were at one time called imaginary, and that infinity is sometimes apologetically placed in that category. No doubt also, in popular speech, a cash balance becomes imaginary when it attains the value zero. Still, in careful mathematical speech, a certain definite usage has become established, and departures from this are for the most part the result of negligence.

It is, for instance, generally understood that "imaginary" is equivalent to "unreal," and that "complex" includes the real. When anyone admits these things on one page, and on the next writes "complex" for "imaginary," he is simply following the bad tradition that mars so many older text-books, of *forgetting to allow for the extreme cases.*

It would be easy, but unnecessary, to name some books of this type in which we can never be certain that the positive number system is not going to include zero, and where the inequality $x > y$ is or is not consistent with $x = y$ according to the caprice of the author. In geometry too a great deal of confusion commonly prevails about the limiting cases. Of the many writers who entreat the reader to accept as a triangle one that has a zero angle, how many take the trouble to warn him of the necessity of re-stating some elementary theorems? For instance, a triangle may have two angles equal, but no two sides equal.

The reader will of course not take too literally the "definition" of i as one of the roots of $x^2 + 1 = 0$. How do we know that this equation has two roots, and not more (as in quaternion theory)? The answer is that we have already at this stage decided that a root i exists, and that it shall obey certain formal algebraic laws. It is a consequence of these assumptions that $x^2 + 1 = 0$ only when $x = \pm i$. It is therefore a matter of choice whether we take i to be a one- or a two valued number.[1]

It may be noted that Professor Allen's last suggestion is exactly the opposite of one that is sometimes followed. For example, Harkness and Morley[2] denote by $\sqrt[n]{a}$ *any* nth root of a, and by a^x the exponential function of $x \log a$, where $\log a$ is the *principal* logarithm and hence a^x is one-valued. This plan has one inconvenient consequence: it makes an odd root of a negative number imaginary. It agrees with the general practice of making e^x one-valued.

The discussion by Professors Cajori and Miller arises from Professor Miller's former article on the same subject, and requires no comment.

I. Definitions of Imaginary and Complex Numbers.[3]

By Edward S. Allen, Iowa State College.

In reading certain parts of about 60 texts on algebra—those of Chrystal, Serret, and Weber among them[4]—I have discovered with surprise that the

[1] For the synthetic treatment of complex numbers as pairs of real numbers, see L. E. Dickson, *Elementary Theory of Equations*, p. 21.

[2] *Introduction to Analytic Functions*, London, 1898, p. 24, 168.

[3] Read at the April meeting of the Ohio Section of the Association.

[4] The list includes also books by all members of that subcommittee of the National Committee on Mathematical Requirements which made the admirable report on "Terms and symbols in elementary mathematics." Dickson's *First Course in the Theory of Equations*, published since this discussion was submitted to the editors of the Monthly, conforms to the policy here recommended in its elegant, brief exposition of the complex number system.

treatment of imaginary and complex numbers is in no instance free from logical inconsistency. A discrepancy is usually found before the following four items have been passed:

1. Definition of imaginary numbers,
2. Definition of complex numbers,
3. If a, b, and c are real, and $b^2 - 4ac$ is negative, then the roots of the equation $ax^2 + bx + c = 0$ are imaginary,
4. The sum of two complex numbers is a complex number.

Probably the most common error, one which persisted in books published as late as 1916, is the definition of an imaginary number as an even root of a negative number. A class of numbers which includes $1 + i$ and $\sqrt{3} + i$, but excludes $2 + i$ and $1 + \sqrt{3}\, i$, is of no value in connection with the usual theorems.

Some authors, realizing the strength of this argument, define imaginary (or pure imaginary) numbers as those whose squares are negative. A complex number s then defined as the sum of a real number and an imaginary (or pure imaginary) one. Real numbers are then not complex; yet the authors using these definitions do not balk at the statement that the sum of two complex numbers is always a complex number. $(2 + 3i) + (4 - 3i) = 6$.

Another group of books handles the subject in the following way. A pure imaginary number is defined as the product of i and of a real number (so that 0 is included). Imaginary and complex numbers are then declared to be identical —sums of real and pure imaginary numbers. The theorem about imaginary (or complex) roots of a quadratic equation loses its meaning, real numbers being, by definition, imaginary. This is Weber's procedure, for instance. Serret, in his *Algèbre Supérieure,* argues in a similar way. He uses the word "imaginaire" on page 86, volume 1, to indicate any number in the complex plane, but on page 269, for instance, to indicate one not on the real axis. Likewise Chrystal, on page 222 of his first volume, intends complex numbers to cover the whole plane; on page 134 ("the coefficients in the factors are complex numbers") he wishes them to avoid the axis of reals. And he uses the word "imaginary" with never a definition.

Two text-books I found, which were so careful as to miss rigor but slightly; perhaps their comparative freedom from error should be specifically mentioned. In Wilczynski and Slaught's *College Algebra* I detected only this flaw—that on page 35 complex numbers must avoid both axes, on page 105 they are excluded only from the real axis, while on page 189 they are allowed to occupy any position on the plane. In Eiesland's *Advanced Algebra* there is but this—on pages 64 and 66 it was forgotten that, according to the definitions used, 0 is real, pure imaginary, and complex, but not imaginary.

Now it is time for "constructive suggestions,"—suggestions which will in no case be new. In the first place let i be defined as one of the roots of the equation $x^2 + 1 = 0$. A pure imaginary number is then the product of i and of any real number (including 0). A complex number is the sum of a real number and of a pure imaginary one. Finally, an imaginary number is a complex number which is not real. Linguistically, we don't like to have imaginaries come later than pure imaginaries. Very well, say "neomonic," if you wish. The important thing is that there must be a name to cover all numbers of the complex plane, a second one for those on the vertical axis, and a third for all numbers not on the horizontal axis.

Now that I am offering suggestions to writers of text-books and dictionaries, I will venture on another remark. $\sqrt{x}$ is always called a single-valued function when x is positive or zero, it is often regarded as such when x is negative, but it is undoubtedly double-valued when x is—if I may use the definition in the last paragraph—imaginary. $\sqrt[3]{x}$ is single-valued if and only if x is real, $\sqrt[4]{x}$ only if x is positive. It is perhaps inevitable, it is surely bewildering, that the same symbol should indicate, now a single-valued, now a multiple-valued function. There is a need for a symbol which shall always indicate that we may take our choice among all the nth roots of x. Should we not agree that $x^{1/n}$ shall be that symbol? Should not future books say that $4^{1/2} = \pm \sqrt{4} = \pm 2$?

II. The Formula $\frac{1}{2}a\,(a+1)$ for the Area of an Equilateral Triangle.

A Reply to Professor Miller by Florian Cajori, University of California.

In this Monthly (*1921,* 257) Professor G. A. Miller writes on Gerbert's explanation of the question why $\frac{1}{2}a(a+1)$ gives too large a value for the area of an equilateral triangle; Professor Miller claims that Cantor's figure is "inaccurate" and then states:

"What is more important is the fact that the corresponding figure found in various histories is still more misleading since it represents according to the explanations in the text an isosceles triangle whose base is equal to the altitude, while the text itself relates to an equilateral triangle. This fact can be verified by consulting either edition of Cajori's *History of Elementary Mathematics,* 1896 or 1917, p. 132."

Professor Miller is in error; the figure in the 1917 edition does not represent "according to the explanations in the text an isosceles triangle whose base is equal to the altitude"; my figure, as well as my explanation, fit (as they should) the case of an equilateral triangle.

One is astonished at Professor Miller's declaration that Bubnov in his edition of Gerbert's *Opera mathematica* gives "a correct figure," and that those of Cantor, Günther and Cajori are all inaccurate or misleading. In the first place, Bubnov gives *two* figures. In the second place, Professor Miller misses completely the essential fact that we do not possess Gerbert's own drawing, and that the drawings in our histories are necessarily conjectural. Neither of Bubnov's two figures agrees with the figure given in Pez's edition of Gerbert's geometry and in the reprint of Pez by Migne in 1880. Pez's figure is due to some unskilful scribe of the Middle Ages whose copy of that geometry contains errors both in the text