very much from mere estimation of the risk. But the government steps in between the insurer and the insured, and imposes a duty on each policy which nearly trebles the annual payment upon it. This has been called a tax upon prudence, and in like manner the stamp duty might be called a tax upon justice. I am afraid that if nothing commendable suffered under an impost, reformation would thrive more than revenue; and a deficiency of means to pay the interest of the debt would be a heavier tax on prudence, justice, and every thing else, than any minister has yet contemplated. But it may not therefore follow that the particular tax in question is politic, still less that its amount is justifiable. The reason of the tax is plainly this: the moral security offered by the fire office is worth so much more than competition will allow them to ask, that the impost is one which does not fall so heavily as it would do if levied in many other quarters. Nobody can question the truth of this; but, nevertheless, the amount of the tax imposed by the legislature must be owned to be excessive, and likely to act as a prohibition in the case of poor persons occupying small premises.

But there is a mode of overthrowing this tax, or, at least, of bringing the government to terms, to which I can see no impediment, practical or moral. It is the application of the principle of mutual insurance by a number of individuals acting in a private capacity, and not opening a public office. Suppose a thousand individuals, registering their names, to appoint three men of undoubted character to receive contributions of one guinea a year each. If the subscribers be occupiers of dwelling-houses in London, there is no doubt that this sum would be amply sufficient to insure a thousand guineas to each. If three years were to elapse without a fire taking place, the subscription might be suspended, until circumstances should diminish the fund; which, improving in the mean time at interest, would become every year more capable of meeting demands upon it. There would be no need of any legal security, if the trustees were well chosen; and a short agreement would explain the understanding on which the parties contribute. As soon as a few such clubs were formed, the inutility of imposing a tax on one particular way of effecting an object would become apparent.

It would be lucky for the preceding plan, if it were the decided opinion of lawyers that the courts of equity would not entertain any application for inquiry into the state or management of such funds; since, in that case, the law of honour would be sufficient. It has always been found, that whenever the law of the land refuses to protect a proceeding which is fair and equal in itself, a stronger law claims jurisdiction. The parties benefited in the end would be the fire offices, since such a method of resisting this excessive tax would inevitably procure its abolition.

There is another tax which, though not so disproportionate in its amount, is much worse in its principle than that on policies of fire insurance; namely, the tax on policies of life insurance. It must be remembered, that the income of which the savings are invested in this manner, has already undergone a considerable amount of taxation. If any investment of such savings be taxed, all should be treated alike.

The abolition of lotteries happily leaves nothing to be said upon the subject of gambling, encouraged and promoted by the government; and the recent decision of the French legislature, by which the public gaming-houses have been suppressed, must be a source of congratulation, both from the excellence of the measure itself, and the prospect of imitation which it opens, on the part of other continental powers. But, at the same time, it cannot be denied that, however desirable it may be that no community should give to gambling that appearance of sanction which is implied in regulation, the refusal of the latter is accompanied by evils, of which it is never possible to say positively that they fall short of those which would be produced by sufferance accompanied by restriction. In this country, there are the means of gambling open to every class of the community, and there can be no doubt that those who avail themselves of them are subject to imposition in a degree which could not be the case if the play were accompanied by publicity. The classes of rank and wealth have the power of forming themselves into clubs, in which illegal games are played without the possibility of detection, and in such institutions there can be no doubt, with rare and occasional instances of exception, the play is conducted at least with fairness. But no such thing can be supposed with regard to the numerous receptacles in London and other large towns, and which are believed to exist in different forms, suited to all classes of society. The difficulty of obtaining legal proof renders conviction next to impossible; and the occurrences which sometimes take place at the sessions, prove that, even when enough of evidence is obtained to hold parties to bail, the accused can generally find the means of preventing the evidence from being forthcoming to sustain the indictment. Under such circumstances, gambling in its worst form thrives in defiance of law. Nevertheless, the good consequences of discouragement are visible throughout the country. There is no people in the world among whom so little of direct gambling is found.

The infatuation which leads persons to suppose that they can ultimately win from a bank, which has chosen a game in which the chances are against the player, is one which can only be cured, if at all, by a quiet study of the theory of probabilities. Perhaps some of our readers may suppose, that the persons who thus court ruin, do it under the notion that the results given by that theory are dubious, or derived from unpractical speculation, or perhaps absolutely false. So far is this from being the case, that though they undoubtedly fall into error by forming their notions from observation unaided by theory, yet their error frequently consists in representing games of chance as being more unfavourable to themselves than they really are. Though the true premises should lead them to the conclusion that success is next to impossible, they cannot learn the truth even from a mistake which should teacn it *à fortiori*. The author of the article "GAMING," in the *Penny Cyclopædia*, states, apparently from his own knowledge, that it is customary to consider the chances of the bank at the game of *rouge-et-noir*, as 7½ per cent. above those of the player. Now, it can be immediately shown, from the first appendix, that when a player puts down a stake, his chances of doubling his stake, of losing it, and of simply recovering it, are as 8903, 9122, and 1975. Now 9122 does not exceed 8903 by 7½ per cent. of 8903, but only by about 2½ per cent. If, however, the preceding assertion meant that the game was considered as a simple one, in which the chances were as 46¼ to 53¾, the error was very large indeed. So far as this one instance goes, it should seem that the warning against this game, as derived from observation of its results, was yet stronger than that which would have been given by the theory of the game. The same author adds, that he heard it frequently asserted by constant frequenters of the Parisian gaming-houses, that it was absolutely *impossible* for any one to win in the long run.

Still, however, to the hopeless attempt of squaring the circle, or of finding perpetual motion, we have to add that of discovering a method of certainly winning at play: the attempt at which has been the ruin of many a speculator. The gaming banks have discovered the secret, which is simply to embark considerable capital, and to play with chances unequally in their favour. To produce in the young mind a conviction that events will happen, in the long run, in a fixed, and not in what is called a fortuitous manner, should be an object of education, in order to produce that soundness of views on the results of gambling which is a sure protection against the temptation. By trying experiments upon what are called chance events, such as might easily be done with a pack of cards, or a few dice, it might easily be made to appear that no large number of events will present any marked deviations from the general average which the knowledge of this theory points out before-hand. Persons aware of the truth of the law just stated, may often be able to apply it advantageously. I received the following anecdote from a distinguished naval officer, who was once employed to bring home a cargo of dollars. At the end of the voyage it was discovered that one of the boxes which contained them had been forced; and on making further search, a large bag of dollars was discovered in the possession of some one on board. The coins in the different boxes were a mixture of all manner of dates and sovereigns; and it occurred to the commander, that if the contents of the boxes were sorted, a comparison of the proportions of the different sorts in the bag with those in the box which had been opened, would be strong presumptive evidence one way or the other. This comparison was accordingly made, and the agreement between the distribution of the several coins in the bag and those in the box, was such as to leave no doubt as to the former having formed a part of the latter.

ADDITION TO CHAPTER X.

The following formulæ will sometimes be found useful.

The present value of an insurance, which is to be £1 if the party die in the first year, £2 if in the second, and so on, is —

$$\frac{1+A-rI}{1+r}$$

where A is the value of a common annuity; I that of an annuity which is £1 at the first payment, £2 at the second, and so on; and r the interest of £1 for one year.

If an office engage to pay £1 at the death of an individual, and also to return all the premiums at the same time, that is, if they guarantee that the interest of his investments shall amount to £1, the *premium* which should be demanded is

$$\frac{E-A}{1+A+I}$$

where A and I are as before, and E is the value of a perpetuity of £1.

APPENDIX.

APPENDIX THE FIRST.

ON THE ULTIMATE CHANCES OF GAIN OR LOSS AT PLAY, WITH A PARTICULAR APPLICATION TO THE GAME OF ROUGE ET NOIR.

THOUGH the first part of the following reasoning is of a mathematical character, I have been induced to insert it by the consideration that the results of page 109. have never yet been introduced into an elementary work, nor even proved to the mathematician except either by incomplete or complicated trains of reasoning. Such being the case, perhaps even a well-informed mathematician might be excused for doubting some of the results of chapter V., and I have therefore digested the following demonstration, that no one who bears such a character may be able to weaken the evidence for the *necessity* of the pernicious results of gambling which that chapter is intended to afford.

De Moivre was the first who gave a solution of the following problem, and by a method of the most striking ingenuity. But his demonstration has the defect of *assuming* that one or other of the players *must* be ruined in the long run. Laplace* and Ampère,—the

* The solution of Laplace gives results for the most part in precisely the same form as those of De Moivre, but, according to Laplace's usual custom, no predecessor is mentioned. Though generally aware that Laplace (and too many others, particularly among French writers) was much given to this unworthy species of suppression, I had not any idea of the extent to which it was carried until I compared his solution of the problem of the duration of play, with that of De Moivre. Having been instru-

former in his *Théorie, &c.*, the latter in a tract entitled, *Considérations sur la Théorie Mathématique du Jeu*, Lyons, 1802.—have also solved the problem: both solutions are of the highest order of difficulty, and cannot be rendered elementary. If my memory be correct, I have seen references to other solutions.

The problem is as follows:—Two players, A and B, the first possessed of m times and the second of n times his stake, play at a game so constituted that it is a to b that A shall win any one game; required the probability which each has of ruining the other, if the game be indefinitely continued.

I shall first take the case where one of the players, A, is possessed of unlimited means, that is, in which m is infinite. Let $B_{n,m}$ represent the probability that B having n counters, shall ruin A who has m counters. Then, if m be infinite, $B_{n,\infty}$ will after the first game, become either $B_{n+1,\infty}$, or $B_{n-1,\infty}$, according as that game is B's or A's; of which the chances are

$$\frac{b}{a+b} \quad \text{and} \quad \frac{a}{a+b}$$

Consequently,

$$B_{n,\infty} = \frac{b}{a+b} B_{n+1,\infty} + \frac{a}{a+b} B_{n-1,\infty} \dots (1)$$

which gives

$$B_{n,\infty} = C'\left(\frac{a}{b}\right)^n + C''$$

which, when C' and C'' are determined, will represent the probability that B will never be ruined, but will continually gain more and more from A. But the

mental (in my mathematical treatise on Probabilities, in the *Encyclopædia Metropolitana*) in attributing to Laplace more than his due, having been misled by the suppressions aforesaid, I feel bound to take this opportunity of requesting any reader of that article to consider every thing there given to Laplace as meaning simply that it is to be found in his work, in which, as in the *Mécanique Céleste*, there is enough originating from himself to make any reader wonder that one who could so well afford to state what he had taken from others, should have set an example so dangerous to his own claims.

same equation (1) is equally true if for $B_{n,\infty}$, we substitute either $1 - B_{n,\infty}$, or $A_{\infty,n}$, which two last may be different, for any thing yet proved to the contrary. In fact, the equation (1) is merely the general expression of the condition that n is changed into $n+1$ or $n-1$ according as one or the other of two events happens, whose chances are $\frac{b}{a+b}$ and $\frac{a}{a+b}$. It may be seen however, immediately, that in the case of $n=0$, in which case the proposed contingency becomes an initial impossibility, we must have $B_{0,\infty}=0$, or $C'+C''=0$. We have then

$$B_{n,\infty} = C''\left\{1 - \left(\frac{a}{b}\right)^n\right\}$$

This result is rational only when a is not greater than b, unless we suppose $C''=0$. But the necessity for investigating what takes place in this case is saved by observing a very simple relation which exists between $B_{n,m}$, and $B_{n,\infty}$. Supposing A to have infinite means, it makes no difference in the state of the question if we take any number of stakes m from the stock of A, and suppose that they shall be lost before the rest are touched. Consequently B cannot win indefinitely from A unless he first ruin A's stock of m stakes, and afterwards, beginning from $n+m$ stakes, win indefinitely from A's remainder. That is

$$B_{n,\infty} = B_{n,m} \times B_{n+m,\infty}$$

$$\text{or } B_{n,m} = B_{n,\infty} \div B_{n+m,\infty}$$

$$= C''\left\{1 - \left(\frac{a}{b}\right)^n\right\} \div C''\left\{1 - \left(\frac{a}{b}\right)^{n+m}\right\}$$

$$= \frac{b^m\left(b^n - a^n\right)}{b^{n+m} - a^{n+m}}$$

Whence, applying the same reasoning to $A_{m,n}$, we find that if two players A and B, possessing m and n

stakes, play at a game which gives a to b in favour of A at each trial, then, in an indefinite number of trials

$$B_{n,\,m}, \text{ the chance that B shall ruin A} = \frac{b^m\left(b^n - a^n\right)}{b^{n+m} - a^{n+m}}$$

$$A_{m,\,n}, \text{ the chance that A shall ruin B} = \frac{a^n\left(b^m - a^m\right)}{b^{n+m} - a^{n+m}}$$

The sum of these two chances is unity, from which it appears that *one or other must be ruined in the long run.* These results agree entirely with those of De Moivre, and all the rules in page 109. may be easily deduced from them.

It appears also, that if the conditions of any game, however complicated, can be reduced, in the case where one of the players (A) has unlimited means, to the equation

$$B_{n,\infty} = \beta\, B_{n+1,\infty} + \alpha\, B_{n-1,\infty} \text{ (where } \alpha + \beta = 1);$$

then the ultimate results of that game exhibit probabilities of precisely the same value as those of a simple game, in which it is α to β for A against B.

If, besides cases in which A or B wins, there be cases in which the game is drawn, no alteration is made in the result (though the number of games in which there is a given probability of either party winning must of course be increased.) Let α, β, and δ, be the chances that A wins, that B wins, and that the game is drawn: then $(\alpha+\beta+\delta=1)$

$$B_{n,\infty} = \beta\, B_{n+1,\infty} + \alpha\, B_{n-1,\infty} + \delta\, B_{n,\infty} \quad \text{or}$$

$$B_{n,\infty} = \frac{\beta}{1-\delta} B_{n+1,\infty} + \frac{\alpha}{1-\delta} B_{n-1,\infty} \left(\frac{\beta}{1-\delta} + \frac{\alpha}{1-\delta} = 1\right)$$

the same as in a game which must be won or lost, and in which it is α to β for A against B.

The following is the problem of the game of *rouge et noir,* which I shall afterwards proceed to explain.

A and B play at a game which presents four cases, A, B, D, and T, of which the chances are α, β, δ, and θ, so that $\alpha+\beta+\delta+\theta=1$. When A happens, the player A wins; when B happens, the player B wins; when D happens, the game is unconditionally drawn; but when T happens, the game is drawn, and in the next game only the player A puts down a stake, and not the player B. If D should follow T, or if T should happen any number of times running, or D, or successions of T and D, still A's stake remains risked, without any from B: nor does B stake again, until the happening of A or B recovers A's stake, or assigns it to B: after which, both parties stake again.

Supposing the means of B to be unlimited, let A_m represent A's chance of winning indefinitely, immediately before a game in which both are to stake, A having m stakes in his possession: and let A'_m represent A's chance immediately before a game in which B does *not* stake. Then, by the preceding method

$$A_m = \alpha A_{m+1} + \beta A_{m-1} + \delta A_m + \theta A'_m$$

$$A'_m = \alpha A_m + \beta A_{m-1} + \delta A'_m + \theta A'_m$$

Eliminate A'_m and we have

$$A_m = \frac{\alpha(\alpha+\beta)}{(\alpha+\beta)^2+\theta\beta} A_{m+1} + \frac{\beta(\alpha+\beta)+\theta\beta}{(\alpha+\beta)^2+\theta\beta} A_{m-1}$$

in which the sum of the two co-efficients is unity. Hence this game is equivalent in its ultimate chances to a simple game in which it is $\alpha(\alpha+\beta)$ to $\beta(\alpha+\beta)+\theta\beta$ for A against B. If $\alpha=\beta$, the last odds are those of 2α to $2\alpha+\theta$.

This game of *rouge et noir* is described in an unintelligible manner, and with material omissions, in the later editions of Hoyle, from which work, and from the testimony of persons who have seen it played, I give the best description I can make of it, observing that the most modern method of playing differs in several particulars from that given in the book referred to.

A number of packs of cards is taken (six, it is said in

Hoyle) and all the cards are well mixed. Each common card counts for the number of spots on it, and the court cards are each reckoned as ten. A table is divided into two compartments, one called *rouge,* the other *noir,* and a player stakes his money in which he pleases. The proprietor of the bank, who risks against all comers, then lays down cards in one compartment until the number of spots exceed 30; as soon as this has happened, he proceeds in the same way with the other compartment. The number of spots in each compartment is then between 31 and 40, both inclusive, and that compartment wins which has the lower number of spots; so that if, for instance, there should be 37 spots in the *rouge,* and 32 in the *noir,* those players who staked upon *noir* would win from the bank sums equal to their stakes. If the number of spots be the same in both (which is called in Hoyle a *refait*) the game is drawn, and the parties withdraw, diminish, or augment their stakes at pleasure, for a new game: except only when the number of spots in both compartments is 31 (called in Hoyle a *refait trente et un*), in which case the bank is allowed to withdraw its stakes, and those of the players, whatever their compartment may be, are impounded (placed *en prison*). In the next game (now called an *après*), the impounded stakes are played for, the players choosing their compartments as before: should the bank win it takes the stake, should the bank lose the player recovers his stake. Should a second *refait trente et un* occur, or a drawn game, the stakes still remain impounded, and are not released until a gain or loss arrives. In the meanwhile new stakes may be put down, before the fate of the old ones is decided.

The chances of this game depend, in a slight degree, upon the number of packs of cards which are mixed together. When, however, there are as many packs as six, it is very nearly * indeed the same thing as if the

* The only ways in which 31, for example, could be obtained from an unlimited number of packs, and which could not equally well be obtained from six packs, are those in which more than 24 aces occur. Now the proba-

number of packs were unlimited. The following tables, computed upon the latter supposition, will represent, with more than sufficient accuracy, the chances of the several arrivals. The first three columns of the first table exhibit the chance of arrival of each number among the several sum-totals which precede the arrival of thirty-one or more. Thus opposite to 4 we see ·0961, which is the chance that the beginning of the sequence drawn shall show one of the following sets of cards (1 standing for ace, &c.):

1, 1, 1, 1	2, 1, 1	2, 2
1, 1, 2	1, 3,	4
1, 2, 1	3, 1	

1	·0769	11	·1200	21	·1398	31	·1481
2	·0828	12	·1247	22	·1424	32	·1379
3	·0892	13	·1293	23	·1449	33	·1275
4	·0961	14	·1340	24	·1472	34	·1169
5	·1035	15	·1386	25	·1493	35	·1060
6	·1114	16	·1432	26	·1511	36	·0950
7	·1200	17	·1476	27	·1527	37	·0838
8	·1292	18	·1519	28	·1541	38	·0723
9	·1392	19	·1559	29	·1552	39	·0607
10	·3806	20	·2129	30	·1683	40	·0518

31	·0219	34	·0137	37	·0070	40	·0027
32	·0190	35	·0112	38	·0052		
33	·0163	36	·0090	39	·0037	D. G.	·0878

The fourth column contains all the chances of those points which lie between 31 and 40, that is to say, the chance of each being the first which arrives. Thus ·1060 is the chance that 35 will appear, and that it will be the first which appears above 30. The second table (containing the squares of the numbers in the last column of the first) shows the chance of each *refait*, that of the

bility that out of 31 cards drawn at hazard, 24 or more shall be aces, is altogether beneath consideration. It is less than one out of a million of million of millions.

*refait** *trente et un* being ·0219. Opposite to D. G. is the sum of the chances of all the *refaits* except the first, which sum is the chance of a drawn game.

If then we say that ·021 is the chance of a *refait trente et un* and ·087 that of a drawn game, there remains ·892 for the chance that either the bank or the player must win; which chances being equal, give ·446 for the player, and the same for the bank (exclusive of the benefit of the *après*). Returning then to the result in page v., we find $\alpha = \beta = \cdot 446$, $\theta = \cdot 021$, and 2α to $2\alpha + \theta$ is ·892 to ·913, or 892 to 913. Consequently; —

At the game of *rouge et noir*, as now played, the *chances of ultimate ruin* to the bank or the player are the same as they would be at a simple game, which must be either won or lost at each throw, and in which the bank has 913 chances of winning, where the player has 892.

The bank, as noticed in page 110., is playing against the whole public, or against a player with unlimited means. Taking 892 to 913, or more correctly 8903 to 9122, and applying the rule in page 110., the following table results, which must be thus used. Opposite to 30 we find ·4824, which is the chance of the ultimate ruin of a bank which risks one-thirtieth of its means at every game: —

10	·7843	110	·0690	210	·0061
20	·6151	120	·0541	220	·0048
30	·4824	130	·0425	230	·0037
40	·3783	140	·0333	240	·0029
50	·2967	150	·0261	250	·0023
60	·2327	160	·0205	260	·0018
70	·1825	170	·0161	270	·0014
80	·1431	180	·0126	280	·0011
90	·1122	190	·0099	300	·0007
100	·0880	200	·0077	400	·0001

* The editor of Hoyle says, or implies, that the chance of the arrival of 31 is one-eighth, or ·125. This is, no doubt, a conclusion drawn from observation. The table in Hoyle, exhibiting the odds (page 147., which refers to 141., edition of 1814), is altogether erroneous.

Hitherto all our results seem in favour of the bank; that is, tending to show that its advantage is not so great as is commonly supposed. No person, granting a bank permission to exist, would grudge it such an advantage as would make it 49 to 1 against its being ruined by the possible fluctuation attendant upon an unlimited duration of play. This chance of being ruined, namely ·02, appears from the table to be exceeded, unless the bank possess 160 times the sum risked at each game: if this were 100*l.*, the bank would need a capital of 16,000*l.* But I must now request attention to the other side of the question; first, considering the bank against the public; and, secondly, the bank against an individual player. One of the most important features in this game (which springs from the old game of Faro, as did the last from the still older game of Basset*) is, that the bank does not risk the whole sum it lays down, but only the difference between those sums which the caprice of the players obliges it to stake on *rouge* and on *noir*. If 20 players have each staked a guinea, 12 on *rouge* and 8 on *noir*, and if *rouge* win, the bank loses 12 guineas and gains 8, and consequently did not risk more than four guineas. It is impossible to say what chance there is of the bank having to risk a given sum in such a case, as this depends on the will of the players. When the cards have several times decided for *rouge*, those players who think the run is not finished will stake on that colour, while others who think differently will stake on *noir*. I am wholly without the means of saying what average exists, but I should incline to think it very unlikely that the bank really risks more than one fourth of its deposits.† But the advantage which it derives from the *refait trente et*

* An assertion of the editor of Hoyle, which is true as to the principle of the game — namely, that besides equal chances for the bank and the player, there are chances for a drawn game, and a case in which the bank has a direct advantage amounting to half the stake — but the details are very different. Both games are described in De Moivre.

† The bank is evidently (its chance of the *après* excepted) merely the means of equalising the sums staked on the two colours.

un, and its consequences, is gained on the whole of the opponent stakes. The following is the method of estimating the mathematical advantage of the bank:—Before a common game, the prospects of the bank lie entirely in the chance of that game being followed by an *après*, since, in all other respects, the chances of gain and loss are the same. After a *refait trente et un*, on whichever colour any player may choose to risk his impounded stake, the bank has the chance α of winning that stake, and none of losing. But besides this, the bank has all the chances of a second *refait* (or $\theta+\delta$ or $1-2\alpha$), for having another trial of the same kind: if then x express the fraction of a pound, which such a chance of 1*l.* is worth, we have

$$x=\alpha + (1-2\alpha)\,x, \text{ or } x=\tfrac{1}{2}.$$

The mathematical advantage of the bank is therefore, the chance θ, or ·0219, of being put in possession of the worth of half the stakes; or ·011*l.* of all the sums it *deposits*. This is 1*l.* 2*s.* *per cent.* * *per deposit*; which, to those who know the rapidity with which the risks succeed one another, will appear to yield, in the course of the year, an ample return, not merely to the deposits, but to the sums which are reserved for security against fluctuation. It is probably $4\frac{1}{2}$ per cent. upon every real *risk*; and the return in the course of a year may be easily guessed at. Imagine 100 different games, played on each of 100 different evenings, the sum risked by the collective players on each game being 50*l.* The total deposits of the bank would be 500,000*l.*, on which 1*l.* 2*s.* per cent. is 5500*l.* The capital required to make this speculation much more safe than any mercantile adventure, would not be larger than its probable return in one year.

* Some time after this was written I chanced to find the following sentence in the lately published Theory of Probabilities of M. Poisson. " Dans les jeux publics de Paris l'avantage à chaque coup est peu considérable : au jeu de *trente-et-quarante* par exemple, il est un peu au-dessous de ·011 de chaque mise. Voyez sur les chances de ce jeu, le mémoire que j'ai inséré dans le journal de M. Gergonne, tome xvi. numéro 6, Décembre, 1825." I have not seen this memoir; the accordance of the result with my own, shows that I hav · described the game correctly, as it was played in Paris; of which, from paucity of information, I was by no means sure.

If any persons, aware that the preceding calculations are new, should imagine that there must be some mis calculation in a result which shows that *cent. per cent.* on the necessary capital might be gained three times in a year, I reply, that the chance of a *refait trente et un*, as given by the editor of Hoyle, produces a result of nearly as surprising a character. For ·0219 read ·0156, or $\frac{1}{64}$, and the 5500*l.* above-mentioned becomes a trifle less than 4000*l.* The preceding results, or either of them, being admitted, it might be supposed hardly necessary to dwell upon the ruin which must necessarily result to individual players against a bank which has so strong a chance of success against its united antagonists. But so strangely are opinions formed upon this subject, that it is not uncommon to find persons who think they are in possession of a specific by which they must infallibly win. The last table given will show the chances which any single player has of ruining the bank, and of being ruined himself, as follows: —If the player stake one mth part of his means * at each throw, and the bank one nth part, from unity subtract the number in the table opposite to n and to $n+m$, and the first result divided by the second shows the chance that the bank will ruin the player. Suppose, for example, that the player risks 1-10th and the bank 1-160th of their respective resources. Then opposite to 160 and 160 + 10, we find ·0205 and ·0161; which, being subtracted from 1, give ·9795 and ·9839, whence $\frac{9795}{9839}$ is the chance for the bank; or it is 9795 to 44, or 223 to 1, that such a player will be ruined. Even if both the player and the bank stake 1-160th parts of their several funds, the bank will still have ·98 or 49 to 1 in its favour *against that one player.*

The last column of the table in page vii, shows that the bankers at *rouge et noir*, by making the *après* de-

* We must not here consider what the bank stakes against the individual player, but the whole sum which it risks.

pend on the *refait trente et un* have chosen the most favourable out of the ten cases. The following table will show the effect of substituting any other *refait;* the first column pointing out the *refait* in question; the second, the simple game to which it is equivalent in the chances of ultimate ruin; the third, the benefice of the bank upon every 100*l.* deposited:—

						£	s.	d.
31	10,000	to		10,246	-	1	2	0
32	-		-	10,213	-	0	19	0
33	-	-	-	10,183	-	0	16	6
34		-	-	10,154	-	0	13	6
35	-	-	-	10,126	-	0	11	0
36	-	-	-	10,101	-	0	9	0
37	-	-	-	10,079	-	0	7	0
38	-	-	-	10,058	-	0	5	0
39	-	-	-	10,042	-	0	4	0
40	-	-	-	10,030	-	0	3	0

The above is a graduated scale of poisons, each one being slower in its operation than the preceding; the first, or quickest of all, being that which is used at present. Of all the illegal games, none that I know of is less likely to lead to ruin than *rouge et noir;* and the results of this investigation give a sufficient notion of the state of the case between the banker and his dupes.

The first table, in page vii., is calculated in the following manner:—The chance of any given card at a given point is $\frac{1}{13}$ for every number which a card can give, excepting 10, the chance of which is $\frac{4}{13}$, on account of the value given to the court cards. Let x be a number greater than 10, and let V_x be the chance of arriving at that number in the laying down of the cards. Then, if (V_x) signify the event of which the chance is V_x, and if by (a) (b), we mean the consecutive happening of the two events whose chances are a and b, it follows that (V_x) when it happens, must happen in one of the following ways:—

$(V_{x-1})(\frac{1}{13})$ or $(V_{x-2})(\frac{1}{13})\ldots$ or $(V_{x-9})(\frac{1}{13})$ or $(V_{x-10})(\frac{4}{13})$
or $V_x = \frac{1}{13}(V_{x-1} + V_{x-2} + \ldots + V_{x-9}) + \frac{4}{13}V_{x-10}$

from which it is readily found that

$$\Delta V_x = \frac{1}{13}(V_x + 4\,\Delta V_{x-10} - V_{x-9}.)$$

The first eleven values of V_x are thus determined: V_1 is evidently $\frac{1}{13}$; V_2 is $\frac{1}{13}V_1 + \frac{1}{13}$ or $\frac{1}{13}\cdot\frac{14}{13}$; V_3 is $\frac{1}{13}V_1 + \frac{1}{13}V_2 + \frac{1}{13}$ or $\frac{1}{13}(\frac{14}{13})^2$; and so on up to V_{10} which is

$\frac{1}{13}(V_1 + V_2 + \ldots + V_9) + \frac{4}{13}$ or $\frac{1}{13}(\frac{14}{13})^9 + \frac{3}{13}$
V_{11} is $\frac{1}{13}(V_{10} + V_9 \ldots + V_2) + \frac{4}{13}V_1$ or
$\frac{1}{13}(\frac{14}{13})^{10} + 6(\frac{1}{13})^2 - \frac{1}{13}$.

The rest were then calculated by the preceding formula for ΔV_x to six places of decimals (by which the accuracy of four was insured) as far as V_{31} inclusive. The remainder were calculated by those which preceded, leaving out the terms which the necessary distinction already mentioned requires to be omitted.

It may perhaps appear to some that a part of the preceding reasoning is inapplicable, since it only calculates the chances of ruin in an indefinite succession of games, whereas any practicable number of games, though great, may not involve the same chances as an infinite number. The objection is valid in principle, but the correction which is rendered necessary by it is not worth consideration, if any large number of games be in question.

The following are the only cases in which a simple approximate rule can be given, connected with finite numbers of games: —

Problem. Both parties have the same number of stakes (which should not be less than 20), say a, and the play is equal, or either has an even chance of winning any one game. What is the chance that one or other shall have been ruined before x games have been played? (x being a large number).

RULE. Divide 30 times x by 56 times the square of a, and from the quotient subtract ·1049. If the result be the common logarithm of z, then $z - 1$ to 1 are the odds in favour of the event.

EXAMPLE. The number of stakes is 45, and the play equal: what is the chance that one or other is ruined before 1520 games have been played?

$$a = 45,\ x = 1520, \quad 56 \times a \times a = 113400$$

$$x \times 30 = 45600,\ \frac{45600}{113400} = \cdot 4021$$

$$\cdot 4021 - \cdot 1049 = \cdot 2972 = \text{logarithm of } 1\cdot 982.$$

Answer, ·982 to 1, or 982 to 1000,

PROBLEM. The stakes being equal, and also the play, as before, what is the number of games in which it is n to 1 that one party or the other will have been ruined?

RULE. To the common logarithm of one more than n add ·1049: multiply the result by 56 times the square of the number of stakes, and divide by 30, which gives the number required, very nearly.

EXAMPLE. Both parties have 50 stakes: in what number of games is it 10 to 1 that one or other will be ruined?

$$a = 50,\ n = 10,\ n + 1 = 11,\ \log. 11 = 1\cdot 0414$$

$$1\cdot 0414 + \cdot 1049 = 1\cdot 1463,\ 56 \times a \times a = 140000$$

$$1\cdot 1463 \times 140000 = 160482: \quad \frac{160482}{30} = 5349$$

Answer: in about 5349 games.

To find the number of games in which it is an even chance that one or other will be ruined, from three-fourths of the square of the number of stakes, subtract its hundredth part. Thus, if both parties have 40 stakes, then 40×40 being 1600, three-fourths of which is 1200, from 1200 subtract 12, which gives 1188 for the number of games (very nearly) in which it is an even chance that one or other will be ruined.

If a player with a stakes play with one of unlimited means, the chances being the same for both, it is an even chance that he is ruined in a number of games which is thus found: take $2\frac{3}{8}$ of the square of a. If greater accuracy be required, add to the result one less than its 760th part, which is sure to make it correct within a single game. Thus if the number of his stakes be 100, $100 \times 100 \times 2\frac{3}{8}$ is 23750, the 760th part of which is 31, whence $23750 + 30$, or 23780, is within one of the number of games required.

APPENDIX THE SECOND.

ON THE RULE FOR DETERMINING THE VALUE OF SUCCESSIVE LIVES, AND OF COPYHOLD ESTATES.

The rule given in the work is in a different form from that of any writer with whom I am acquainted, though it agrees with that given by Mr. Milne, as will be shown. This Appendix has been rendered necessary by the fact that no writer has solved the question of the value of copyhold estates with absolute correctness except Mr. Milne, whose solution is in a form of unnecessary difficulty. The writers with whom I am acquainted, who give the old rule, or one involving an omission of the same kind, are De Moivre, Dodson, Thomas Simpson, Stonehouse, Morgan, Baily, and the French translator of the latter. Mr. Milne stands alone in proposing a somewhat different rule, which like many results of independent investigation, differs more than need have been the case from the form of preceding results.

Let there be an estate held on a single life, and renewable for ever upon payment of a fine of 1*l.* Let it be a condition, that each renewal is to be made on the 1st of January next following the extinction of the previous life; it is required to find the present value of all the fines.

Firstly, To find the value on the 1st of January, the moment after a fine has been received, and the best life which can be found has been put in. Let P be the value of the life, or the value of an annuity upon it; r the rate of interest *per pound*, and F the value of all the fines. Then upon the new year's day next following the extinction of that life, the whole value of the fines will be $1+F$, because the person who claims the fines will have one pound to receive, and will then have remaining the interest which we have called F. It follows therefore that F is the present value of $1+F$ to be received at the end of the year in which the life drops: or, by the well known formula

$$F=\frac{1-rP}{1+r}(1+F) \text{ or } F=\frac{E-P}{1+P}$$

where E is $\frac{1}{r}$, the value of a perpetuity of 1*l.*

Secondly, Let the life already in possession be of the value A, the lives at each renewal having the value P, as before. Consequently, the present value of the fines is that of $1+F$ to be received at the end of the year in which the present life drops: and this is

$$\frac{1-rA}{1+r}.(1+F) \text{ or } \frac{E-A}{E+1}.\frac{1+E}{1+P} \text{ or } \frac{E-A.}{1+P}$$

If an estate be held on any number of lives, with a fine of 1*l.* on renewing each, it is precisely the same thing as if a similar number of estates were held on single lives, and the present value of all the fines, the present lives being worth A, B, C, &c., n in number, is

$$\frac{nE-(A+B+C+\ldots.)}{1+P}$$

tne common rule divides by P instead of $1+P$.

A mathematical rule, when erroneous, is best exposed in its extreme cases; let us then suppose the life P certain to drop in the year, that is, worth nothing. There will consequently be a fine to pay every 1st of January, and the present value of what I called F in the preceding investigation, is $\frac{1}{r}$ or E. The rule I have given, shows that $F=E$ when $P=0$; that of De Moivre, Simpson, &c., makes F infinite. Again, suppose the life P certain to last one year and to drop in the second; in which case its value is $\frac{1}{1+r}$, or $\frac{E}{E+1}$, and F is a perpetuity of 1*l.* receivable at the expiration of every two years, or $\frac{E^2}{2E+1}$. If $A=P=\frac{E}{E+1}$, the new formula becomes $\frac{E^2}{2E+1}$, the old one becomes E, the value of a perpetuity of 1*l.* receivable at the end of *every* year.

I shall now show that the preceding rule agrees with that of Mr. Milne; which is as follows:—Let P be worth an annuity certain of t years, and let v be the present value of 1*l.* to be received a year hence. Then the present value of all the fines, according to Mr. Milne, is

$$\frac{A'+B'+C'+\ldots\;\;..}{1-v^{t+1}}$$

where A′, B′, C′, &c. mean the present value of 1*l.*, to be received at the end of the years in which the lives severally drop. Since P is the value of an annuity certain for t years, we have

$$P=E-v^tE,\ 1+P=\frac{1+r}{r}-v^tE=\frac{E}{v}-v^tE,$$

or $$1+P=\frac{E}{v}(1-v^{t+1})=(E+1)(1-v^{t+1})$$

and $$A'=\frac{E-A}{E+1},\text{ whence }\frac{A'}{1-v^{t+1}}=\frac{E-A}{1+P}$$

from which the coincidence of the two rules is manifest. The error of the old rule, by the Northampton Tables, and at 4 per cent. (the best life being worth 17·25 years purchase) is, that the result is $5\frac{1}{2}$ per cent. too great.

The old rule, as Mr. Milne justly observes, is derived from the supposition that the new life is put in at the beginning of the year in which the old one drops, instead of at the end; which last was in the intention of those who formed the rule. It may be said however, that the old rule is nearer the truth than the new; since one time with another, the renewal is made before the end of the year in which the old life drops. This objection must be valid to some extent; and I proceed to inquire how much weight must be allowed to it.

Let α be the fraction of a year allowed for renewal: it is clear then that $\frac{1}{\alpha}$ renewals (each accompanied by a fine) *may* take place in the year. Let the life A drop at the expiration of the fraction θ of a year, or between θ and $\theta + d\theta$, the chance of which is $d\theta$ itself, if A be supposed equally likely to drop at any period of the year. At $\theta + \alpha$, then, the new life is put in; and if this new life drop before $1 - \alpha$ of the year is gone, another fine must be paid, and another renewal is made, which again may drop before $1 - \alpha$, and so on. But since the chance of each additional renewal is very much smaller than that of the preceding, it will be sufficient to take the first only into consideration. Let it be supposed, then, that not more than one renewal shall take place within the year in which A drops.

Let a be the chance that the life P drops in a year after nomination, in which case we may call $x\,a$ the chance that it drops in any fraction x of a year. Then $d\,\theta\,(1 - \theta - \alpha)\,a$ is the chance that the life A drops between θ and $\theta + d\,\theta$, and that the next life drops within the year, in which case another fine is to be paid at the beginning of the next year. Consequently, neglecting the interest of the fine in a fraction of a year, the lessee has the chance $d\,\theta\,(1 - \theta - \alpha)\,a$ of having a second fine to pay, upon the contingency of A dropping between θ and $\theta + d\,\theta$. Integrate this expression from $\theta = 0$ to $\theta = 1 - \alpha$, and we have $\frac{1}{2}a\,(1 - \alpha)^2$ for the chance of a second fine: which, with the fine certain upon the death of A, shows that $1 + \frac{1}{2}a\,(1 - \alpha)^2$ is the mathematical expectation of the fines to be paid, when the probability of one renewal within the year is contemplated, and another at the end, if necessary.

A more complicated process, proceeding on the supposition that any given number of renewals may take place within the year in which A dies, gives the following terms for the total probability of one or more renewals taking place before the end of the year: —

α, the chance that there shall be no renewal;

$1-\alpha \quad - \quad \frac{\alpha(1-2\alpha)^2}{2}$ the chance of one only;

$\frac{\alpha(1-2\alpha)^2}{2} - \frac{\alpha^2(1-3\alpha)^3}{2\,.\,3}$ the chance of two only;

and so on, the series being continued as long as the requisite multiples of α are less than unity. Hence the chance that the number of renewals shall not exceed n, is

$$1 - \frac{\alpha^n\left(1-(n+1)\alpha\right)^{n+1}}{2\,.\,3\,.\,.\quad n\,.\,n+1}$$

which, in the case most against us, that is, supposing instantaneous renewals, or $\alpha=0$, is

$$1 - \frac{a^n}{2\,.\,3\,.\,.\,.\,.\,n\,.\,n+1}$$

Let the life P be one of seven years old, in which case the Northampton table gives $\alpha = \frac{110}{5925} = \cdot 0186$. Neglecting interest for the fractions of the year, and remembering that the number of renewals is the number of fines, the mathematical expectation of all the fines is the sum of

$$1-\alpha-\frac{\alpha(1-2\alpha)^2}{2}, \qquad 2\left(\frac{\alpha(1-2\alpha)^2}{2}-\frac{\alpha^2(1-3\alpha)^3}{2\,.\,3}\right), \text{ \&c.}$$

To this add α, the chance that the renewal fine certain on the death of A, shall outrun the year, and we find

$$1+\frac{\alpha(1-2\alpha)^2}{2}+\frac{\alpha^2(1-3\alpha)^3}{2\,.\,3}+\cdots\cdot$$

for the mathematical expectation of all the fines which shall be paid in the year in which the present life drops, including the chance of that life dropping too late to renew it within the year, and of its being therefore renewed within a period not exceeding α of the year following. If $\alpha=0$, this becomes

$$\frac{\epsilon^a - 1}{a}, \quad \epsilon \text{ being } 2 \cdot 7182818 \cdots$$

and when $a = \cdot 0186$, $\epsilon^a = 1 \cdot 019$

Hence $\cdot 019 \div \cdot 0186$ being $1 \cdot 02$, it appears that $1 \cdot 02(1+r)$ is an enormously exaggerated representation of the fine which must be substituted for $1l$ in the amended rule to make the correction which might be suggested by the advocates of the old rule. Allowing $r = \cdot 035$, it may be granted that the old rule is correct, in the particular case before cited, if we suppose: 1. That no time whatever is allowed for renewal; 2. That the best life which can be found is such, that 1 out of 59 of such lives drop in a year, and; 3. That the lessor is to receive his fines at the beginning of the year succeeding that in which A drops, with $3\frac{1}{2}$ per cent. interest, reckoned from the beginning of the preceding year. To take a more rational supposition, let us allow six months for renewal. Here $\alpha = \frac{1}{2}$ and only the first term of the series can be taken, which, with the interest, gives $1 + \frac{r}{2}$.

It appears from the preceding, that the advantage of the lessor over that which the rule gives him, is trivial, except in this, that at the time when the rule supposes him to receive one pound, he may have received and improved one pound during a fraction of a year. There is also another advantage which he has, and which neither rule allows him. The renewal may have been made before the time supposed in the rule, in which case the existing life will be somewhat worse than that supposed. To take all these circumstances into account, suppose the life A to drop at the end of θ of the year, in which case the lessor is in possession of $1 + r - (\theta + \alpha)r$ at the end of the year, and the lessee has a life * worth $P - \{1 - (\theta + \alpha)\} \triangle P$, where $\triangle P$ is the decrement

* Let it be remembered that the renewal may take place after the beginning of the year, which is equivalent to having a better life than that supposed. Algebra, as in other cases, strikes the balance of positive and negative quantities without the necessity of introducing several formulæ.

of the value of the life in one year. Multiplying by $d\theta$, the chance of this occurrence, and integrating from $\theta=0$ to $\theta=1$, we find $1+\frac{1}{2}r-\alpha r$ for the mean sum, and $P-\frac{1}{2}\Delta P+\alpha\Delta P$ for the mean life. If the time allowed for renewal be more than six months, the rule (without this correction) gives an advantage to the *lessor*; if six months, to neither; if less than six months, to the *lessee*. Substitute $1+\frac{1}{2}r-\alpha r$ and $P-\frac{1}{2}\Delta P+\alpha\Delta P$ for 1*l* fine and a life P, and we have

$$\frac{E-A}{1+P-\frac{1}{2}\Delta P+\alpha\Delta P}\times\left\{1+\tfrac{1}{2}r-\alpha r\right\}:$$

which I believe to be the most correct rule that can be given for the present value of all the fines upon a single life renewable for ever, the value of the present life being A, the tabular value of the best life P, that of a life one year older $P-\Delta P$, the fraction of a year allowed for renewal α, r the interest of one pound for one year, and E the perpetuity. By substituting $nE-(A+B+C+\cdots)$ for $E-A$, the value of all the fines from an estate held on any number of lives is found. Similar considerations apply to the present value of 1*l.* to be received in α of a year after the death of a life A. The offices frequently pay in three months after the death is proved, whereas the tables are calculated for the end of the year of death. Again, they rate all lives as they will be at the next birthday, the parties being one with another half a year younger. To a party who dies at the end of θ of a year, the office has paid by the end of the year $1+(1-\alpha-\theta)r$ which, treated as before, gives $1+\frac{1}{2}r-\alpha r$. And the present value of 1*l.*, which is computed by the office from

$$\frac{E-A}{E+1}(1+k)$$

(where k is the proportion of profit demanded) may be more strictly computed from

$$\frac{E-A-\frac{1}{2}\Delta A}{E+1}\times(1+\tfrac{1}{2}r-\alpha r)$$

where $A + \Delta A$ is the value of a life one year younger than the *office age* of the party at entry.

When an annuity is granted upon condition that the executors of the party are to receive such a proportion of payment for the year in which the annuitant dies, as corresponds to the portion of the year during which he is alive, the addition to A the value of the annuity is the present value of $\frac{1}{2}+\frac{1}{6}r$ of a year's purchase, payable at the end of the year of death. The formula which then very nearly represents the result of the preceding correction is

$$A\left(1-\frac{r}{2}\right)+\frac{1}{2}-\frac{r}{3}$$

or to the tabular value of the annuity add the excess of half a year's purchase over half a year's interest of the tabular value, together with one-third of the interest of 1*l.*

APPENDIX THE THIRD.

ON THE RULE FOR DETERMINING THE PROBABILITIES OF SURVIVORSHIP.

It does not seem to have been noticed, that this rule is considerably more correct than its framers could have anticipated, supposing them to have contemplated no higher degree of exactness than their demonstration entitled them to assert. Let ϕt and ψt represent the probabilities that A and B, now alive, shall be alive at the end of t years, t being whole or fractional. Then $-\psi' t . dt$ is the probability that B shall die between t and $t+dt$ and $-\int \phi t . \psi' t \, dt$ from $t=n$ to $t=n+1$ is the chance of a survivorship of A beginning to take place somewhere in the $(n+1)$ *th* year after the present time. Let α and $\alpha+\Delta\alpha$ and β and $\beta+\Delta\beta$ be the chances that A and B shall be alive at the end of n and $n+1$ years from this time: then, in the demonstration of the rule, ϕt is assumed $= \alpha + \Delta\alpha . t$ and $\psi t = \beta + \Delta\beta . t$, t being measured from the beginning of the $(n+1)th$ year. Hence

$$-\int \phi t . \psi' t . dt \text{ (from } t=0 \text{ to } t=1) = -\alpha\Delta\beta - \tfrac{1}{2}\Delta\alpha\Delta\beta$$

which is the common rule in a different form.

Let us now suppose (t being measured from the beginning of the $(n+1)$ th year)

$$\phi t = \alpha + \Delta\alpha . t + \Delta^2\alpha . t\frac{t-1}{2}$$

$$\psi t = \beta + \Delta\beta . t + \Delta^2\beta . t\frac{t-1}{2}$$

where the differences constitute series of rapidly diminishing terms. The only term of the second order which this addition to the hypothesis introduces into $-\int \phi t \, \psi' t . dt$ is $\alpha\Delta^2\beta \int (t-\frac{1}{2}) \, dt$ which is $=0$, when taken from $t=0$ to $t=1$. Consequently the errors of the rule are all of the third order.

To give a notion of the amount of error, extend the preceding formulæ to terms of the third order, and form the integral, reserving only the terms of the third order. The final result is as follows: — If x and y be the number of the living at the age of A and B, and if $a, b, c, \ldots$ be the numbers alive, at $n, n+1, \ldots$ years older than A, and $p, q, r, \ldots$ at $n, n+1, \ldots$ years older than B, then the probability that A shall begin to survive B in the course of the $(n+1)$ th year of the calculation, is

$$\frac{(a+b)(p-q)}{2\,xy} + \frac{(b-c)(p-q)-(a-b)(q-r)}{12\,xy}$$

the first term being that generally used, and the second a correction which ought always to be applied in those parts of the table in which the yearly decrements are not equal.

The demonstration of the preceding will be easily arrived at by the indication which I have given, by any one acquainted with the integral calculus. To those who have not that advantage, reason may be shown *in* the result, though not *for* the result. The preceding correction will have the positive sign when $b-c$ bears a greater proportion to $a-b$ than does $q-r$ to $p-q$: that is, when the mortality in A's part of the table is increasing faster than in B's. Now, *ceteris paribus*, the larger the comparative mortality of the year succeeding a given year, the more likely are the deaths of the latter part of that given year to predominate over those of the former; consequently, the more likely is the death of A, if it happen in that year, to be towards the end of it. But any thing which shows that the death of A is more likely than before to take place later in its year, increases the probability that a survivorship commencing in that year shall be in favour of A, and not of B.

APPENDIX THE FOURTH.

ON THE AVERAGE RESULT OF A NUMBER OF OBSERVATIONS.

THAT I might not further embarrass the most abstruse chapter of this work, by the introduction of an isolated point of difficulty, I have chosen here to mention some considerations connected with the value of the average of observations. There is a remarkable difference of principle between two problems which at first sight appear identical; namely, where it is required to invent a method of treating observations before they are made, and after they are made. Positive and negative errors being equally likely, and no observations having been made, it is easily proved that there is a high probability in favour of a large number of observations giving exactly or nearly the same total amount of one as of the other. The case is analogous to that of an urn filled with black and white balls in equal numbers, out of which, in a large number of drawings, both sorts will come in nearly equal proportions

But in this, as in every other question of probabilities, any additional knowledge of the circumstances which may happen, or have happened, changes the problem, and is equivalent to an extension or limitation of its conditions. When the observations *have been made*, the position of the observer is altered, since though the law of facility of error be not determined, yet more probability is given to some laws than to others, by inspection of the observations themselves. For instance; if the observations give results of very little discordance, it is immediately obvious that a law of facility which makes the probability of large errors very small, is more likely to have been that which actually existed than one of a different character. The problem now presents an analogy with that of an urn, from which drawings have been made and registered, so that the contents of the urn are to be guessed at from the drawings.

In the first problem, and supposing that a method of combining the observations is to be chosen *before observations made*, it is demonstrable that the average of the results is more likely to be true than any other magnitude. And the same conclusion seems probable in the second case, since unassisted common sense would never draw any distinction between the two problems. But the results of calculation applied to the development of the distinction just drawn, show that the average of observations made is not necessarily the most probable result, nor can be such for more than two observations, unless one particular law of facility of error be supposed, which law is the standard law described in Chapter VII. But it is also shown, as mentioned in page 142, that the results of any law of facility, when applied to tolerably large numbers of observations, are nearly identical with those of some variety or other of the standard law; so that, practically, the average of observations is either the result which the strictest application of sound principles would declare to be the most probable truth, or else very near to it.

It is, in the meanwhile, a most remarkable circumstance that a method so simple, and so conformable to common sense, as that of averaging, should first turn out to be incorrect, except upon a supposition never contemplated in thinking of the evidence of this rule, and should afterwards prove to be always nearly correct, for large numbers of observations, on account of the tendency of all admissible suppositions to confound themselves, as the number of observations increases, with that one particular supposition, which makes the common notion absolutely correct. My own impression, derived from this and many other circumstances connected with the analysis of probabilities, is, that mathematical results have outrun their interpretation: and that some simple explanation of the force and meaning of the celebrated integral, whose values are tabulated at the end of this work, will one day be found to connect the higher and lower parts of the subject with a degree of simplicity which will at once render useless (except to the historian) all the works hitherto written.

APPENDIX THE FIFTH.

ON THE METHOD OF CALCULATING UNIFORMLY DECREASING OR INCREASING ANNUITIES.

An authority from which I rarely differ has spoken thus, "A few writers on these subjects, of late years, have employed the differential and integral calculus in their investigations. We have not yet seen any fruits of this application of the calculus, which appear to us of much value, nor are we at all sanguine in expecting any." The tendency of such an assertion is to encourage those who study the subject, to stop short of the differential calculus in their mathematical studies. Now I assert, 1. That the calculus aforesaid may, as evidenced in the results of chapter IV., lead to most valuable rules in the estimation of complicated probabilities. 2. That if the calculus be not serviceable in the deduction of the law of mortality, it is from defect of observed *data*. As soon as larger and more correct tables of the numbers living are obtained, the differential calculus is ready to furnish methods for correcting those now in use. 3. That the differential calculus may be made to give important simplifications of processes, and to render the tables already constructed immediately available for purposes to which no one now dreams of applying them.

If v be the present value of 1*l*., to be received at the end of a year, and ϕv be the present value of a contingent annuity of 1*l*., then that of an annuity which is to be 1*l*. at the end of the first year, 2*l*., 3*l*., &c., at the end of the second, third, &c. years, is $v\phi' v$, where $\phi' v$ is the differential coefficient of ϕv. Now $1+r$ being the amount of 1*l*. in one year, we have

$$\frac{dv}{dr} = -\frac{1}{(1+r)^2}, \quad \frac{d\phi v}{dv} = -\frac{d\phi v}{dr}(1+r)^2$$

and the annuity $$v\frac{d\phi v}{dv} = -\frac{d\phi v}{dr}(1+r)$$

Now tables of annuities of 1*l*. being calculated for a succession of values of r differing by ·01, we have

$$\cdot 01 \times \frac{d\phi v}{dr} = \Delta\phi v - \tfrac{1}{2}\Delta^2\phi v + \tfrac{1}{3}\Delta^3\phi v - \tfrac{1}{4}\Delta^4\phi v + \&c.$$

Substitute the value of $\frac{d\phi v}{dr}$ thence obtained, and we have a method of finding the value of the required annuity, which may be described in the following

RULE.

Take out the value of an annuity of 1*l.* at the given rate of interest, and at several successive higher rates: take the successive differences, the difference of the differences, and so on. To the first difference add half of the second difference, one-third of the third, and so on: the sum of these, multiplied by the amount of 100*l.* in one year at the first named rate, is the value of the annuity required.

I take examples from the Northampton tables, at 4 per cent., because Mr. Morgan has given a table of the annuities required, which will serve to find verifications. First suppose the age to be 5 years.

Annuity	4 p. c.	17·248	2·421	·556	·164	·060
——	5 —	14·827	1·865	·392	·104	
——	6 —	12·962	1·473	·288		
——	7 —	11·489	1·185			
——	8 —	10·304				

2·421 + $\frac{1}{2}$ of ·556 + $\frac{1}{3}$ of ·164 + $\frac{1}{4}$ of ·060 = 2·769
2·769 × 104 = 288·0 answer: in Morgan 288·4

Next suppose the age to be 80 years.

Annuity	4 p. c.	3·643	·128	·007
——	5 —	3·515	·121	·008
——	6 —	3·394	·113	·006
——	7 —	3·281	·107	
——	8 —	3·174		

·128 + $\frac{1}{2}$ of ·007 = ·132 ; ·132 × 104 = 13·7 answer
13·8 in Morgan.

PROBLEM. A life annuity is £m at the end of the first year, and diminishes £n every year, until nothing is due, after which it ceases entirely. Required its present value.

RULE. When n is so small, that the annuity cannot be extinguished during the tabular life of the party, from the value of an annuity of £$(m+n)$ subtract n times that of an increasing annuity of 1*l.* found as already described. But when the annuity can be extinguished during the life of the party (say in t years exactly, so that $m=nt$), then to the preceding result *add* n times the value of an increasing annuity of 1*l.* on a life $t+1$ years older than the party, multiplied by the chance of his living $t+1$ years, and by the present value of £1 due $t+1$ years hence.

Problem. Required the present value of £m to be received at the end of the year in which A dies, if in a year, or £$(m-n)$ if in the second year, and so on.

Rule. When n is so small that the sum insured cannot be extinguished during the tabular life of the party, to the value of a perpetuity of £m, add that of an increasing annuity of £n, £$2n$, £$3n$, &c., and subtract the value of a simple life annuity, of which the yearly payment is £m, increased by the product of a perpetuity due, and the value of a simple annuity of £n: divide the difference by the value of a perpetuity due, and the quotient is the present value required. But if the insurance be extinguished in t years, or if $m=nt$: find the product of an annuity due of £1 on a life $t+1$ years older than the given life, and of a perpetuity; subtract the value of an increasing annuity of 1*l.* on that life, and having multiplied the difference by the chance of the first life surviving $t+1$ years, and by the present value of £n due $t+1$ years hence, *add* the result to the dividend in the first part of the rule, before dividing by the value of a perpetuity due of £1.

Various other questions will present themselves, which can be easily reduced to practice by aid of the expeditious rule for finding the values of increasing annuities. This rule may be applied to the Carlisle tables (for which Mr. Milne has deduced the values of annuities on single lives, at rates of interest from 3 to 8 per cent., both included), and also to joint lives, though not with so much correctness, on account of the tables not containing so many rates of interest.

The following is an instance in which a deduction from the *calculus of differences* will supply in a rough manner the deficiencies of tables. There are none of these for determining the mean duration of the joint existence of two lives, but the defect may be supplied with sufficient accuracy for many purposes, and particularly at the middle and older ages, by the following RULE. Let (3), (4), &c. stand for the values of an annuity on a single life, or on two joint lives, at 3, 4, &c. per cent.: from twice (3) subtract (6) and reserve the remainder: from (4) subtract (5), and having halved the remainder, to it add the tenth part of (5), and multiply the result by 9. Subtract the last product from the reserved remainder, and multiply the difference by 10. The result increased by ·5 in the case of a single life, or by ·25 in that of two joint lives, will be something under the mean duration required. For example, and to take a very unfavourable case, let the Carlisle table be used, the life being 10 years old.

(3) =	23·512	(4) =	19·585
	2	(5) =	16·669
	47·024	2)	2·916
(6) =	14·448		1·458
	32·576	$\frac{1}{10}$ (5)	1·667
	28·125		3·125
	4·451		9
	10		28·125
	44·51		
	·5		
	45·01 the truth being 48·82		

The error diminishes as the age increases, as the following table will show : —

Age.	Approx.	Truth.	Age.	Approx.	Truth.
0	34·78	38·72	60	14·13	14·34
10	45·01	48·82	70	9·33	9·18
20	39·07	41·46	80	5·57	5·51
30	33·09	34·34	90	3·43	3·28
40	27·05	27·61	100	2·36	2·28
50	20·96	21·11			

APPENDIX THE SIXTH.

ON A QUESTION CONNECTED WITH THE VALUATION OF THE ASSETS OF AN INSURANCE OFFICE.

If an insurance office were about to close its doors, and to buy up all the policies of its members, the process of valuation would only require the assets to be expressed by the amount of money which they would actually produce at the time of valuation. In this case, the profit or surplus is properly expressed by $A + P - C$, or the amount of assets increased by the present value of all the premiums, and diminished by that of all the claims.

But valuation is not usually made with reference to an immediate settlement; but for the purpose of ascertaining what sum can be set apart as profit, and declared to belong to existing policies, without anticipatory injustice to future members. The preceding formula, with allowance for expences of management, still represents the sum which may be called profit, *provided that the stock belonging to the office can really be improved at the rate of interest assumed in the valuation.* For the sufficiency of this stock to answer all demands depends upon its increasing at that rate of interest upon which the values of P and C were found.

Now, it generally happens, that the property of an insurance office consists of funds invested at different rates of interest, the consequence of which is, that there is no absolutely rigorous method of determining the profit, except by prospective calculation of the state of the office for every year of the tabular duration of the life of its youngest member. Supposing the insured to die precisely in the manner indicated in the table, and assigning the order in which the different principals are to be touched, when necessary, it is then possible to calculate the amount which will remain when all claims are paid. The present value of this amount (the species of stock in which it is to be left being known) is all that can be called profit at the time of the valuation. This process, however, is exceedingly laborious; and, in all probability, where yearly valuations are made, the expence of making the calculation would be greater than the loss prevented by taking the more simple, but less accurate, method.

If money made only simple interest, and computations were performed accordingly, no difficulty would arise: for £S improving at r per pound, and £S′ at r' per pound, is at all times equivalent to £(S + S′), improving at $(Sr+S'r') \div (S+S')$ per pound: so that all the different stocks might be considered as lying at one average rate of interest. Such, however, is not the case with compound interest.

To introduce the question in a simple form, let us suppose that all the stock of the office makes r per pound, the rate assumed in the valuation, except only one sum, H, which makes r' (less than r) per pound. If, then, this sum were set down as H in the item A, the profit would be overrated; nor can we answer the question, how much should it be estimated at, without some reference to the time at which H, with its accumulations, is to become necessary. If this will not be wanted for n years, then $H \times (1+r')^n \div (1+r)^n$ is the value at which it must be estimated.

The best method of treating this case is to suppose H to stand for such a sum, that there will be no loss arising from a lower rate of interest before the next valuation. Accordingly, in the preceding formula, n must be the number of years intervening between two valuations. If such a process should give too little profit at one valuation, the same item will be larger in the next, and *vice versâ*: so that there will be a continual tendency to correctness. If, for instance, the valuations be made yearly (for which this very circumstance is one reason among many), then $H(1+r') \div (1+r)$ should be taken for H, and the existing policies may have the benefit when r' is greater than r.

TABLE I.

t.	H.	Δ	Δ²	t.	H	Δ	Δ²
0·00	0·00000 00	1128 33	22	0·40	0·42839 22	957 68	7 82
0·01	0·01128 33	1128 11	45	0·41	0·43796 90	949 86	7 95
0·02	0·02256 44	1127 66	67	0·42	0·44746 76	941 91	8 07
0·03	0·03384 10	1126 99	90	0·43	0·45688 67	933 84	8 17
0·04	0·04511 09	1126 09	1 12	0·44	0·46622 51	925 67	8 30
0·05	0·05637 18	1124 97	1 35	0·45	0·47548 18	917 37	8 40
0·06	0·06762 15	1123 62	1 58	0·46	0·48465 55	908 97	8 51
0·07	0·07885 77	1122 04	1 79	0·47	0·49374 52	900 46	8 61
0·08	0·09007 81	1120 25	2 01	0·48	0·50274 98	891 85	8 69
0·09	0·10128 06	1118 24	2 24	0·49	0·51166 83	883 16	8 78
0·10	0·11246 30	1116 00	2 46	0·50	0·52049 99	874 38	8 88
0·11	0·12362 30	1113 54	2 67	0·51	0·52924 37	865 50	8 96
0·12	0·13475 84	1110 87	2 88	0·52	0·53789 87	856 54	9 03
0·13	0·14586 71	1107 99	3 10	0·53	0·54646 41	847 51	9 10
0·14	0·15694 70	1104 89	3 31	0·54	0·55493 92	838 41	9 17
0·15	0·16799 59	1101 58	3 52	0·55	0 56332 33	829 24	9 23
0·16	0·17901 17	1098 06	3 72	0·56	0·57161 57	820 01	9 30
0·17	0·18999 23	1094 34	3 93	0·57	0·57981 58	810 71	9 35
0·18	0·20093 57	1090 41	4 14	0·58	0·58792 29	801 36	9 40
0·19	0·21183 98	1086 27	4 34	0·59	0·59593 65	791 96	9 45
0·20	0·22270 25	1081 93	4 53	0·60	0·60385 61	782 51	9 49
0·21	0·23352 18	1077 40	4 73	0·61	0·61168 12	773 02	9 53
0·22	0·24429 58	1072 67	4 92	0·62	0·61941 14	763 49	9 55
0·23	0·25502 25	1067 75	5 12	0·63	0·62704 63	753 94	9 59
0·24	0·26570 00	1062 63	5 29	0·64	0·63458 57	744 35	9 62
0·25	0·27632 63	1057 34	5 49	0·65	0·64202 92	734 73	9 63
0·26	0·28689 97	1051 85	5 67	0·66	0·64937 65	725 10	9 65
0·27	0·29741 82	1046 18	5 84	0·67	0·65662 75	715 45	9 66
0·28	0·30788 00	1040 34	6 01	0·68	0·66378 20	705 79	9 68
0·29	0·31828 34	1034 33	6 19	0·69	0·67083 99	696 11	9 67
0·30	0·32862 67	1028 14	6 36	0·70	0·67780 10	686 44	9 68
0·31	0·33890 81	1021 78	6 52	0·71	0·68466 54	676 76	9 68
0·32	0·34912 59	1015 26	6 67	0·72	0·69143 30	667 08	9 66
0·33	0·35927 85	1008 59	6 84	0·73	0·69810 38	657 42	9 66
0·34	0·36936 44	1001 75	6 98	0·74	0·70467 80	647 76	9 65
0·35	0·37938 19	994 77	7 14	0·75	0·71115 56	638 11	9 62
0·36	0·38932 96	987 63	7 29	0·76	0·71753 67	628 49	9 61
0·37	0·39920 59	980 34	7 42	0·77	0·72382 16	618 88	9 57
0·38	0·40900 93	972 92	7 55	0·78	0·73001 04	609 31	9 56
0·39	0·41873 85	965 37	7 69	0·79	0·73610 35	599 75	9 52

t.	H.	Δ	Δ²	t.	H	Δ	Δ²
0·80	0·74210 10	590 23	9 48	1·30	0·93400 80	205 52	5 32
0·81	0·74800 33	580 75	9 45	1·31	0·93606 32	200 20	5 22
0·82	0·75381 08	571 30	9 41	1·32	0·93806 52	194 98	5 11
0·83	0·75952 38	561 89	9 36	1·33	0·94001 50	189 87	5 02
0·84	0·76514 27	552 53	9 31	1·34	0·94191 37	184 85	4 93
0·85	0·77066 80	543 22	9 26	1·35	0·94376 22	179 92	4 82
0·86	0·77610 02	533 96	9 21	1·36	0·94556 14	175 10	4 74
0·87	0·78143 98	524 75	9 16	1·37	0·94731 24	170 36	4 63
0·88	0·78668 73	515 59	9 09	1·38	0·94901 60	165 73	4 55
0·89	0·79184 32	506 50	9 04	1·39	0·95067 33	161 18	4 45
0·90	0·79690 82	497 46	8 97	1·40	0·95228 51	156 73	4 35
0·91	0·80188 28	488 49	8 91	1·41	0·95385 24	152 38	4 27
0·92	0·80676 77	479 58	8 83	1·42	0·95537 62	148 11	4 18
0·93	0·81156 35	470 75	8 77	1·43	0·95685 73	143 93	4 09
0·94	0·81627 10	461 98	8 70	1·44	0·95829 66	139 84	3 99
0·95	0·82089 08	453 28	8 61	1·45	0·95969 50	135 85	3 91
0·96	0·82542 36	444 67	8 55	1·46	0·96105 35	131 94	3 82
0·97	0·82987 03	436 12	8 46	1·47	0·96237 29	128 12	3 74
0·98	0·83423 15	427 66	8 39	1·48	0·96365 41	124 38	3 65
0·99	0·83850 81	419 27	8 30	1·49	0·96489 79	120 73	3 57
1·00	0·84270 08	410 97	8 22	1·50	0·96610 52	117 16	3 49
1·01	0·84681 05	402 75	8 13	1·51	0·96727 68	113 67	3 40
1·02	0·85083 80	394 62	8 05	1·52	0·96841 35	110 27	3 32
1·03	0·85478 42	386 57	7 96	1·53	0·96951 62	106 95	3 25
1·04	0·85864 99	378 61	7 86	1·54	0·97058 57	103 70	3 16
1·05	0·86243 60	370 75	7 77	1·55	0·97162 27	100 54	3 09
1·06	0·86614 35	362 97	7 68	1·56	0·97262 81	97 45	3 01
1·07	0·86977 32	355 29	7 60	1·57	0·97360 26	94 44	2 94
1·08	0·87332 61	347 69	7 49	1·58	0·97454 70	91 50	2 86
1·09	0·87680 30	340 20	7 40	1·59	0·97546 20	88 64	2 79
1·10	0·88020 50	332 80	7 31	1·60	0·97634 84	85 85	2 73
1·11	0·88353 30	325 49	7 21	1·61	0·97720 69	83 12	2 64
1·12	0·88678 79	318 28	7 12	1·62	0·97803 81	80 48	2 59
1·13	0·88997 07	311 16	7 01	1·63	0·97884 29	77 89	2 51
1·14	0·89308 23	304 15	6 91	1·64	0·97962 18	75 38	2 45
1·15	0·89612 38	297 24	6 82	1·65	0·98037 56	72 93	2 38
1·16	0·89909 62	290 42	6 72	1·66	0·98110 49	70 55	2 31
1·17	0·90200 04	283 70	6 61	1·67	0·98181 04	68 24	2 26
1·18	0·90483 74	277 09	6 52	1·68	0·98249 28	65 98	2 20
1·19	0·90760 83	270 57	6 42	1·69	0·98315 26	63 78	2 12
1·20	0·91031 40	264 15	6 31	1·70	0·98379 04	61 66	2 08
1·21	0·91295 55	257 84	6 22	1·71	0·98440 70	59 58	2 01
1·22	0·91553 39	251 62	6 11	1·72	0·98500 28	57 57	1 96
1·23	0·91805 01	245 51	6 02	1·73	0·98557 85	55 61	1 90
1·24	0·92050 52	239 49	5 91	1·74	0·98613 46	53 71	1 85
1·25	0·92290 01	233 58	5 81	1·75	0·98667 17	51 86	1 79
1·26	0·92523 59	227 77	5 71	1·76	0·98719 03	50 07	1 75
1·27	0·92751 36	222 06	5 61	1·77	0·98769 10	48 32	1 68
1·28	0·92973 42	216 45	5 52	1·78	0·98817 42	46 64	1 65
1·29	0·93189 87	210 93	5 41	1·79	0·98864 06	44 99	1 59

t.	H.	Δ	Δ²	t.	H.	Δ	Δ²
1·80	0·98909 05	43 40	1 54	2·30	0·99885 68	5 56	25
1·81	0·98952 45	41 86	1 50	2·31	0·99891 24	5 31	24
1·82	0·98994 31	40 36	1 44	2·32	0·99896 55	5 07	23
1·83	0·99034 67	38 92	1 41	2·33	0·99901 62	4 84	23
1·84	0·99073 59	37 51	1 36	2·34	0·99906 46	4 61	20
1·85	0·99111 10	36 15	1 33	2·35	0·99911 07	4 41	21
1·86	0·99147 25	34 82	1 27	2·36	0·99915 48	4 20	19
1·87	0·99182 07	33 55	1 24	2·37	0·99919 68	4 01	19
1·88	0·99215 62	32 31	1 20	2·38	0·99923 69	3 82	18
1·89	0·99247 93	31 11	1 16	2·39	0·99927 51	3 64	17
1·90	0·99279 04	29 95	1 12	2·40	0·99931 15	3 47	16
1·91	0·99308 99	28 83	1 08	2·41	0·99934 62	3 31	16
1·92	0·99337 82	27 75	1 06	2·42	0·99937 93	3 15	15
1·93	0·99365 57	26 69	1 01	2·43	0·99941 08	3 00	14
1·94	0·99392 26	25 68	99	2·44	0·99944 08	2 86	14
1·95	0·99417 94	24 69	95	2·45	0·99946 94	2 72	12
1·96	0·99442 63	23 74	91	2·46	0·99949 66	2 60	14
1·97	0·99466 37	22 83	89	2·47	0·99952 26	2 46	11
1·98	0·99489 20	21 94	85	2·48	0·99954 72	2 35	11
1·99	0·99511 14	21 09	84	2·49	0·99957 07	2 24	12
2·00	0·99532 23	20 25	78	2·50	0·99959 31	2 12	10
2·01	0·99552 48	19 47	79	2·51	0·99961 43	2 02	10
2·02	0·99571 95	18 68	73	2·52	0·99963 45	1 92	9
2·03	0·99590 63	17 95	72	2·53	0·99965 37	1 83	10
2·04	0·99608 58	17 23	69	2·54	0·99967 20	1 73	8
2·05	0·99625 81	16 54	67	2·55	0·99968 93	1 65	8
2·06	0·99642 35	15 87	65	2·56	0·99970 58	1 57	7
2·07	0·99658 22	15 22	61	2·57	0·99972 15	1 50	9
2·08	0·99673 44	14 61	61	2·58	0·99973 65	1 41	7
2·09	0·99688 05	14 00	57	2·59	0·99975 06	1 34	7
2·10	0·99702 05	13 43	55	2·60	0·99976 40	1 27	6
2·11	0·99715 48	12 88	54	2·61	0·99977 67	1 21	6
2·12	0·99728 36	12 34	51	2·62	0·99978 88	1 15	6
2·13	0·99740 70	11 83	50	2·63	0·99980 03	1 09	6
2·14	0·99752 53	11 33	47	2·64	0·99981 12	1 03	5
2·15	0·99763 86	10 86	47	2·65	0·99982 15	98	5
2·16	0·99774 72	10 39	43	2·66	0·99983 13	93	5
2·17	0·99785 11	9 96	44	2·67	0·99984 06	88	4
2·18	0·99795 07	9 52	39	2·68	0·99984 94	84	5
2·19	0·99804 59	9 13	41	2·69	0·99985 78	79	4
2·20	0·99813 72	8 72	36	2·70	0·99986 57	75	4
2·21	0·99822 44	8 36	38	2·71	0·99987 32	71	4
2·22	0·99830 80	7 98	34	2·72	0·99988 03	67	4
2·23	0·99838 78	7 64	33	2·73	0·99988 70	63	2
2·24	0·99846 42	7 31	33	2·74	0·99989 33	61	4
2·25	0·99853 73	6 98	30	2·75	0·99989 94	57	3
2·26	0·99860 71	6 68	30	2·76	0·99990 51	54	3
2·27	0·99867 39	6 38	29	2·77	0·99991 05	51	3
2·28	0·99873 77	6 09	27	2·78	0 99991 56	48	2
2·29	0·99879 86	5 82	26	2·79	0·99992 04	46	3

t.	H.	Δ	Δ²
2·80	0·99992 50	43	2
2·81	0·99992 93	41	3
2·82	0·99993 34	38	1
2·83	0·99993 72	37	3
2·84	0·99994 09	34	1
2·85	0·99994 43	33	2
2·86	0·99994 76	31	2
2·87	0·99995 07	29	2
2·88	0·99995 36	27	1
2·89	0·99995 63	26	2
2·90	0·99995 89	24	1
2·91	0·99996 13	23	1
2·92	0·99996 36	22	1
2·93	0·99996 58	21	2
2·94	0·99996 79	19	1
2·95	0·99996 98	18	1
2·96	0·99997 16	17	0
2·97	0·99997 33	17	2
2·98	0·99997 50	15	1
2·99	0·99997 65	14	-
3·00	0·99997 79	—	-

*** This table has been continued to 3·00 from the data in Kramp's Treatise on Astronomical Refractions. The description of it in the work was written before this addition was made.

TABLE II.

t.	K	Δ	t.	K	Δ	t.	K	Δ
0·00	0·00000	538	0·45	0·23851	513	0·90	0·45618	446
0·01	0·00538	538	0·46	0·24364	512	0·91	0·46064	445
0·02	0·01076	538	0·47	0·24876	512	0·92	0·46509	443
0·03	0·01614	538	0·48	0·25388	510	0·93	0·46952	441
0·04	0·02152	538	0·49	0·25898	509	0·94	0·47393	439
0 05	0·02690	538	0·50	0·26407	508	0·95	0·47832	438
0·06	0·03228	538	0·51	0·26915	506	0·96	0·48270	435
0·07	0·03766	537	0·52	0·27421	506	0·97	0·48705	434
0·08	0·04303	537	0·53	0·27927	504	0·98	0·49139	431
0·09	0·04840	538	0·54	0·28431	503	0·99	0·49570	430
0·10	0·05378	536	0·55	0·28934	502	1·00	0·50000	428
0·11	0·05914	537	0·56	0·29436	500	1·01	0·50428	425
0·12	0·06451	536	0·57	0·29936	499	1·02	0·50853	424
0·13	0·06987	536	0·58	0·30435	498	1·03	0·51277	422
0·14	0·07523	536	0·59	0·30933	497	1·04	0·51699	420
0·15	0·08059	535	0·60	0·31430	495	1·05	0·52119	418
0·16	0·08594	535	0·61	0·31925	494	1·06	0·52537	415
0·17	0·09129	534	0·62	0·32419	492	1·07	0·52952	414
0·18	0·09663	534	0·63	0·32911	491	1·08	0·53366	412
0·19	0·10197	534	0·64	0·33402	490	1·09	0·53778	410
0·20	0·10731	533	0·65	0·33892	488	1·10	0·54188	407
0·21	0·11264	532	0·66	0·34380	486	1·11	0·54595	406
0·22	0·11796	532	0·67	0·34866	486	1·12	0·55001	403
0·23	0·12328	532	0·68	0·35352	483	1·13	0·55404	402
0·24	0·12860	531	0·69	0·35835	482	1·14	0·55806	399
0·25	0·13391	530	0·70	0·36317	481	1·15	0·56205	397
0·26	0·13921	530	0·71	0·36798	479	1·16	0·56602	396
0·27	0·14451	529	0·72	0·37277	478	1·17	0·56998	393
0·28	0·14980	528	0·73	0·37755	476	1·18	0·57391	391
0·29	0·15508	527	0·74	0·38231	474	1·19	0·57782	389
0·30	0·16035	527	0·75	0·38705	473	1·20	0·58171	387
0·31	0·16562	526	0·76	0·39178	471	1·21	0·58558	384
0·32	0·17088	526	0·77	0·39649	469	1·22	0·58942	383
0·33	0·17614	524	0·78	0·40118	468	1·23	0·59325	380
0·34	0·18138	524	0·79	0·40586	466	1·24	0·59705	378
0·35	0·18662	523	0·80	0·41052	465	1·25	0·60083	377
0·36	0·19185	522	0·81	0·41517	462	1·26	0·60460	373
0·37	0·19707	522	0·82	0·41979	461	1·27	0·60833	372
0·38	0·20229	520	0·83	0·42440	459	1·28	0·61205	370
0·39	0·20749	519	0·84	0·42899	458	1·29	0·61575	367
0·40	0·21268	519	0·85	0·43357	456	1·30	0·61942	366
0·41	0·21787	517	0·86	0·43813	454	1·31	0·62308	363
0·42	0·22304	517	0·87	0·44267	452	1·32	0·62671	361
0·43	0·22821	515	0·88	0·44719	450	1·33	0·63032	359
0·44	0·23336	515	0 89	0·45169	449	1·34	0·63391	356

t.	K	Δ	t.	K	Δ	t.	K	Δ
1·35	0·63747	355	1·85	0·78790	246	2·35	0·88705	152
1·36	0·64102	352	1·86	0·79036	244	2·36	0·88857	151
1·37	0·64454	350	1·87	0·79280	242	2·37	0·89008	149
1·38	0·64804	348	1·88	0·79522	239	2·38	0·89157	147
1·39	0·65152	346	1·89	0·79761	238	2·39	0·89304	146
1·40	0·65498	343	1·90	0·79999	236	2·40	0·89450	145
1·41	0·65841	341	1·91	0·80235	234	2·41	0·89595	143
1·42	0·66182	339	1·92	0·80469	231	2·42	0·89738	141
1·43	0·66521	337	1·93	0·80700	230	2·43	0·89879	140
1·44	0·66858	335	1·94	0·80930	228	2·44	0·90019	138
1·45	0·67193	333	1·95	0·81158	225	2·45	0·90157	136
1·46	0·67526	330	1·96	0·81383	224	2·46	0·90293	135
1·47	0·67856	328	1·97	0·81607	221	2·47	0·90428	134
1·48	0·68184	326	1·98	0·81828	220	2·48	0·90562	132
1·49	0·68510	323	1·99	0·82048	218	2·49	0·90694	131
1·50	0·68833	322	2·00	0·82266	215	2·50	0·90825	129
1·51	0·69155	319	2·01	0·82481	214	2·51	0·90954	128
1·52	0·69474	317	2·02	0·82695	212	2·52	0·91082	126
1·53	0·69791	315	2·03	0·82907	210	2·53	0·91208	124
1·54	0·70106	313	2·04	0·83117	207	2·54	0·91332	124
1·55	0·70419	310	2·05	0·83324	206	2·55	0·91456	122
1·56	0·70729	309	2·06	0·83530	204	2·56	0·91578	120
1·57	0·71038	306	2·07	0·83734	202	2·57	0·91698	119
1·58	0·71344	304	2·08	0·83936	201	2·58	0·91817	118
1·59	0·71648	301	2·09	0·84137	198	2·59	0·91935	116
1·60	0·71949	300	2·10	0·84335	196	2·60	0·92051	115
1·61	0·72249	297	2·11	0·84531	195	2·61	0·92166	114
1·62	0·72546	295	2·12	0·84726	193	2·62	0·92280	112
1·63	0·72841	293	2·13	0·84919	190	2·63	0·92392	111
1·64	0·73134	291	2·14	0·85109	189	2·64	0·92503	110
1·65	0·73425	289	2·15	0·85298	188	2·65	0·92613	108
1·66	0·73714	286	2·16	0·85486	185	2·66	0·92721	107
1·67	0·74000	285	2·17	0·85671	183	2·67	0·92828	106
1·68	0·74285	282	2·18	0·85854	182	2·68	0·92934	104
1·69	0·74567	280	2·19	0·86036	180	2·69	0·93038	103
1·70	0·74847	277	2·20	0·86216	178	2·70	0·93141	102
1·71	0·75124	276	2·21	0·86394	176	2·71	0·93243	101
1·72	0·75400	274	2·22	0·86570	175	2·72	0·93344	99
1·73	0·75674	271	2·23	0·86745	172	2·73	0·93443	98
1·74	0·75945	269	2·24	0·86917	171	2·74	0·93541	97
1·75	0·76214	267	2·25	0·87088	170	2·75	0·93638	96
1·76	0·76481	265	2·26	0·87258	167	2·76	0·93734	94
1·77	0·76746	263	2·27	0·87425	166	2·77	0·93828	94
1·78	0·77009	261	2·28	0·87591	164	2·78	0·93922	92
1·79	0·77270	258	2·29	0·87755	163	2·79	0·94014	91
1·80	0·77528	257	2·30	0·87918	160	2·80	0·94105	90
1·81	0·77785	254	2·31	0·88078	159	2·81	0·94195	89
1·82	0·78039	252	2·32	0·88237	158	2·82	0·94284	87
1·83	0·78291	251	2·33	0·88395	155	2·83	0·94371	87
1·84	0·78542	248	2·34	0·88550	155	2·84	0·94458	85

t.	K	Δ	t.	K	Δ	t.	K	Δ
2·85	0·94543	84	3·10	0·96346	60	3·35	0·97615	42
2·86	0·94627	84	3·11	0·96406	60	3·36	0·97657	41
2·87	0·94711	82	3·12	0·96466	58	3·37	0·97698	40
2·88	0·94793	81	3·13	0·96524	58	3·38	0·97738	40
2·89	0·94874	80	3·14	0·96582	56	3·39	0·97778	39
2·90	0·94954	79	3·15	0·96638	56	3·40	0·97817	359
2·91	0·95033	78	3·16	0·96694	55	3·50	0·98176	306
2·92	0·95111	76	3·17	0·96749	55	3·60	0·98482	261
2·93	0·95187	76	3·18	0·96804	53	3·70	0·98743	219
2·94	0·95263	75	3·19	0·96857	53	3·80	0·98962	185
2·95	0·95338	74	3·20	0·96910	52	3·90	0·99147	155
2·96	0·95412	73	3·21	0·96962	51	4·00	0·99302	129
2·97	0·95485	72	3·22	0·97013	51	4·10	0·99431	108
2·98	0·95557	71	3·23	0·97064	50	4·20	0·99539	88
2·99	0·95628	70	3·24	0·97114	49	4·30	0·99627	73
3·00	0·95698	69	3·25	0·97163	48	4·40	0·99700	60
3·01	0·95767	68	3·26	0·97211	48	4·50	0·99760	48
3·02	0·95835	67	3·27	0·97259	47	4·60	0·99808	40
3·03	0·95902	66	3·28	0·97306	46	4·70	0·99848	31
3·04	0·95968	65	3·29	0·97352	45	4·80	0·99879	26
3·05	0·96033	65	3·30	0·97397	45	4·90	0·99905	21
3·06	0·96098	63	3·31	0·97442	44	5·00	0·99926	
3·07	0·96161	63	3·32	0·97486	44			
3·08	0·96224	62	3·33	0·97530	43			
3·09	0·96286	60	3·34	0·97573	42			

THE END.

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De Moivre p. 165.