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ELEMENTARY AND ADVANCED TRIGONOMETRY

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ELEMENTARY AND ADVANCED TRIGONOMETRY

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PREFACE

The subject of trigonometry is an old one, and the number of books in the field is legion. Thus when an author decides to write a new text on the subject he must justify his action. Our main justification lies in Part II of this book, which will be discussed below. We also believe we have some useful features in Part I.

Part I covers all the usual topics of elementary trigonometry. However, we have kept the findings of the Commission on Mathematics in mind, as well as our own experience in college teaching. In Chapter 1 we mention angles, the sine, and the cosine on page 3. After all, if this is to be a book on trigonometry, the student is entitled to see (at least in restricted form) the definitions of the basic trigonometric functions at as early a stage as possible. This introductory section, however, is followed immediately by a discussion of coordinate systems, functions, the distance formula, and the important subject of inequalities. It is not our intention to give a course in analytic geometry. But we believe that the basic notions of cartesian geometry, straight lines, and circles, as well as the definition of the absolute value, may be advantageously introduced at a very early stage.

Chapter 2 is devoted to the trigonometric functions, both general and special, and to a treatment of the basic identities. The most important result of this chapter is the graphs of the six trigonometric functions in all quadrants. We exhort the student to keep these plots in the forefront of his mind at all times. Chapter 3 is concerned with trigonometric terminology, radian measure, and frequently used terms such as "sinusoidal," "periodic," "phase," "amplitude," "frequency," "wave form." We also discuss small angles and give a proof of the result: $\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1$.

A careful proof of the addition formula and the usual half-angle and double-angle formulas appear in Chapter 4. Also, as an optional topic we discuss at some length the Axiom of Induction—which is used at various points later throughout the book Inverse functions receive a careful and detailed treatment in Chapter 5 The addition formulas for the inverse sine, cosine, and tangent are derived We also include a section on polar coordinates

Trigonometric identities and trigonometric equations are treated in Chapters 6 and 7 respectively Identities involving inverse functions and equations involving both trigonometric *and* algebraic functions are included The solution of triangles is de-emphasized Nevertheless, the complete repertoire is contained in Chapter 8, unencumbered by the use of logarithms In Chapter 9 we discuss practical problems and techniques The techniques are the use of logarithms and the slide rule The practical problems include applications to surveying, navigation, and mechanics We also include a brief introduction to vector analysis in two dimensions

Throughout all the chapters we give numerous examples (worked out in detail) illustrating the theory as it is developed Exercises are listed at the end of almost every section The authors believe that, while the theoretical or functional approach to trigonometry should be pursued, no student can become an expert in trigonometry without incessant drill in the manipulation of trigonometric functions We reserve the more meaty problems (word problems) for the end of each chapter Thus the student is exposed to copious examples, exercises, and problems

After a student has mastered the elements of trigonometric theory he may naturally ask the question What can we do with trigonometry except manipulate identities and solve triangles? In Part II we try to introduce the advanced student to some of the deeper properties of the trigonometric functions Chapter 10 on sums, areas, and tangents solves some interesting problems from the domain of finite differences, integral calculus, and differential calculus respectively We do not try to give a course in calculus We recognize the limitations of the background of the average student Yet we feel that the inquisitive student can understand and appreciate the results we obtain and the methods we use

Complex numbers complete with De Moivre's theorem and the roots of complex numbers appear in Chapter 11 Perhaps the conceptually most difficult chapter is on limits, Chapter 12 We warn the student in advance and try to deduce the essential features with a minimum of abstract language Yet at the same time we try never to be sloppy in our proofs and clearly mention all the subtleties involved Chapter 12 is a prelude to the analytic definitions of the trigonometric functions which appear in Chapter 13 The construction of tables and Euler's formula are also part of this chapter

The analogy of the hyperbolic functions to the circular functions leads us to a discussion of hyperbolic trigonometry in Chapter 14 The inverse hyperbolic functions are also treated as well as such problems as finding an "angle" whose sine is a given complex number

It is interesting to note that certain elementary properties of Fourier series can be developed without recourse to the calculus. Using the results of Chapter 10 we actually formally derive some elementary Fourier series. Numerical methods, in particular the Runge scheme, do not seem too advanced to be presented at this level. Finally, the Tschebyscheff polynomials are easily defined in terms of the inverse cosine. We explain some of their elementary properties in Chapter 16. We also include a proof of the approximation theorem to the effect that the normalized Tschebyscheff polynomial of degree n has smallest maximum absolute value of any other normalized polynomial of degree n.

The authors express their sincere appreciation to Mr. W. A. Miller for the preparation of the many detailed diagrams, to Mr. B. G. Stephan who solved all the exercises, and to Professor C. A. Hutchinson for meticulously reading and correcting the entire manuscript.

K. S. M.
J. B. W.

November, 1961

I

ELEMENTARY THEORY

1
INTRODUCTION

Let A, B, C be the vertices of a right triangle with C at the right angle (see Fig. 1.1). The angle $\measuredangle ABC$ will, for convenience, simply be called the angle B. Similarly, we define A as $\measuredangle BAC$, while C is the right angle. We shall also let a, b, c be the lengths of the sides opposite the angles A, B, C respectively. Then the "sine of A" is defined as the ratio a/c and is written

$$\sin A = \frac{a}{c}; \tag{1}$$

the "cosine of A" is defined as the ratio b/c and is written

$$\cos A = \frac{b}{c}. \tag{2}$$

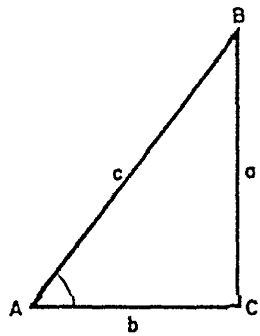


Fig. 1.1

Thus with this elementary picture of a right triangle, which the reader learned about in arithmetic courses long before he studied plane geometry, we can introduce the *trigonometric functions* sine and cosine. A study of trigonometry can be summed up as a study of the trigonometric functions sine and cosine—and their ramifications.

After reading the above introductory paragraph, the student might well flip through the 343 pages of this book and ask himself: If the authors are only going to discuss (1) and (2), what could possibly be in the remaining 340 pages? We ask the reader to be patient and bear with us as the exciting and useful properties of the trigonometric functions, from the most basic to the most esoteric, are carefully unfolded.

Looking at Fig. 1.1, one recalls that a simple relation exists among the sides of a right triangle, namely, the familiar Pythagorean theorem,

$$a^2 + b^2 = c^2.$$

If we divide by c^2, we obtain

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1. \tag{3}$$

But from (1) and (2), a/c is nothing more than $\sin A$, and b/c is $\cos A$ Thus (3) leads immediately to the identity*

$$\sin^2 A + \cos^2 A = 1 \tag{4}$$

This is perhaps the most basic and most widely used identity in the whole of trigonometry

Equation (4) is indeed a remarkable formula from which we may deduce a number of facts For example, we can conclude that†

$$-1 \leqq \sin A \leqq +1 \tag{5}$$

First we write (4) as

$$\sin^2 A = 1 - \cos^2 A,$$

and, since $\cos^2 A \geqq 0$ (for the square of any real number is nonnegative and by definition $\cos A$ is a real number), we see that

$$\sin^2 A \leqq 1$$

Thus the truth of (5) is evident We see, therefore, that the sine of an angle can never be less than -1 nor greater than $+1$

So far we seem to be making progress in our study of trigonometric functions, and one might suppose that we should continue, immediately, with the further investigation of their properties Although this would plunge us into the subject matter of trigonometry, it is not the most efficient method of attack It is true we could deduce many more fascinating properties of the trigonometric functions by proceeding along the lines outlined above But we would meet, at various stages, rather fundamental obstacles because of our meager background Thus, in order to pave the way for a smooth approach, we shall backtrack a bit and consider certain cognate material indispensable to a thorough understanding of trigonometry

Before embarking on these "digressions," we would like to make two more observations First, the student who is interested in applications of trigonometry must wait a few chapters After all, how are we to use the tools of

---

* Perhaps we should explain the distinction between an *identity* and an *equation* (A detailed analysis of identities and equations appears in Chapters 6 and 7) When we write, for example $x^2 + 3x + 2 = 0$ and say it is an equation we mean that we are searching for a number x such that $x^2 + 3x + 2$ is zero For example, $x = -1$ is such a number since $(-1)^2 + 3(-1) + 2$ *is* zero In this case -2 is also a suitable value for x since $(-2)^2 + 3(-2) + 2$ is also zero Thus we see that $x^2 + 3x + 2$ will equal zero if x is -1 or -2, but that in general $x^2 + 3x + 2 \neq 0$ However, $\sin^2 x + \cos^2 x = 1$ is an *identity* since no matter what value of x we choose, $\sin^2 x + \cos^2 x - 1$ is *always* equal to zero Thus $\sin^2 x + \cos^2 x - 1 = 0$ is called an *identity* and $x^2 + 3x + 2 = 0$ is called an *equation* For emphasis when writing identities we frequently write $\equiv$ rather than $=$, that is, the identity of (4) is often written

$$\sin^2 A + \cos^2 A \equiv 1$$

† The notation $x > a$ is read "x is greater than a" Thus, for example $x > 0$ means that x is positive Similarly, $x < a$ is read x is less than a ' For example $x < 0$ means x is negative The symbol $x \geqq a$ means that x is greater than or equal to a, for example, $x \geqq 0$ means x is nonnegative The symbol $x \leqq a$ is defined in the expected fashion

trigonometry until they have been forged? Second, and most important, trigonometry has little to do with angles. This may come as a shock to some. For we defined the trigonometric functions sine and cosine in terms of angles; but we also said that trigonometry was a study of *trigonometric functions.* It will turn out that, except for the solution of triangles (see Chapter 8) and certain elementary applications to surveying and navigation, the applications of the trigonometric functions in mathematics and science do not involve angles at all. For some of these uses we refer the reader to Part II of this book. In particular, we shall be able to give (cf. Chapter 13) a definition of the trigonometric functions that makes no mention of angles. However, since the student has some familiarity with angles, circles, etc., it seems best, from a pedagogical point of view, to introduce first the trigonometric functions through the easily visualized notion of angle.

EXERCISE 1-0

Find the cosines of the angles whose sines are given below:

1. 0.7.
2. 0.3.
3. 0.8.
4. 0.99.
5. 0.1.
6. 0.6.
7. 0.5.
8. 0.866.
9. 0.707.
10. 0.4.

Determine the hypotenuse of each right triangle when the lengths of the legs are as follows:

11. 3, 4.
12. 12, 5.
13. 7, 9.
14. 6, 6.
15. 5, 11.
16. 0.5, 1.2.
17. 0.3, 0.4.
18. 15, 18.
19. 4, 1.
20. 1.000, 1.732.

Compute the length of the remaining leg of each right triangle:

21. $a = 5 \quad c = 7.$
22. $a = 1 \quad c = 4.$
23. $b = 8 \quad c = 9.$
24. $a = 2 \quad c = 3.$
25. $b = 1 \quad c = 2.$
26. $b = 1.000 \quad c = 1.414.$
27. $a = 2.000 \quad c = 2.236.$
28. $b = 12 \quad c = 13.$
29. $b = 0.1 \quad c = 0.2.$
30. $b = 1.1 \quad c = 1.2.$

1.1. Angles

We recall a few facts from plane geometry. The opening between two intersecting straight lines is called an *angle*—denoted by the Greek letter theta (θ) in Fig 1 2a Of course θ' of Fig 1 2b also represents an angle We intuitively see that the angle θ' is greater than the angle θ How did we arrive at this conclusion? We simply displaced Fig 1 2b until $O'A'$ coincided with the line OA as in Fig 1 2c Clearly, then, θ' is larger than θ Now the next

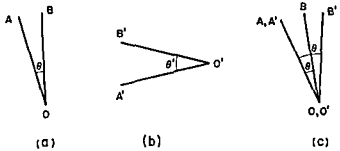


Fig 1 2

question we ask is Do we have a *quantitative measure* of the size of an angle? Again we recall that the answer is yes, we measure an angle in *degrees* A degree is $\frac{1}{360}$ of a circle That is, it is customary to divide the circumference

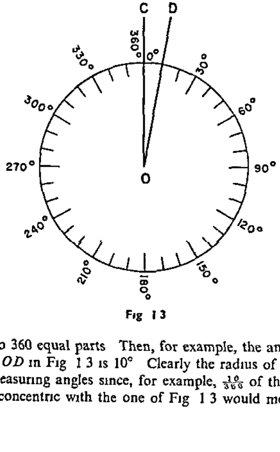


Fig 1 3

of a circle into 360 equal parts Then, for example, the angle between the lines OC and OD in Fig 1 3 is 10° Clearly the radius of the circle is immaterial in measuring angles since, for example, $\frac{10}{360}$ of the circumference of any circle concentric with the one of Fig 1 3 would measure the same opening. Thus if we wished to find how many degrees the angle θ of Fig. 1.2a represented, we could take a calibrated circle such as indicated in Fig. 1.3 and superimpose it on Fig. 1.2a with OC coinciding with OA (see Fig. 1.4). We see in this case that θ is a little more than 15°. A mechanical device that performs this feat is called a *protractor*. With a protractor we can find, for example, that the angle θ' of Fig. 1.2b is approximately 27°.

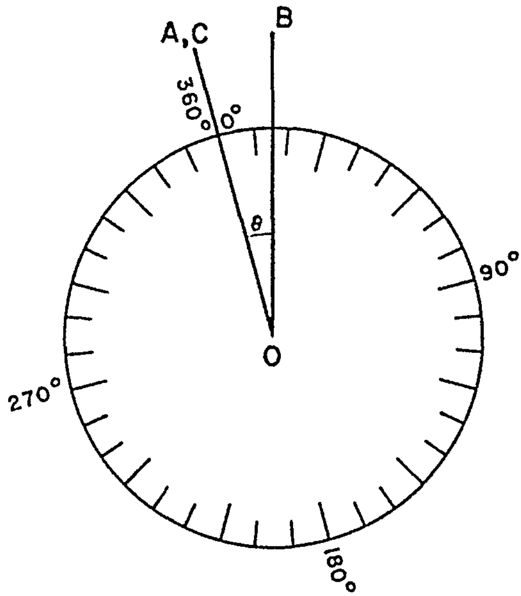


Fig. 1.4

Theoretically, one can measure an angle to any desired degree of accuracy. One way would be to use decimal fractions of a degree. For example, the angle θ of Fig. 1.2a might be 15.235°. Another standard convention is to measure the angle in *minutes* and *seconds*. A minute is defined as $\frac{1}{60}$ of a degree and a second is defined as $\frac{1}{60}$ of a minute. There are no accepted units for additional subdivisions, such as $\frac{1}{60}$ of a second. If we desire to measure an angle more accurately than to the nearest second, we use decimal fractions of a second. For instance, suppose the angle θ of Fig. 1.2a were exactly 15.235°. Let us write it in terms of minutes and seconds. If x represents the number of minutes, then

$$x = (0.235)(60) = 14.1 \text{ minutes},$$

and we could write

$$\theta = 15° \ 14.1'.$$

But 0.1 minutes is $(0.1)(60) = 6$ seconds. Thus

$$\theta = 15° \ 14' \ 6''.$$

Conversely if we were given, say, the angle

$$\theta = 27° \; 18 \; 25'',$$

then

$$18 = \frac{18}{60} = 0\;3°,$$

and

$$25'' = \frac{25}{(60)(60)} = 0\;00694°,$$

and

$$\theta = 27\;30694°$$

Note that if two lines are perpendicular to each other they form a *right angle* which is 90° (see Fig 1 5) If the lines OA and OB are collinear they

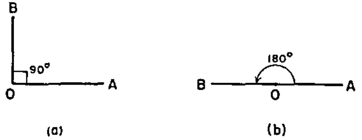


Fig 1 5

form a *straight angle*, which is 180° Angles less than 90° are called *acute angles* and angles greater than 90° are called *obtuse angles*

EXERCISE 1 1

Convert the following angles to degrees and decimal parts of a degree

1 14 15 36″
2 87° 45 9″
3 61° 14 16″
4 175° 11 2″
5 315° 3 14″
6 201° 22 10″
7 255° 51 48″
8 140 37 39″
9 77° 59 57
10 328° 45 26″

Express the following angles in degrees minutes and seconds

11 62.515°
12 107 255°
13 309 215°
14 281 672°
15 341 093°
16 176 32
17 209 44°
18 63 88°
19 94 19°
20 11 76°

1.2. Coordinate Systems

A convenient method of describing points, angles, curves, and regions makes use of *coordinate systems.* Coordinate systems are used throughout mathematics from arithmetic onward. The fundamental idea, probably already familiar to many readers, is very simple. Suppose we consider two perpendicular lines Ox and Oy drawn on a piece of graph paper (see Fig. 1.6). The line Ox is called the *x-axis* and the line Oy the *y-axis.* The point O is called the *origin.* Suppose we label distances along the x-axis using some convenient

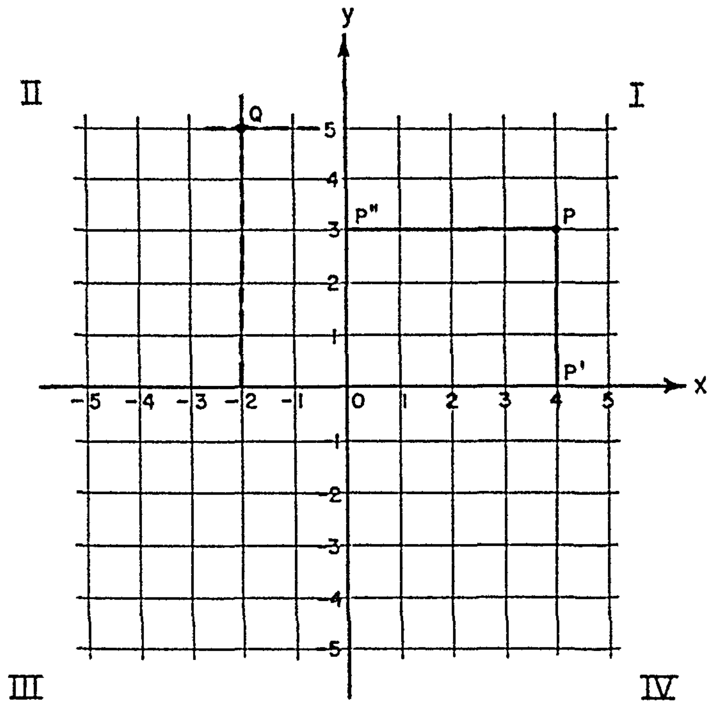


Fig. 1.6

scale; for example, let each square represent one unit. Distances to the right of the origin will be labeled "plus" and distances to the left will be labeled "minus." Similarly, distances above the x-axis will be labeled positively, and those below, negatively.

Now suppose P is any point in the plane. If we drop perpendiculars from P to the x-axis and y-axis, then the distance OP' is called the *abscissa* of P and the distance OP'' is called the *ordinate* of P. In our example, $OP' = 4$ and $OP'' = 3$. The pair of numbers (4,3) thus uniquely describes the point P. This pair of numbers is called "the coordinates of P." One *always* writes the abscissa first: The pair (4,3) does *not* represent the same point as the pair (3,4). (Query: What point *does* (3,4) represent?)

Suppose, on the other hand, we are given the pair of numbers $(-2,5)$ and told that it represents a point in the plane Where does this point appear, geometrically, with respect to the coordinate axes of Fig 1 6? This is easy At the point -2 on the x-axis draw a line perpendicular to the x-axis and at the point 5 on the y axis draw a line perpendicular to the y-axis They intersect at the point Q Thus Q is the point with coordinates $(-2,5)$ Thus we see, given any point in the plane, that we can represent it by a pair of numbers with respect to some coordinate system Conversely, given any pair of numbers with respect to some coordinate system we can determine a point in the plane

The coordinate axes divide the plane into four regions which we have labeled I, II, III, IV as in Fig 1 6 These regions are called *quadrants* Thus I is the first quadrant, II is the second quadrant, etc Note that they are labeled *counter*clockwise The description we have just given is geometric Let us now give an algebraic definition of quadrant The first quadrant consists of all points (a,b) where a and b are both positive The second quadrant consists of all points (a,b) where a is negative and b is positive The third quadrant consists of all points (a,b) where a is negative and b is negative The fourth quadrant consists of all points (a,b) where a is positive and b is negative If $a = 0$, then $(0,b)$ consists of all points on the y-axis between the first and second quadrants (if b is positive) and all points on the y-axis between the third and fourth quadrants (if b is negative) Similarly, $(a,0)$ consists of all points on the x-axis between the first and fourth quadrants (if a is positive) and all points on the x-axis between the second and third quadrants (if a is negative) If a and b are both zero, then $(0,0)$ is simply the origin

EXERCISE 1 2

Each of the following problems lists the coordinates of three points Plot each point on a sheet of graph paper Draw a line connecting the first point with the second, draw one connecting the second with the third Measure the angle formed by these lines

1. (1,1), (0,0), (0,1)
2. (−1,2), (0 1), (1,1)
3. (4,3), (−3,2), (5,−1)
4. (5,7), (4 5), (6,7)
5. (−1,1) (4,−3), (5 2)
6. (−4,−8), (−2,−5), (2,−7)
7. (−3,8) (−3,5), (2,3)
8. (−2,−4) (4,3), (−2,−2)
9. (3,−1) (−3 −3) (4,−2)
10. (−3,−2) (−4,5) (4 −1)

1.3. Functions

By a *function* of x, written $y = f(x)$, we shall mean a rule that associates with a numerical value of x (the *independent* variable) a new number which we call y (the *dependent* variable). For example, suppose the rule is: Associate with every number its cube decreased by four times itself. Then we write

$$y = f(x) = x^3 - 4x.$$

If, for example, $x = 1$, then $y = f(1) = (1)^3 - 4(1) = -3$. If $x = -2.1$, $y = f(-2.1) = (-2.1)^3 - 4(-2.1) = -9.261 + 8.4 = -0.861$. In general, x cannot be unrestricted. For example,

$$y = \sqrt{x}$$

is defined only for $x \geqq 0$. The set of admissible values that x may take on is called the *domain* of the function. Generally, it is fairly clear what our domain of definition must be in any given problem.

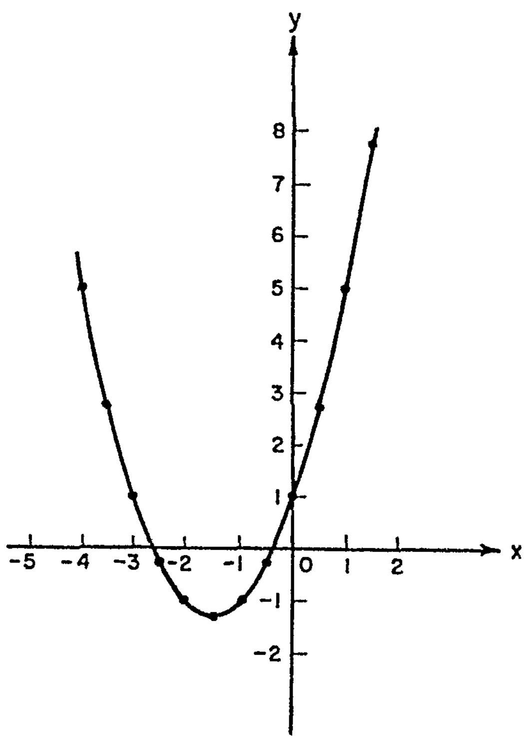


Fig. 1.7

One way to represent functions is to *plot* them on a piece of graph paper with respect to a set of coordinate axes. For example, consider the function

$$y = x^2 + 3x + 1. \tag{6}$$

If we construct a table of values as illustrated on page 12 and plot the pairs of points (x,y) as in Fig. 1.7 we obtain the *graph* of the function.

We sometimes call it a *curve*, or the *locus* of the equation $y = x^2 + 3x + 1$

x	y
−4	5
−3 5	2 75
−3	1
−2 5	−0 25
−2	−1
−1 5	−1 25
−1	−1
−0 5	−0 25
0	1
0 5	2 75
1	5
1 5	7 75

One can also use our coordinate system to find the equations of certain curves For example, consider a circle of radius r with center at the origin drawn on the coordinate system of Fig 1 8 Then if (x,y) represents the coordinates of any point on the circumference, we see by the Pythagorean theorem that

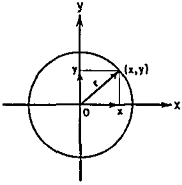


Fig 1 8

$$x^2 + y^2 = r^2, \tag{7}$$

which is the equation of the *circle*

Note that the function of (6) is *single valued* that is, for every value of x we get only *one* value of y However, if we consider (7) as defining y as a function of x, then this function is *not* single valued since for every value of x we get two values of y, namely, $y = +\sqrt{r^2 - x^2}$ and $y = -\sqrt{r^2 - x^2}$ The geometric interpretation of these remarks is If every line parallel to the y-axis cuts the curve in at most one point, then the function is single valued Clearly this is the case in Fig 1 7, but certainly not in Fig 1 8 (In connection with the circle we see how naturally the domain of the function is restricted to the values of x that lie between $-r$ and $+r$)

Perhaps the simplest curve we can imagine is the *straight line* Can we find its equation? We know, geometrically, that two points determine a line Thus, for example, let us attempt to find the equation of the line passing through the points (2,1) and (4,3) (see Fig 1 9) Let (x,y) be the coordinates of any point R on the straight line Then from the similar triangles PQS and QRQ we have the proportion

$$\frac{QR}{SQ} = \frac{QQ}{PS} \tag{8}$$

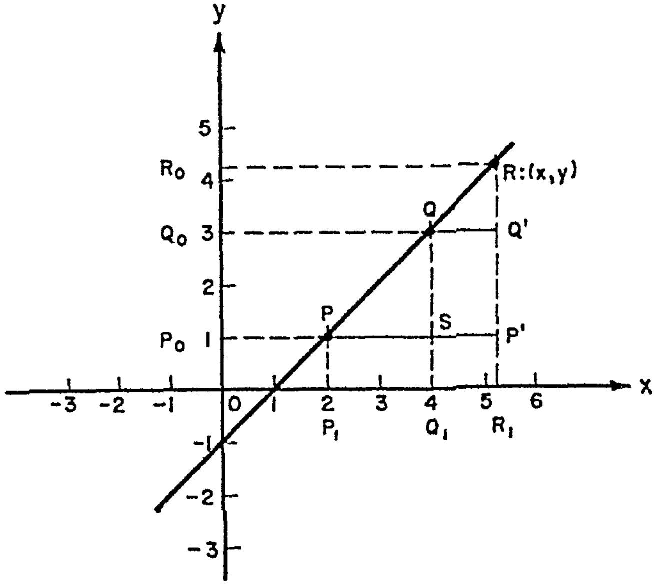


Fig. 1.9

But

$$Q'R = R_1R - R_1Q' = y - 3,$$
$$SQ = Q_1Q - Q_1S = 3 - 1 = 2,$$
$$QQ' = Q_0Q' - Q_0Q = x - 4,$$
$$PS = P_0S - P_0P = 4 - 2 = 2.$$

Thus (8) becomes

$$\frac{y-3}{2} = \frac{x-4}{2},$$

or

$$y = x - 1 \tag{9}$$

is the equation of the straight line.

We leave it to the student to show that if a straight line passes through the two points (a,b) and (c,d), then its equation is

$$\frac{y-b}{b-d} = \frac{x-a}{a-c} \tag{10}$$

(provided the line is not parallel to one of the coordinate axes), or

$$y = \frac{b-d}{a-c}x + \frac{ad-cb}{a-c} \tag{11}$$

(provided the line is not parallel to the y-axis).

EXERCISE 1-3

Graph the following functions of x for the range of values indicated

1 $f(x) = x^2$ $\quad -4 < x < +4$
2 $f(x) = x^3$ $\quad -3 < x < +3$
3 $f(x) = x^2 - 3x + 2$ $\quad 0 < x < 4$
4 $f(x) = x^2 - x - 2$ $\quad -3 < x < 4$
5 $f(x) = x^2 - \frac{1}{4}$ $\quad -1 < x < +1$

Determine the equation of the straight line which passes through the following pairs of points

6 (1 1) (8 8)
7 (−1 4) (7 2)
8 (3 −2) (6 5)
9 (3 5) (6 −3)
10 (4 1) (−3 2)

1 4 The Distance Formula

Suppose $(a\ b)$ and $(c\ d)$ are the coordinates of two points What is the distance between these points? This is an easy question to answer yet the result we obtain (13) has wide applications Suppose that we consider the

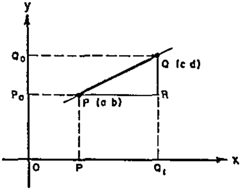


Fig 1 10

coordinate system of Fig 1 10 The distance D between P and Q is evidently given by the Pythagorean theorem as

$$D = \sqrt{\overline{PR}^2 + \overline{RQ}^2} \tag{12}$$

But just as in the previous section

$$PR = P_1Q_1 = OQ_1 - OP_1 = c - a$$
$$RQ = P_0Q_0 = OQ_0 - OP_0 = d - b$$

and from (12),

$$D = \sqrt{(c-a)^2 + (d-b)^2}. \tag{13}$$

One easily sees that the distance formula is valid regardless of the quadrant or quadrants in which P and Q lie. For example, the distance between the points $(-0.5,-2)$ and $(1,-0.3)$ is

$$D = \sqrt{(-0.5-1)^2 + (-2-(-0.3))^2} = \sqrt{2.25 + 2.89}$$
$$= \sqrt{5.14} = 2.267.$$

EXERCISE 1-4

Determine the distance between each of the following pairs of points:

1. (3,4), (7,7).
2. (1,2), (6,14).
3. (−2,−3), (6,5).
4. (−4,−7), (5,−6).
5. (3,−2), (−4,1).
6. (−3,4), (5,6).
7. (4,7), (5,−5).
8. (1,−7), (3,1).
9. (5,2), (−5,2).
10. (6,8), (4,−3).

1.5. Inequalities

We mentioned at the beginning of Section 1.2 that coordinate systems could also be used to represent *regions*. For example, the *right-half plane* (the shaded area of Fig. 1.11) is the set of all points such that $x > 0$. The third quadrant may be described by the inequalities $x < 0$, $y < 0$ (Fig. 1.12).

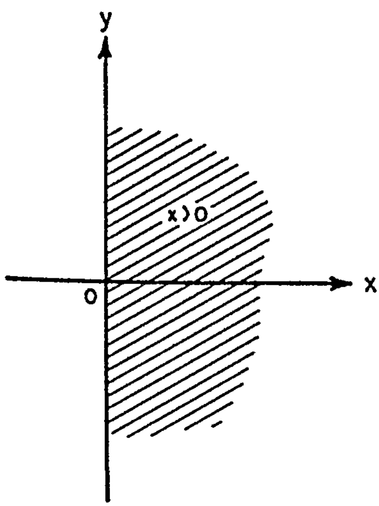


Fig. 1.11

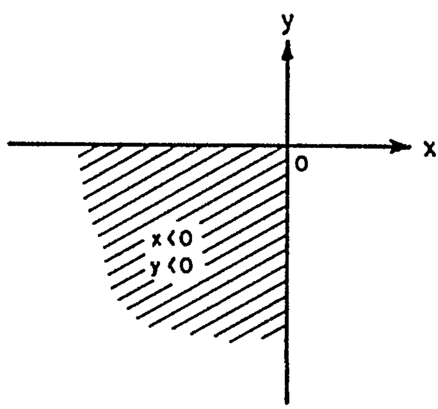


Fig. 1.12

Let us consider some nontrivial examples What region in the plane is described by the inequality $y > x + 2$? First we plot the line $y = x + 2$ (Fig 1 13) Then if we choose any point *above* this line, say (x_0,y_0), we see

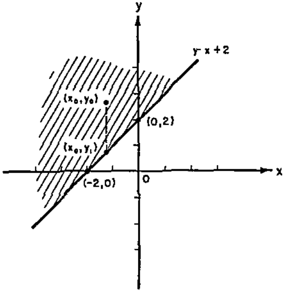


Fig. 1.13

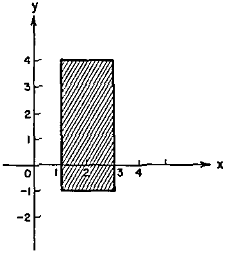


Fig. 1.14

that $y_0 > y_1$ But $y_1 = x_0 + 2$ Hence $y_0 > x_0 + 2$ The region $y > x + 2$ is therefore represented by the shaded area of Fig 1 13 The region described by $1 < x < 3$, $-1 < y < 4$ is illustrated in Fig 1 14 The set of points that simultaneously satisfies the three inequalities $y < x$, $y > 1$, $y < -x + 4$ is the shaded region of Fig 1 15

If x is any real number, then the *absolute value* of x is written as $|x|$ and is defined by the equations

$$|x| = x \quad \text{if} \quad x \geqq 0,$$
$$|x| = -x \quad \text{if} \quad x < 0.$$

Thus $|x|$ is always nonnegative and $|x| = |-x|$. For example,

$$|3| = 3,$$
$$|-2| = 2,$$
$$|0| = 0.$$

The region of Fig. 1.16 can be succinctly described by the inequality $y > |x|$. The shaded square of Fig. 1.17 is the set of all points (x,y) such that $|x| < 1$ and $|y| < 1$.

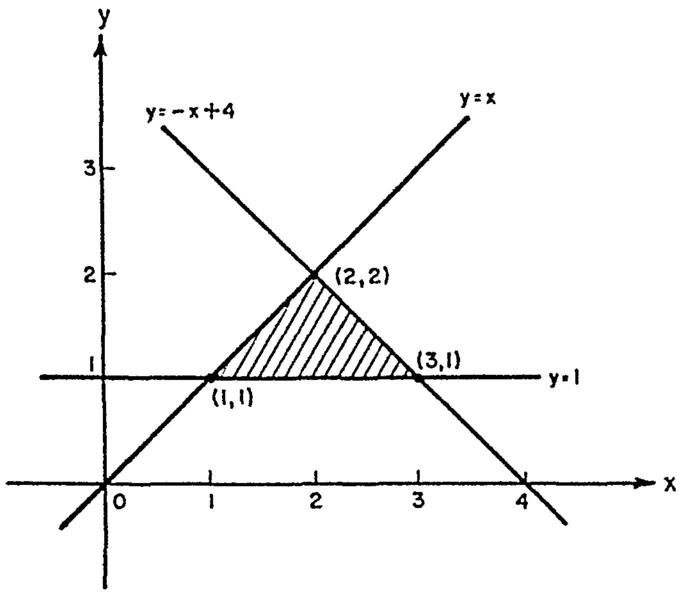


Fig. 1.15

A fundamental inequality in elementary mathematics is the *triangle* inequality which states that for any real numbers x and y

$$|x + y| \leqq |x| + |y|. \tag{14}$$

If either x or y (or both) are zero, or if x and y are both positive or both negative, then the truth of (14) is clear. Two other cases need be considered: $x > 0$ and $y < 0$; $x < 0$ and $y > 0$. Suppose $x > 0$ and $y < 0$. Then

$$x + y < x + (-y). \tag{15}$$

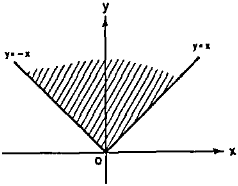


Fig. 1.16

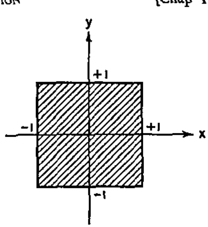


Fig. 1.17

But $|x| = x$, $|y| = -y$, and $x + y = \pm|x + y|$. (The plus sign is used if $x > -y$ and the minus sign if $x < -y$ Then (15) becomes

$$\pm|x + y| \leqq |x| + |y|$$

Clearly, (14) is still valid Similarly, we establish (14) if x is negative and y positive

EXERCISE 1-5

Indicate the regions described by the following inequalities

1. $x > -1$
2. $y \geqq 2$
3. $x < y - 7$
4. $y > x^2$
5. $x^2 + y^2 > 1$

Indicate the region described by each of the following sets of inequalities

6. $y < 7$ and $y \geqq 3$
7. $x^2 + y^2 > 1$ and $x > 0$
8. $x + y < 0$ and $x^2 < 1 + y$
9. $4x > y$ and $3 + 2x < y$ and $x < 3$
10. $x^2 > y$ and $y^2 > x$

Describe by elementary inequalities the following regions:

11. the upper half plane
12. the second quadrant
13. the interior of the triangle with vertices (0,0), (1,7), and (−2,3)
14. the exterior of the square with vertices (0,0), (1,0), (0,1), and (1,1)
15. the exterior of a circle of radius 5 with center at the origin
16. the region exterior to a circle of radius 2 and interior to a circle of radius 4, both with centers at the origin

17. the region interior to a square of side two with center at the origin and exterior to a circle of radius one with center at the origin.
18. the region obtained by deleting the second quadrant from the plane.
19. the region consisting of all points in plane except the origin.
20. the region consisting of all points whose abscissas are less than twice their ordinates.

Sketch the region described by each of the following inequalities:

21. $|x| > 2$.
22. $|x| \leqq 3$ and $|y| > 7$.
23. $6y - |y| > 5$.
24. $2 < |x| < 4$.
25. $|x| + |y| < 1$.
26. $3y < |x|$.
27. $|x| < y < |x + 4|$.
28. $|y| \leqq x^2$.
29. $|x| + x < 1$.
30. $|x + y| < |x| + |y|$.

PROBLEMS

1. Show that $|\cos \theta| \leqq 1$.

2. With the aid of a protractor, draw triangles two of whose angles are the following. In each case, let the distance AB be three inches:

   (a) $A = 80°$ $B = 40°$.
   (b) $A = 30°$ $B = 40°$.
   (c) $A = 60°$ $C = 50°$.
   (d) $B = 120°$ $C = 20°$.
   (e) $A = 100°$ $C = 95°$.

3. Determine the coordinates of each of the points shown in Fig. 1.18.

4. Prove Equation (10).

5. Prove that a straight line not parallel to the y-axis may be represented by the equation

$$y = mx + b$$

where m is the slope of the line and b is the y-intercept (that is, the value of y at which the line intercepts the y-axis—the value of y when $x = 0$). This is known as the *slope-intercept* form of the equation of a straight line.

6. Write the equations of the straight lines of Exercise 1-3 (nos. 6–10) in *slope-intercept* form (Problem 5).

7. Prove that a straight line neither parallel to the x- or y-axis nor passing through the origin may be represented by the equation

$$\frac{x}{a} + \frac{y}{b} = 1$$

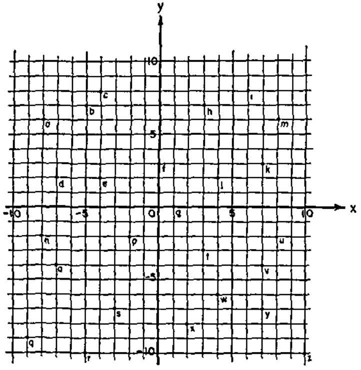


Fig 1.18

where a is the x-intercept and b is the y-intercept of the line That is, a is the value of x at which the line crosses the x-axis ($y = 0$), and b is the value of y at which the line crosses the y-axis ($x = 0$) This is the *intercept* form of the equation of a straight line

8. Write the equations of the straight lines of Exercise 1-3 (nos 6–10) in *intercept* form (Problem 7)

9. Consider a line joining the points P_1 and P_2 The coordinates of P_1 are (x_1,y_1), those of P_2 are (x_2,y_2) Let the point P lie on the line in such a position that it divides P_1P_2 in the ratio k_1 k_2 That is

$$\frac{P_1P}{PP_2} = \frac{k_1}{k_2}$$

Show that the coordinates (x,y) of P are

$$x = \frac{k_2x_1 + k_1x_2}{k_1 + k_2},$$

$$y = \frac{k_2y_1 + k_1y_2}{k_1 + k_2}$$

2

THE TRIGONOMETRIC FUNCTIONS

2.1. Definition of the Trigonometric Functions for Acute Angles

After the brief preliminaries of Chapter 1, we can now give a systematic discussion of the trigonometric functions. Consider a right triangle such as illustrated in Fig. 2.1 with sides a, b, c, the side c being opposite the right angle. We call c the *hypotenuse*. The side a opposite θ is called the *opposite side*, and the remaining side b is called the *adjacent side*. There are six trigonometric functions. The three most common are the sine, cosine, and tangent, which are defined as

$$\sin\theta = \frac{\text{opposite side}}{\text{hypotenuse}} = \frac{a}{c}, \tag{1}$$

$$\cos\theta = \frac{\text{adjacent side}}{\text{hypotenuse}} = \frac{b}{c}, \tag{2}$$

$$\tan\theta = \frac{\text{opposite side}}{\text{adjacent side}} = \frac{a}{b} \tag{3}$$

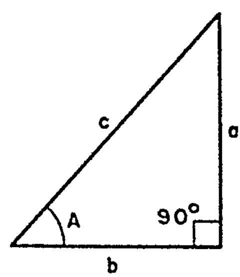


Fig. 2.1

respectively. The other three trigonometric functions are the cotangent, secant, and cosecant, which are the reciprocals of the tangent, cosine, and sine respectively:

$$\cot\theta = \frac{1}{\tan\theta} = \frac{\text{adjacent side}}{\text{opposite side}} = \frac{b}{a}, \tag{4}$$

$$\sec\theta = \frac{1}{\cos\theta} = \frac{\text{hypotenuse}}{\text{adjacent side}} = \frac{c}{b}, \tag{5}$$

$$\csc\theta = \frac{1}{\sin\theta} = \frac{\text{hypotenuse}}{\text{opposite side}} = \frac{c}{a}. \tag{6}$$

We emphasize again that the trigonometric functions of an angle are independent of the length of the sides of the triangle since only ratios are involved For example consider the angle θ as belonging to both the right triangles ABC and $AB C$ of Fig 2 2 Then according to triangle ABC

$$\sin \theta = \frac{BC}{BA} \tag{7}$$

and according to triangle $AB C$

$$\sin \theta = \frac{B C}{B A} \tag{8}$$

But these two triangles are similar Hence

$$\frac{BC}{BA} = \frac{B C}{B A}$$

Thus the value of $\sin \theta$ as given by (7) and (8) are identical Of course similar remarks apply to the other five trigonometric functions (On the other hand

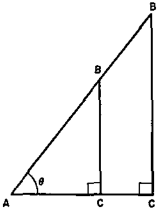


Fig 2 2

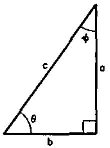


Fig 2 3

it must be kept in mind that the trigonometric functions have been defined only in terms of right triangles)

Now consider a right triangle Fig 2 3 where we have labeled both acute angles (ϕ is the Greek letter phi) Since the sum of the angles of any triangle is 180°

$$\phi + \theta = 90°$$

or

$$\phi = 90° - \theta \tag{9}$$

From the definition of the trigonometric functions,

$$\begin{aligned}
\sin \theta &= \frac{a}{c} = \cos \phi = \cos (90^\circ - \theta),\\
\cos \theta &= \frac{b}{c} = \sin \phi = \sin (90^\circ - \theta),\\
\tan \theta &= \frac{a}{b} = \cot \phi = \cot (90^\circ - \theta),\\
\cot \theta &= \frac{b}{a} = \tan \phi = \tan (90^\circ - \theta),\\
\sec \theta &= \frac{c}{b} = \csc \phi = \csc (90^\circ - \theta),\\
\csc \theta &= \frac{c}{a} = \sec \phi = \sec (90^\circ - \theta).
\end{aligned} \tag{10}$$

We therefore have the interesting result: *A trigonometric function of an acute angle is equal to the cofunction of its complementary acute angle.*

Let us consider a few examples.

Example 1. Suppose we are given a right triangle whose sides are 3 and 4 (Fig. 2.4). What are the values of the trigonometric functions of the angle θ?

Solution: First we see by the Pythagorean theorem that the hypotenuse is $\sqrt{3^2 + 4^2} = 5$. Thus

$$\sin \theta = \tfrac{3}{5}, \quad \cos \theta = \tfrac{4}{5}, \quad \tan \theta = \tfrac{3}{4},$$
$$\cot \theta = \tfrac{4}{3}, \quad \sec \theta = \tfrac{5}{4}, \quad \csc \theta = \tfrac{5}{3}.$$

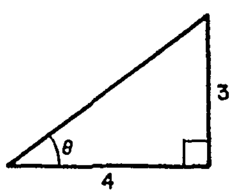


Fig. 2.4

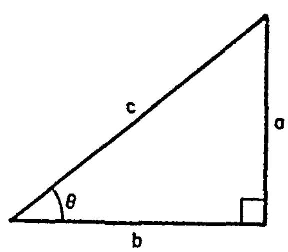


Fig. 2.5

Example 2. An acute angle θ has its sine equal to 0.2. What are the other trigonometric functions of this angle?

Solution: Draw a right triangle as in Fig. 2.5 and label one acute angle θ. Then

$$\sin \theta = \frac{a}{c} = 0.2$$

and $a = 0.2c$. By the Pythagorean theorem,

$$b = \sqrt{c^2 - a^2} = \sqrt{c^2 - 0.04c^2} = c\sqrt{0.96} = 0.4c\sqrt{6}$$

Thus

$$\cos\theta = \frac{b}{c} = \frac{0.4c\sqrt{6}}{c} = \frac{2\sqrt{6}}{5}$$

$$\tan\theta = \frac{b}{a} = \frac{0.2c}{0.4c\sqrt{6}} = \frac{1}{2\sqrt{6}} = \frac{\sqrt{6}}{12},$$

$$\sec\theta = \frac{c}{b} = \frac{5}{2\sqrt{6}} = \frac{5\sqrt{6}}{12},$$

$$\csc\theta = \frac{1}{0.2} = 5$$

Of course, as remarked earlier, since only the *ratios* of the sides of the triangle come into play we can let c have any numerical value, for example, let $c = 1$, the results are not affected.

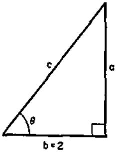


Fig. 2.6

Example 3. One side of a triangle has length 2. The tangent of one acute angle is 3/4. What is the hypotenuse of the triangle?

Solution. Draw a sketch of the triangle as indicated in Fig. 2.6. Now

$$\tan\theta = \frac{a}{b} = \frac{a}{2} = \frac{3}{4},$$

or

$$a = \tfrac{3}{2}$$

Thus

$$c = \sqrt{a^2 + b^2} = \sqrt{(\tfrac{3}{2})^2 + (2)^2} = \sqrt{\tfrac{9}{4} + 4} = \tfrac{5}{2}$$

(Query. Would it have made any difference if we had let $a = 2$ rather than $b = 2$ in Fig. 2.6?)

EXERCISE 2.1

Find the six trigonometric functions of angle θ (Fig. 2.1) in the triangles whose sides are the following.

1. $a = 3$	$b = 4$	6. $b = 3$	$a = 5$
2. $b = 12$	$c = 13$	7. $c = 8$	$b = 7$
3. $c = 5$	$a = 4$	8. $a = 6$	$c = 8$
4. $b = 3$	$a = 4$	9. $b = 1$	$a = 2$
5. $a = 1$	$b = 2$	10. $a = 4$	$b = 5$

2.2. Some Special Angles

From elementary geometry the student recalls certain triangles that were of special interest: the equilateral triangle and the isosceles triangle. An equilateral triangle has all of its sides equal and all of its angles equal. Since the sum of the angles of a triangle is 180°, $A = B = C = 60°$ (Fig. 2.7a). Now let $AB = BC = CA = 1$. We know that this will have no effect on the value of the trigonometric functions of the angles. We also recall from plane

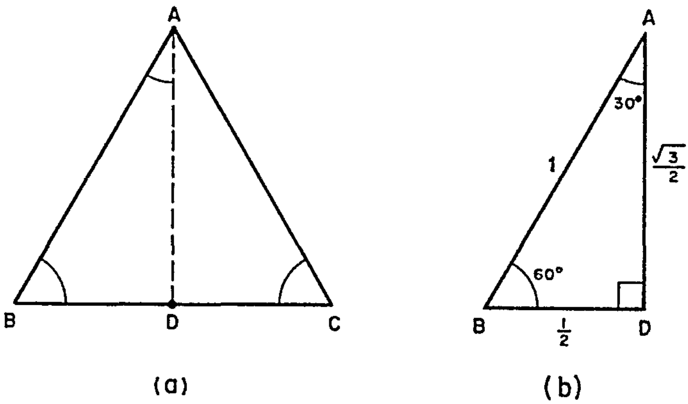


Fig. 2.7

geometry that the perpendicular AD bisects the line BC. Thus the length of the line BD is $\frac{1}{2}$. In the right triangle ABD (Fig. 2.7b), the third side AD is given by the Pythagorean theorem as

$$AD = \sqrt{(1)^2 - \left(\frac{1}{2}\right)^2} = \frac{\sqrt{3}}{2}.$$

We see immediately that

$$\begin{aligned}
\sin 60° &= \frac{\sqrt{3}}{2} = \cos 30°, \\
\cos 60° &= \frac{1}{2} = \sin 30°, \\
\tan 60° &= \sqrt{3} = \cot 30°, \\
\cot 60° &= \frac{\sqrt{3}}{3} = \tan 30°, \\
\sec 60° &= 2 = \csc 30°, \\
\csc 60° &= \frac{2\sqrt{3}}{3} = \sec 30°.
\end{aligned} \tag{11}$$

The triangle ABD is often referred to as a 30-60-90 triangle because of the size of its angles.

Another special triangle besides the 30-60-90 triangle is the isosceles right triangle. Referring to Fig. 2.8 where we have let $AC = CB = 1$, the common value of the angles A and B is 45°. The hypotenuse of this triangle is

$$AB = \sqrt{(1)^2 + (1)^2} = \sqrt{2},$$

and

$$\sin 45^\circ = \cos 45^\circ = \frac{\sqrt{2}}{2},$$

$$\tan 45^\circ = \cot 45^\circ = 1, \tag{12}$$

$$\sec 45^\circ = \csc 45^\circ = \sqrt{2}.$$

Suppose now we are given an angle of, say, 20° and wish to find its sine. One might proceed as follows. Lay off the desired angle with a protractor as

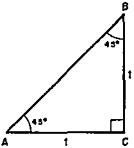


Fig. 2.8

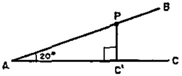


Fig. 2.9

in Fig. 2.9. At some arbitrary point P on the line AB, drop a perpendicular to the line AC. Then we can measure the lengths of the lines PC' and AP and compute sin 20° from the formula

$$\sin 20^\circ = \frac{PC'}{AP}.$$

Thus in principle we could find the sine (and similarly the other five trigonometric functions) of any angle. Actually, we shall learn more analytical methods later in this Part (Chapter 4) and in Part II (Section 13.4 of Chapter 13). In Table I, we list the numerical values of the trigonometric functions. A more extensive table appears at the end of the book, pages 306–310.

We call the table "natural values" to distinguish them from the logarithms of the trigonometric functions to be introduced in Chapter 9. Note that by virtue of (10) we need only construct the table from 0° to 45° rather than from 0° to 90°.

TABLE I. Natural Values of the Trigonometric Functions

Angle	sin	cos	tan	cot	sec	csc	
0°	0.0000	1.0000	0.0000	—	1.0000	—	90°
1°	0.0175	0.9998	0.0175	57.290	1.0002	57.299	89°
2°	0.0349	0.9994	0.0349	28.636	1.0006	28.654	88°
3°	0.0523	0.9986	0.0524	19.08	1.001	19.11	87°
4°	0.0698	0.9976	0.0699	14.30	1.002	14.34	86°
5°	0.0872	0.9962	0.0875	11.43	1.004	11.47	85°
6°	0.1045	0.9945	0.1051	9.514	1.006	9.567	84°
7°	0.1219	0.9925	0.1228	8.144	1.008	8.206	83°
8°	0.1392	0.9903	0.1405	7.115	1.010	7.185	82°
9°	0.1564	0.9877	0.1584	6.314	1.012	6.392	81°
10°	0.1736	0.9848	0.1763	5.671	1.015	5.759	80°
11°	0.1908	0.9816	0.1944	5.145	1.019	5.241	79°
12°	0.2079	0.9781	0.2126	4.705	1.022	4.810	78°
13°	0.2250	0.9744	0.2309	4.331	1.026	4.445	77°
14°	0.2419	0.9703	0.2493	4.011	1.031	4.134	76°
15°	0.2588	0.9659	0.2679	3.732	1.035	3.864	75°
16°	0.2756	0.9613	0.2867	3.487	1.040	3.628	74°
17°	0.2924	0.9563	0.3057	3.271	1.046	3.420	73°
18°	0.3090	0.9511	0.3249	3.078	1.051	3.236	72°
19°	0.3256	0.9455	0.3443	2.904	1.058	3.072	71°
20°	0.3420	0.9397	0.3640	2.747	1.064	2.924	70°
21°	0.3584	0.9336	0.3839	2.605	1.071	2.790	69°
22°	0.3746	0.9272	0.4040	2.475	1.079	2.669	68°
23°	0.3907	0.9205	0.4245	2.356	1.086	2.559	67°
24°	0.4067	0.9135	0.4452	2.246	1.095	2.459	66°
25°	0.4226	0.9063	0.4663	2.145	1.103	2.366	65°
26°	0.4384	0.8988	0.4877	2.050	1.113	2.281	64°
27°	0.4540	0.8910	0.5095	1.963	1.122	2.203	63°
28°	0.4695	0.8829	0.5317	1.881	1.133	2.130	62°
29°	0.4848	0.8746	0.5543	1.804	1.143	2.063	61°
30°	0.5000	0.8660	0.5774	1.732	1.155	2.000	60°
31°	0.5150	0.8572	0.6009	1.664	1.167	1.942	59°
32°	0.5299	0.8480	0.6249	1.600	1.179	1.887	58°
33°	0.5446	0.8387	0.6494	1.540	1.192	1.836	57°
34°	0.5592	0.8290	0.6745	1.483	1.206	1.788	56°
35°	0.5736	0.8192	0.7002	1.428	1.221	1.743	55°
36°	0.5878	0.8090	0.7265	1.376	1.236	1.701	54°
37°	0.6018	0.7986	0.7536	1.327	1.252	1.662	53°
38°	0.6157	0.7880	0.7813	1.280	1.269	1.624	52°
39°	0.6293	0.7771	0.8098	1.235	1.287	1.589	51°
40°	0.6428	0.7660	0.8391	1.192	1.305	1.556	50°
41°	0.6561	0.7547	0.8693	1.150	1.325	1.524	49°
42°	0.6691	0.7431	0.9004	1.111	1.346	1.494	48°
43°	0.6820	0.7314	0.9325	1.072	1.367	1.466	47°
44°	0.6947	0.7193	0.9657	1.036	1.390	1.440	46°
45°	0.7071	0.7071	1.000	1.000	1.414	1.414	45°
	cos	sin	cot	tan	csc	sec	Angle

EXERCISE 2-2

Evaluate each of the following expressions Express the result in fractions and *square roots, not as decimals*

1. $2 \cos 60^\circ + \sin 45^\circ$
2. $\cos 0^\circ + \sin^2 30^\circ + \cos 60^\circ$
3. $\sin^2 45^\circ + \cos^2 30^\circ$
4. $\sin^5 60^\circ + \sin 45^\circ + \sin 90^\circ$
5. $\cos^2 30^\circ + \sin 45^\circ$
6. $\sin 60^\circ + \cos^2 60^\circ + \tan 0^\circ$
7. $\sin^3 45^\circ + \tan 45^\circ$
8. $\cos^4 45^\circ + \sin^4 45^\circ + \sin 45^\circ$
9. $\cos 0^\circ + \tan^2 30^\circ$
10. $\tan 45^\circ + \cos 90^\circ + \sin 0^\circ$

Determine graphically (i e , by drawing appropriate triangles) the following functions

11. $\sin 70^\circ$
12. $\cos 10^\circ$
13. $\tan 80^\circ$
14. $\cot 20^\circ$
15. $\sin 55^\circ$
16. $\cos 25^\circ$
17. $\sin 40^\circ$
18. $\sin 65^\circ$
19. $\cos 50^\circ$
20. $\tan 35^\circ$

Using Table 1, on pp 306–310, determine the values of the following functions

21. $\sin 10^\circ 10$
22. $\cos 72^\circ 30$
23. $\tan 21^\circ 30$
24. $\sin 3^\circ 50$
25. $\sin 89^\circ 20$
26. $\cos 38^\circ 40$
27. $\sin 72^\circ 50$
28. $\tan 10^\circ 30$
29. $\sec 85^\circ 40$
30. $\csc 21^\circ 10$
31. $\sin 5^\circ 10$
32. $\sec 57^\circ 40$
33. $\cos 66^\circ 40$
34. $\tan 39^\circ 0$
35. $\sec 48^\circ 10$
36. $\sin 44^\circ 20$
37. $\sin 67^\circ 30$
38. $\cos 55^\circ 50$
39. $\sin 86^\circ 50$
40. $\cot 87^\circ 0$

With the aid of Table 1, pp 306–310, determine the angles whose functions have the values listed

41. $\cos \alpha = 0.3907$
42. $\sin \theta = 0.5736$
43. $\sin \beta = 0.5100$
44. $\cos x = 0.6626$
45. $\sin y = 0.8274$
46. $\cos \theta = 0.8039$
47. $\sin \alpha = 0.7173$
48. $\sin \beta = 0.9890$
49. $\sin \theta = 0.1074$
50. $\tan x = 1.0176$
51. $\cos x = 0.9696$
52. $\tan y = 0.7265$
53. $\sin \theta = 0.2532$
54. $\cos \alpha = 0.0785$
55. $\cos \alpha = 0.9051$
56. $\cos \beta = 0.8718$
57. $\cos \theta = 0.4975$
58. $\cot \theta = 0.9057$
59. $\tan x = 0.3346$
60. $\tan \theta = 1.4281$

2.3. General Angles

So far, the trigonometric functions have been defined only for acute angles, that is, for angles less than 90°. However, there exist obtuse angles—even greater than 180°. How shall we define the trigonometric functions of such angles? Note that even at this early stage, we must abandon our geometric picture of a right triangle—for how could we draw a right triangle containing an obtuse angle, say of 210°?

To make the appropriate definitions, let us consider a coordinate system as in Fig. 2.10. It is conventional to measure angles *counterclockwise* from the

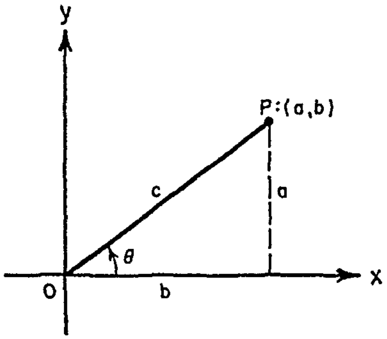


Fig. 2.10

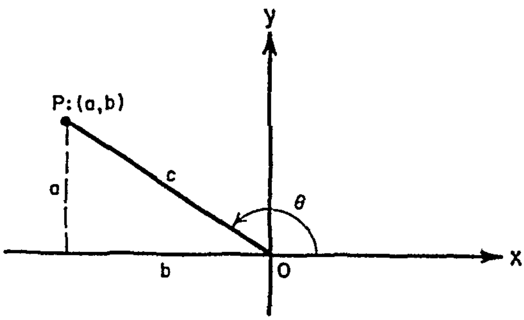


Fig. 2.11

positive direction of the x-axis, as we have done on the diagram. In Fig. 2.10, θ is in the first quadrant, that is, it is an acute angle, and no difficulties about the definition of the trigonometric functions of θ arise. That is,

$$c = \sqrt{a^2 + b^2}$$

and

$$\sin\theta = \frac{a}{c}, \qquad \cos\theta = \frac{b}{c}, \qquad \tan\theta = \frac{a}{b}, \qquad \text{etc.} \tag{13}$$

Now consider an angle in the *second* quadrant as illustrated in Fig. 2.11. The abscissa b is now a negative number, although a is a positive number and c (being the square root of the sum of squares) is also positive We define the trigonometric functions of θ as

$$\sin\theta = \frac{a}{c}, \qquad \cos\theta = \frac{b}{c}, \qquad \tan\theta = \frac{a}{b}, \qquad \text{etc} \tag{14}$$

These are just the definitions of (13) But since b is negative ($b < 0$) and a and c are positive, the sine and cosecant are *positive*, and the cosine, tangent,

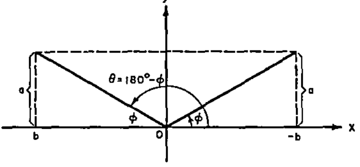


Fig 2 12

cotangent, and secant are negative It can be seen, for example, that the cosine of 150° is the negative of the cosine of 30°,

$$\cos 150^\circ = -\cos 30^\circ = -\frac{\sqrt{3}}{2},$$

and, in general, if θ is any angle in the second quadrant, there exists an acute angle ϕ such that $\theta = 180^\circ - \phi$ and

$$\cos\theta = \cos(180^\circ - \phi) = -\cos\phi \tag{15}$$

(see Fig 2 12) A similar line of argument from (14) and Fig 2 12 leads to the results

$$\sin\theta = \sin(180^\circ - \phi) = \sin\phi, \tag{16}$$

$$\tan\theta = \tan(180^\circ - \phi) = -\tan\phi \tag{17}$$

Similarly, we define the trigonometric functions of an angle in the third or fourth quadrant (Fig 2 13) by (13) or (14) In particular, if θ is in the third quadrant, there exists an acute angle ϕ such that $\theta = 180^\circ + \phi$ and

$$\begin{aligned} \sin\theta &= \sin(180^\circ + \phi) = -\sin\phi, \\ \cos\theta &= \cos(180^\circ + \phi) = -\cos\phi, \\ \tan\theta &= \tan(180^\circ + \phi) = \tan\phi \end{aligned} \tag{18}$$

For θ in the fourth quadrant, there exists an acute angle ϕ such that $\theta = 360° - \phi$ and

$$\sin\theta = \sin(360° - \phi) = -\sin\phi,$$
$$\cos\theta = \cos(360° - \phi) = \cos\phi,$$
$$\tan\theta = \tan(360° - \phi) = -\tan\phi. \qquad (19)$$

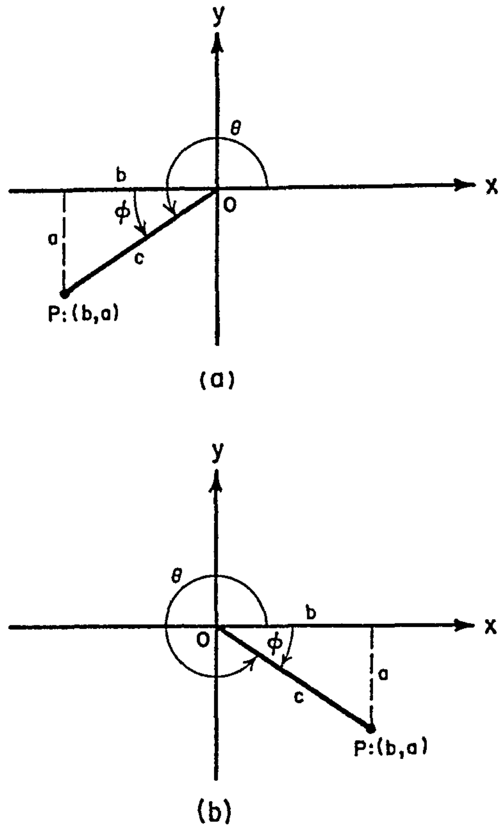


Fig. 2.13

Thus we see that if we are given an angle θ in any quadrant, we can find, by using (15) through (19), an acute angle ϕ such that any trigonometric function of θ differs from the corresponding trigonometric function of ϕ by at most its sign. Furthermore, we can determine this sign.

For convenience, we list the signs of the various trigonometric functions in the various quadrants in Table II below. Since $\csc\theta = 1/\sin\theta$, the sign of $\sin\theta$ and $\csc\theta$ are identical. Similarly, the sign of tangent and cotangent are the same in the same quadrant as well as the sign of cosine and secant.

A convenient way of remembering (15) through (19) follows. Suppose we wish to find the trigonometric functions of an angle θ not in the first quadrant.

TABLE II

Trigonometric Function	Quadrant I	II	III	IV
sin	+	+	−	−
cos	+	−	−	+
tan	+	−	+	−
cot	+	−	+	−
sec	+	−	−	+
csc	+	+	−	−

First lay off θ (which we shall suppose for concreteness to be in the third quadrant) as in Fig. 2.14a. Call ϕ the acute angle between the negative direction of the x-axis and OP. Then ϕ is the acute angle with the property that $\sin\phi = \pm\sin\theta$, $\cos\phi = \pm\cos\theta$, $\tan\phi = \pm\tan\theta$. The appropriate signs to use are determined by noting that both the abscissa and ordinate of P are negative Hence

$$\sin\phi < 0, \qquad \cos\phi < 0, \qquad \tan\phi > 0,$$

and

$$\sin\theta = -\sin\phi,$$
$$\cos\theta = -\cos\phi,$$
$$\tan\theta = \tan\phi$$

In practice, the authors find the following variation of the above scheme more convenient First lay off the angle θ as in Fig 2 14a, but then add the dotted lines as indicated in Fig 2 14b This immediately defines ϕ (an angle in the first quadrant) regardless of which quadrant θ was in originally The appropriate sign is then determined by Table II—which we find convenient to commit to memory

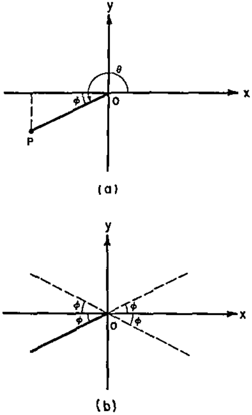


Fig 2.14

Example 1 Find the sine of 210°

Solution

$$\sin 210^\circ = \sin(180^\circ + 30^\circ)$$
$$= -\sin 30^\circ = -\tfrac{1}{2}$$

Example 2 Find tan 300°

Solution Again from Fig 2 14b,

$$\tan 300^\circ = \pm\tan 60^\circ$$

From Table II the minus sign is chosen

$$\tan 300^\circ = -\tan 60^\circ = -\sqrt{3}$$

Example 3 An angle θ in the second quadrant has a sine equal to 4/5 What are the other trigonometric functions of this angle?

Solution Since the angle θ is in the second quadrant, $180^\circ - \theta$ is acute

$$\sin\theta = \sin(180^\circ - \theta) = \frac{a}{c} = \frac{4}{5}$$

Thus

$$a = \tfrac{4}{5}c$$

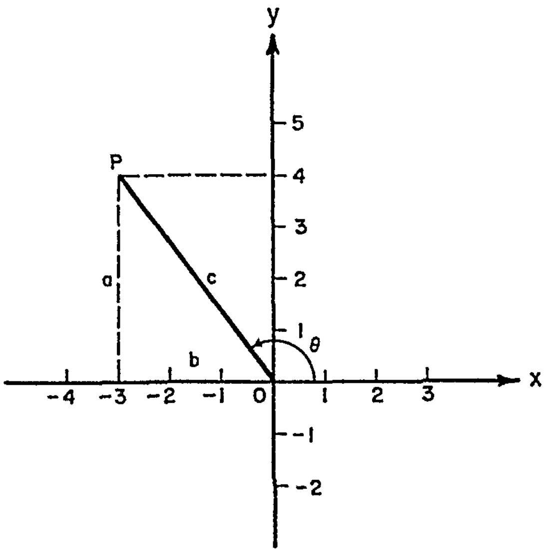


Fig. 2.15

and the third side of the triangle is

$$b = \sqrt{c^2 - a^2} = \sqrt{c^2 - \tfrac{16}{25}c^2} = \tfrac{3}{5}c.$$

Since the numerical values of the sides of the triangle are immaterial, only their ratios being of interest, we may conveniently set $c = 5$. Then $a = 4$, $b = 3$ and the coordinates of P are $(-3,4)$ (Fig. 2.15). Thus

$$\cos\theta = \frac{-3}{5} = -\frac{3}{5},$$

$$\tan\theta = \frac{4}{-3} = -\frac{4}{3},$$

$$\cot\theta = \frac{-3}{4} = -\frac{3}{4},$$

$$\sec\theta = \frac{5}{-3} = -\frac{5}{3},$$

$$\csc\theta = \frac{5}{4}.$$

Example 4. Find all angles between 0° and 360° whose cosine is -0.3.

Solution: If the cosine of an angle is negative, the angle must be in either the second or third quadrants. Thus the angles whose cosines are -0.3 are the angles θ and θ' of Fig. 2.16. If ϕ is an *acute* angle with the property that

$$\cos\phi = 0.3,$$

then

$$\theta = 180° - \phi,$$

and

$$\theta = 180° + \phi$$

(see Fig 2 14b)

For convenience we sometimes use *negative angles* By a negative angle we simply mean an angle measured clockwise For example, the angle in Fig 2 17 is $-30°$ Of course, one quickly sees that the position of the line OP could equally well be specified by $+330°$ In general,

$$\sin(-\theta) = \sin(360° - \theta) \qquad (20)$$

and (20) is true if sine is replaced by any of the other five trigonometric functions For instance,

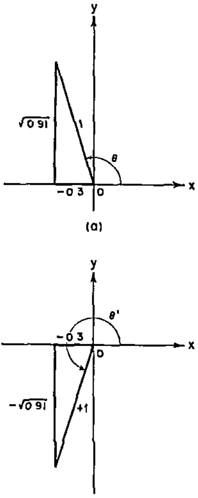
$$\sin(-150°) = \sin(360° - 150°)$$
$$= \sin 210° = -\sin 30° = -\tfrac{1}{2},$$

and

$$\tan(-60°) = \tan 300°$$
$$= -\tan 60° = -\sqrt{3}$$

The representation of Fig 2 14b is also convenient in achieving the above reductions In fact we see immediately [see also (19)] that if θ is an acute angle

$$\begin{aligned} \sin(-\theta) &= -\sin\theta, \\ \cos(-\theta) &= \cos\theta, \\ \tan(-\theta) &= -\tan\theta, \qquad (21) \\ \cot(-\theta) &= -\cot\theta, \\ \sec(-\theta) &= \sec\theta, \\ \csc(-\theta) &= -\csc\theta \end{aligned}$$



(a)

(b)

Fig 2 16

What about angles greater than 360°? What, for instance, is the cosine of 1000°? This is an easy question We note that any two angles that differ by 360° are identical, as in Fig 2 18 Thus for instance,

$$\sin(\theta + 720°) = \sin(\theta + 360°) = \sin\theta,$$

and, in general,

$$\sin(\theta + 360^\circ n) = \sin\theta, \tag{22}$$

where n is an integer, positive, negative, or zero. Equation (22) also remains valid if sine is replaced by any of the other five trigonometric functions. We see now that (20) is merely a special case of (22).

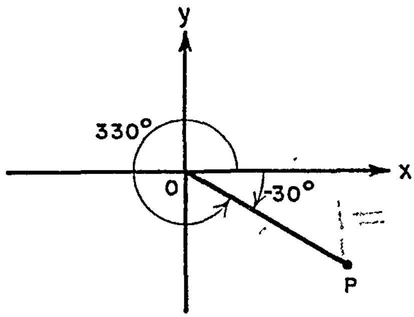


Fig. 2.17

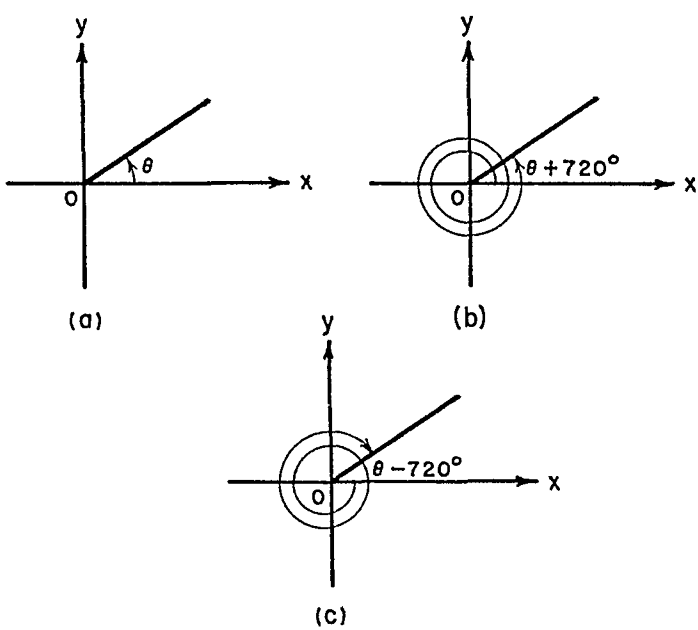


Fig. 2.18

In general, given any angle, we can reduce it to an angle between 0° and 360° by adding or subtracting multiples of 360°. The resulting angle can be reduced to an angle in the first quadrant by applying (15) through (19) (see Fig. 2.14b). If desired, the angle may further be reduced to an angle not greater than 45° by means of (10).

Example 5. Reduce cos 1000° to a trigonometric function of a positive angle less than 45°

Solution·

$$\cos 1000° = \cos [280° + 2(360°)] = \cos 280°$$
$$= \cos 80° = \sin 10°$$

(which equals 0 1736 from Table I, p 27)

Example 6. Reduce tan (−405°) to the tangent of an angle which is in the first quadrant

Solution

$$\tan (-405°) = \tan (360° - 405°) = \tan (-45°)$$
$$= -\tan 45° = -1$$

EXERCISE 2-3

Determine the values of the following functions without using tables

1. sin 120°
2. cos 120°
3. tan 315°
4. cos 210°
5. sin 240°
6 sin 135°
7. cos 330°
8. tan 225°
9. sin 150°
10. tan 300°

Express the following functions as functions of angles less than 45° (e g , sin 230° = −cos 40°)

11. sin 53°
12. cos 268°
13. tan 304°
14. sin 291°
15. tan 178°
16. cos 328°
17. sin 187°
18 cos 164°
19. tan 349°
20. sin 195°
21. tan 279° 11
22 cos 256° 43
23 sin 156° 24
24 sin 72° 42
25. cos 147° 30
26 tan 247° 38
27. sin 62° 25
28 cos 204° 17
29. sin 135° 56
30. tan 233° 59
31. cos 316° 11 59″
32. sin 95° 33 27″
33 sin 288° 21′ 48″
34. tan 220° 33 10″
35. sin 89° 35 26″
36 cos 216° 32 50″
37. sin 119° 45 24″
38. tan 81° 14 59″
39. cos 123° 46 48″.
40. sin 104° 57′ 12″

Determine all values of θ which satisfy each equation

41. $\sin \theta = -0\ 0523$
42. $\cos \theta = 0\ 8192$
43 $\sin \theta = 0\ 2419$
44. $\cos \theta = 0\ 8434$
45. $\tan \theta = 0\ 3541$
46 $\sin \theta = 0\ 9681$
47. $\sin \theta = 0\ 8175$
48 $\cos \theta = 0\ 9304$
49. $\tan \theta = 0\ 5658$
50. $\cos \theta = 0\ 5544$
51. $\cot \theta = 1\ 3190$
52 $\sin \theta = 0\ 6561$
53. $\cos \theta = 0\ 5225$
54. $\cos \theta = 0\ 7412$
55. $\sin \theta = 0\ 5807$
56 $\cos \theta = 0\ 5901$
57. $\tan \theta = 3\ 0475$
58. $\sin \theta = 0\ 4874$
59. $\cos \theta = 0\ 3611$
60. $\cot \theta = 18\ 075$

Determine the values of the functions indicated:

61. $\sin 35° 10'$.
62. $\cos 200°$.
63. $\cos 109° 20'$.
64. $\sin 238° 40'$.
65. $\sin 352° 30'$.
66. $\tan 125° 30'$.
67. $\sin 276° 50'$.
68. $\cos 73° 10'$.
69. $\sin 160° 40'$.
70. $\tan 317° 20'$.

Express the following functions as functions of angles less than 45°:

71. $\sin 405°$.
72. $\cos 760°$.
73. $\tan 511°$.
74. $\sin 493°$.
75. $\cos 372°$.
76. $\sin 680°$.
77. $\sin 760°$.
78. $\cos 900°$.
79. $\sin 441°$.
80. $\cos 368°$.

2.4. Some Basic Relations

In Chapter 1, we deduced the formula

$$\sin^2 \theta + \cos^2 \theta = 1, \tag{23}$$

for any acute angle θ. We assert that this identity is true regardless of the quadrant in which θ lies. For suppose θ is not in the first quadrant. Then there exists an angle ϕ in the first quadrant such that $\sin \theta$ equals either $+\sin \phi$ or $-\sin \phi$ and also such that $\cos \theta$ equals either $+\cos \phi$ or $-\cos \phi$. If θ is in the second quadrant, we take $+\sin \phi$ and $-\cos \phi$. If θ is in the third quadrant, we take $-\sin \phi$ and $-\cos \phi$. If θ is in the fourth quadrant, we take $-\sin \phi$ and $+\cos \phi$. In any case,

$$\sin^2 \theta = \sin^2 \phi \qquad \text{and} \qquad \cos^2 \theta = \cos^2 \phi.$$

Hence

$$\sin^2 \theta + \cos^2 \theta = \sin^2 \phi + \cos^2 \phi.$$

Since ϕ is in the first quadrant, $\sin^2 \phi + \cos^2 \phi = 1$ by (23). Hence the above equation implies

$$\sin^2 \theta + \cos^2 \theta = 1, \tag{24}$$

no matter in which quadrant θ lies.

Consider a right triangle now, such as in Fig. 2.19, and write

$$a^2 + b^2 = c^2, \tag{25}$$

by the Pythagorean theorem. If we divide (25) by c, we have

$$\left(\frac{a}{c}\right)^2 + \left(\frac{b}{c}\right)^2 = 1$$

and

$$\sin^2 \theta + \cos^2 \theta = 1$$

by definition of the trigonometric functions sine and cosine If we divide (25) by b, we have

$$\left(\frac{a}{b}\right)^2 + 1 = \left(\frac{c}{b}\right)^2$$

and

$$\tan^2 \theta + 1 = \sec^2 \theta \tag{26}$$

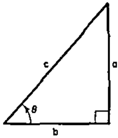


Fig 2.19

by definition of the trigonometric functions tangent and secant If we divide (25) by a, we have

$$1 + \left(\frac{b}{a}\right)^2 = \left(\frac{c}{a}\right)^2$$

and

$$1 + \cot^2 \theta = \csc^2 \theta \tag{27}$$

by definition of the trigonometric functions cotangent and cosecant

Equations (26) and (27) have just been established for acute angles Using the same line of reasoning as we did to establish the validity of (24), we may show (26) and (27) to be true for *any* angle

Perhaps too trivial to mention, we also list the following identities among the trigonometric functions of any angle θ

$$\begin{aligned} &\sin \theta \csc \theta = 1, && \tan \theta = \frac{\sin \theta}{\cos \theta}, \\ &\cos \theta \sec \theta = 1, && \\ &\tan \theta \cot \theta = 1, && \cot \theta = \frac{\cos \theta}{\sin \theta} \end{aligned} \tag{28}$$

From the formulas deduced in this section, we may express any five of the trigonometric functions in terms of the remaining one Suppose, for example, we wish to express $\cos \theta$, $\tan \theta$, $\cot \theta$, $\sec \theta$, $\csc \theta$ in terms of $\sin \theta$ If θ is acute,

$$\cos \theta = \sqrt{1 - \sin^2 \theta}, \tag{29}$$

and from (28),

$$\begin{aligned} \tan \theta &= \frac{\sin \theta}{\cos \theta} = \frac{\sin \theta}{\sqrt{1 - \sin^2 \theta}}, \\ \cot \theta &= \frac{1}{\tan \theta} = \frac{\sqrt{1 - \sin^2 \theta}}{\sin \theta}, \\ \sec \theta &= \frac{1}{\cos \theta} = \frac{1}{\sqrt{1 - \sin^2 \theta}}, \\ \csc \theta &= \frac{1}{\sin \theta} \end{aligned} \tag{30}$$

Another approach is to draw a triangle as in Fig. 2.20a. Then we may write

$$\sin\theta = \frac{\sin\theta}{1} \tag{31}$$

and label the opposite side $\sin\theta$ and the hypotenuse 1. Then by the Pythagorean theorem the third side is

$$\sqrt{1 - \sin^2\theta},$$

and the formulas for $\cos\theta$, $\tan\theta$, $\cot\theta$, $\sec\theta$, $\csc\theta$ in terms of $\sin\theta$ may be immediately read from the diagram.

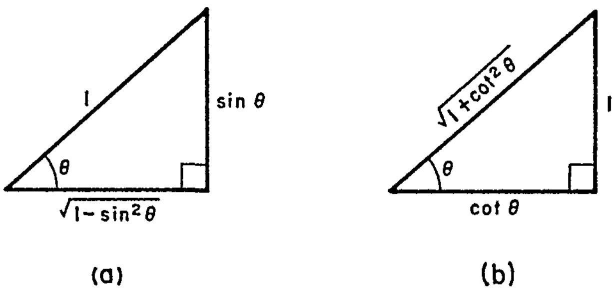


Fig. 2.20

Let us determine the trigonometric functions in terms of $\cot\theta$ by this method. The appropriate diagram is easily seen to be Fig. 2.20b. Thus from the diagram,

$$\begin{aligned}
\sin\theta &= \frac{1}{\sqrt{1 + \cot^2\theta}}, \\
\cos\theta &= \frac{\cot\theta}{\sqrt{1 + \cot^2\theta}}, \\
\tan\theta &= \frac{1}{\cot\theta}, \\
\sec\theta &= \frac{\sqrt{1 + \cot^2\theta}}{\cot\theta}, \\
\csc\theta &= \sqrt{1 + \cot^2\theta}.
\end{aligned} \tag{32}$$

If the angle θ is not in the first quadrant, an appropriate plus or minus sign must be affixed before (29), (30), and (32) since the square roots could be plus or minus For example, if θ is in the third quadrant, (29) and (30) become

$$\cos\theta = -\sqrt{1-\sin^2\theta},$$

$$\tan\theta = \frac{-\sin\theta}{\sqrt{1-\sin^2\theta}},$$

$$\cot\theta = -\frac{\sqrt{1-\sin^2\theta}}{\sin\theta},$$

$$\sec\theta = -\frac{1}{\sqrt{1-\sin^2\theta}},$$

$$\csc\theta = \frac{1}{\sin\theta}$$

(Since θ is in the third quadrant, $\sin\theta$ is negative Hence $-\sin\theta$ is positive and the above expression for $\tan\theta$ is positive)

Similarly, if θ is in the fourth quadrant, (32) becomes

$$\sin\theta = -\frac{1}{\sqrt{1+\cot^2\theta}},$$

$$\cos\theta = \frac{-\cot\theta}{\sqrt{1+\cot^2\theta}},$$

$$\tan\theta = \frac{1}{\cot\theta},$$

$$\sec\theta = -\frac{\sqrt{1+\cot^2\theta}}{\cot\theta},$$

$$\csc\theta = -\sqrt{1+\cot^2\theta}$$

EXERCISE 2-4

Determine the values of the remaining five trigonometric functions of the angles for which the value of one function is given (Assume angles are in either the first or second quadrant, as appropriate) Leave results in radical form

1. $\sin\theta = \frac{3}{4}$
2. $\cos\alpha = \frac{2}{3}$
3. $\tan\beta = \frac{4}{3}$
4. $\csc\theta = \frac{8}{5}$
5. $\sec x = -\frac{4}{3}$
6. $\sin\theta = \frac{1}{2}$
7. $\sec\alpha = -3$
8. $\cos\theta = \frac{4}{5}$
9. $\tan x = -4$
10. $\cos\beta = \frac{7}{8}$
11. $\cot A = -7$
12. $\sin B = \frac{5}{6}$
13. $\cos\beta = \frac{5}{9}$
14. $\sec\alpha = 4$
15. $\tan\theta = \frac{1}{4}$
16. $\csc\phi = 3$
17. $\cos x = -\frac{1}{3}$
18. $\sin\theta = \frac{8}{9}$
19. $\cos\phi = -\frac{7}{9}$
20. $\sin x = \frac{1}{5}$

2.5. Graphs of the Trigonometric Functions

Let us consider the equation

$$y = \sin \theta.$$

Certain values of $\sin \theta$ are immediately available to us, namely, $\sin 30° = 0.500$, $\sin 45° = \frac{1}{2}\sqrt{2} = 0.707$, $\sin 60° = \frac{1}{2}\sqrt{3} = 0.866$. These points are plotted in Fig. 2.21a.

Now what happens when θ is very small? We see from Fig. 2.22a that

$$\sin \theta = \frac{a}{1} = a.$$

If θ is very small and the hypotenuse remains fixed, then a becomes smaller and smaller as θ becomes smaller and smaller. Thus as θ approaches zero, a and hence $\sin \theta$ also approach zero. Thus

$$\sin 0 = 0.$$

Now consider what happens as θ gets close to 90°. From Fig. 2.22b,

$$\sin \theta = a,$$

and as θ approaches 90°, a approaches 1. Hence

$$\sin 90° = 1.$$

We also see from Fig. 2.22c that as θ increases from 0° to 90°, the sine of the angle steadily increases from 0 to 1.

Figure 2.21b illustrates the points we have calculated. Using additional values of $\sin \theta$ from Table I (p. 27) enables us to plot the smooth curve of Fig. 2.21c. Using the relations

$$\sin \theta = \sin (180° - \theta) = -\sin (180° + \theta) = -\sin (360° - \theta) \qquad (33)$$

enables us to plot the sine curve throughout all four quadrants as in Fig. 2.23. If we further recall that the sine curve repeats itself every 360 degrees, we can extend the sine curve to negative values and positive values of arbitrarily large value. For example, see Fig. 2.24.

An exactly similar analysis gives us the cosine curve of Fig. 2.25.

To determine the tangent, we recall that

$$\tan 30° = \frac{\sqrt{3}}{3} = 0.577,$$

$$\tan 45° = 1,$$

$$\tan 60° = \sqrt{3} = 1.732.$$

Also, from Fig 2 26a,

$$\tan \theta = \frac{a}{b},$$

and as θ approaches zero, a approaches 0 and b approaches 1. Hence

$$\tan 0 = 0$$

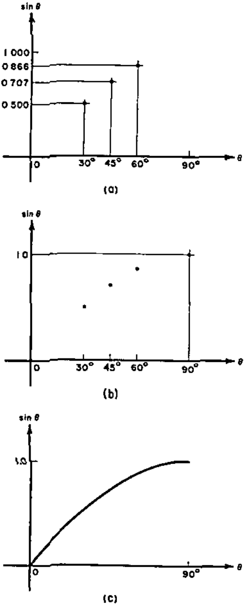


Fig 2.21

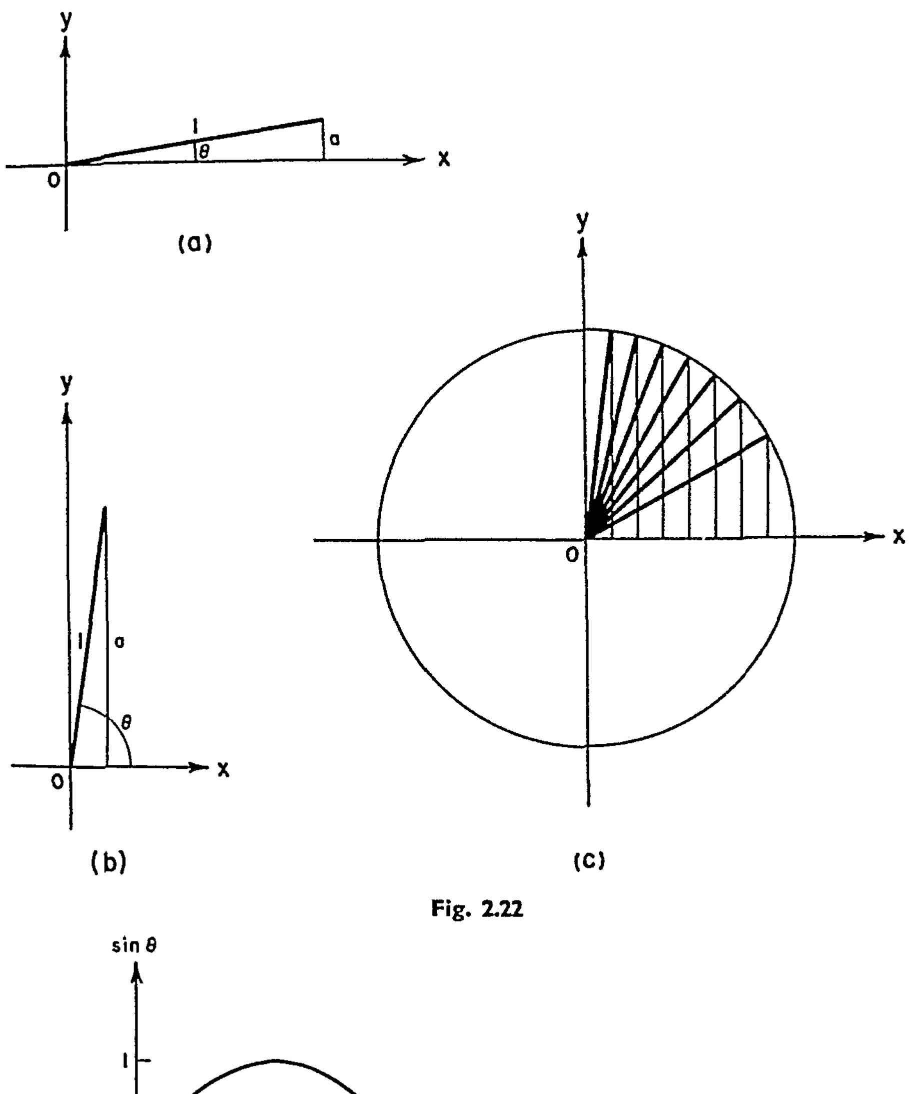


Fig. 2.22

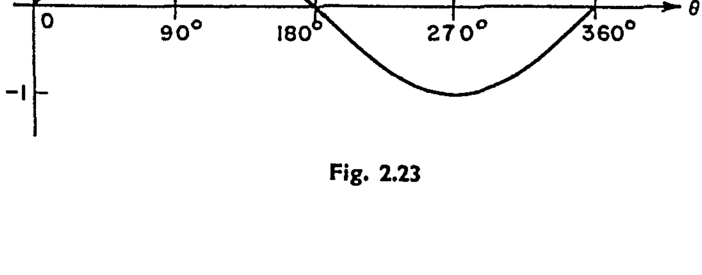


Fig. 2.23

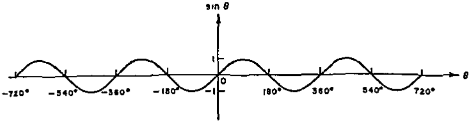


Fig. 2.24

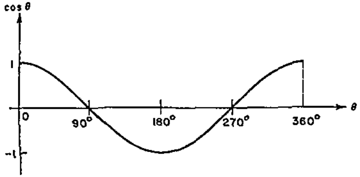


Fig. 2.25

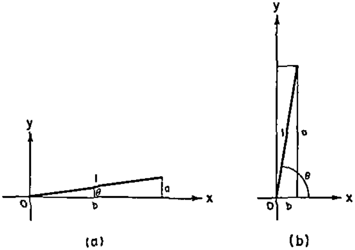


Fig 2.26

From Fig 2.26b,

$$\tan \theta = \frac{a}{b},$$

and as θ approaches 90°, a approaches 1 and b approaches zero Thus $\tan \theta$ becomes arbitrarily large as θ approaches 90°, that is, if M is any positive number, no matter how large, we can always choose a θ_0 so close to 90° that $\tan \theta_0 > M$. We express this result in symbols by writing

$$\lim_{\theta \to 90^\circ} \tan \theta = \infty.$$

This is read: tan θ becomes infinite (or increases without limit) as θ approaches 90°. Also, as θ increases from 0° to 90°, tan θ steadily increases from 0 without limit. Thus, using additional values from Table I (p. 27), we may plot tan θ between 0° and 90° as in Fig. 2.27.

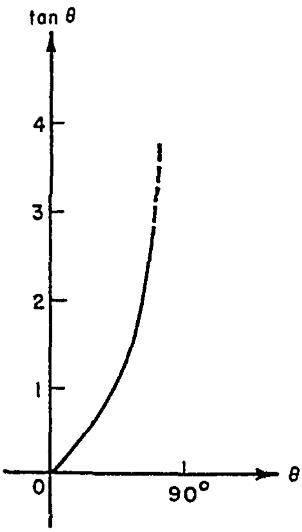


Fig. 2.27

In order to find tan θ in the other quadrants, we use the relations

$$\begin{aligned} \tan \theta &= -\tan (180^\circ - \theta) \\ &= +\tan (180^\circ + \theta) = -\tan (360^\circ - \theta), \end{aligned}$$

which leads to Fig. 2.28.

The cotangent, secant, and cosecant appear in Figs. 2.29, 2.30, and 2.31 respectively. Since they are just the reciprocals of the tangent, cosine, and sine respectively, their graphs present no new difficulties. (Note that as θ approaches zero, sin θ approaches zero and hence csc θ increases without limit. Similar remarks apply to sin 180°, sin 360°, cos 90°, cos 270°, tan 0°, tan 180°, and tan 360°.)

The graphs of the trigonometric functions which appear in Figs. 2.23, 2.25, 2.28, 2.29, 2.30, and 2.31 are undoubtedly the most important results of this chapter. It would be well for the student to keep at all times a mental picture of these curves. Whenever someone mentions a trigonometric function, for example, tan θ, the student should immediately conjure up Fig. 2.28. Note that among other things, the results of Table II (p. 31) are subsumed in these illustrations.

EXERCISE 2-5

Sketch the graphs of the following trigonometric functions:

1. $\sin 2x$.
2. $2 \sin x$.
3. $\sin^2 x$.
4. $2 \cos x$.
5. $2 \cos 2x$.
6. $\cos^2 x$.
7. $\sin 3x$.
8. $\sin^3 x$.
9. $\tan x/2$.
10. $\tan 2x$.

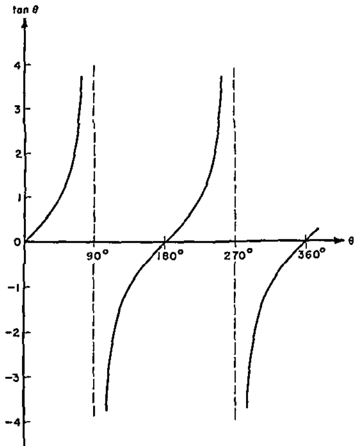


Fig. 2.28

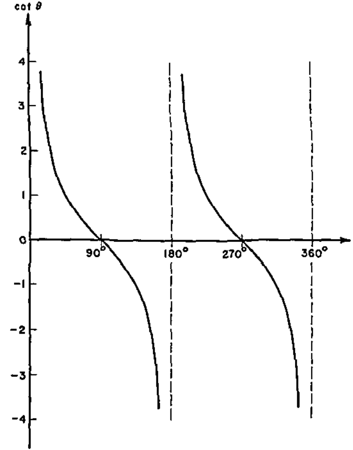


Fig. 2.29

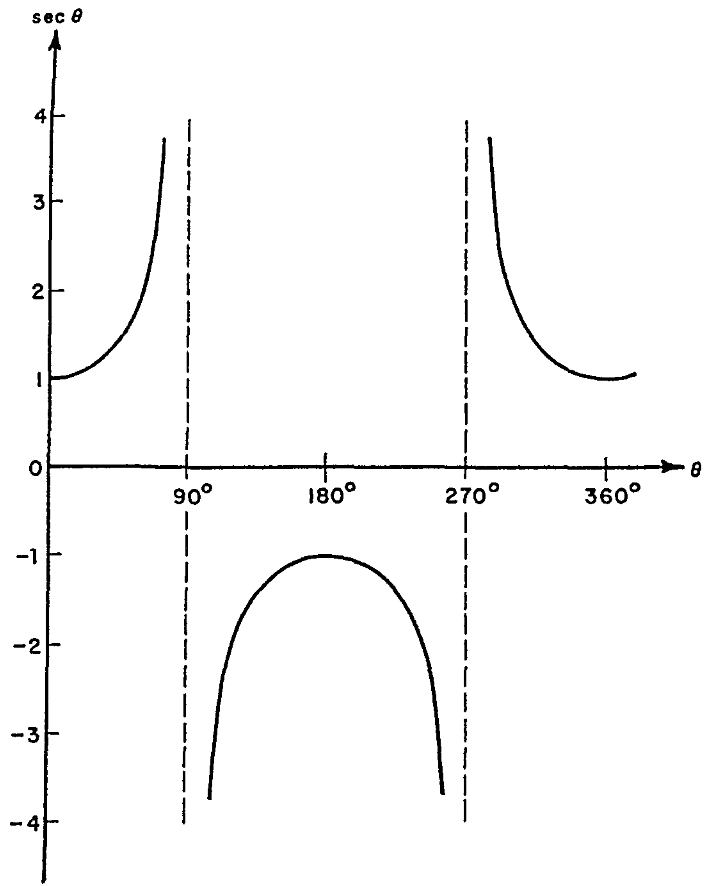


Fig. 2.30

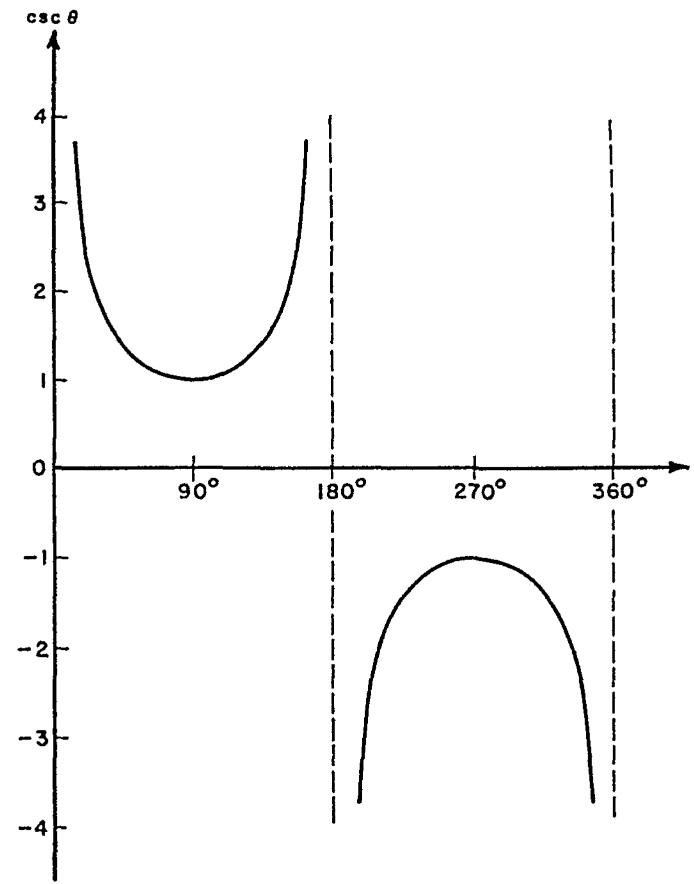


Fig. 2.31

PROBLEMS

1. Show that the equation of a straight line may be written in the form

$$x \cos \alpha + y \sin \alpha = p$$

where p is the distance of the line from the origin and α is the angle which the perpendicular (or *normal*) to the line makes with the x axis (Fig 2 32) This is the *normal* form of the equation of a straight line

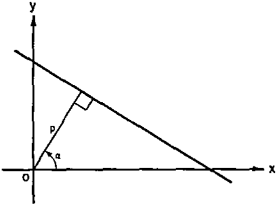


Fig 2 32

2. Write in normal form the equations of the straight lines of Exercise 1-3, nos 6–10

3. A ladder 30 feet long is placed against the wall of an alley It comes to rest 25 feet above the ground Without moving the foot of the ladder, it is swung so it touches the other wall at a point 20 feet above the ground What is the width of the alley?

4. Show that

$$\frac{\sin (180° - x)}{\sin (270° - x)} \tan (90° + x) + \frac{1}{\sin^2 (270° - x)} = 1 + \sec^2 x$$

5. Construct the graph of $y = \sin x \cos x$

3

TRIGONOMETRIC TERMINOLOGY

Why do we measure angles in degrees? Why not divide the circumference of a circle into 100 parts or 17 parts (or any other number) rather than into 360 equal parts? The answer to this question is lost in antiquity—perhaps it was because the Babylonians thought the year had 360 days. Regardless of how it originated, the sexagesimal system for measuring angles is here to stay. However, there are also other convenient measures of angles. The most important of these uses the *radian* as the basic unit. Radian measure is used in all advanced mathematical work and is actually no more difficult to understand and use than the degree, minute, second system. In fact, it rests on a more logical basis and, as we shall see (Section 3.2), makes certain formulas much simpler.

Before defining a radian, we might mention one other unit of angle that is used in ordnance and artillery work. This unit is called the *mil* and is defined as $\frac{1}{6400}$ of a circle. Thus

$$6400 \text{ mils} = 360°,$$

or

$$1 \text{ mil} = 0.05625°,$$

and

$$1° = 17\tfrac{7}{9} \text{ mils}.$$

However, we shall have no occasion to use such a unit in this book.

3.1. Radian Measure

Consider a circle of unit radius (Fig. 3.1). Then the circumference of the circle is 2π times the radius of the circle. Since the radius is one, the circumference C is

$$C = 2\pi.$$

We recall from plane geometry that the central angle θ of a circle is proportional to the subtended arc s,

$$\theta = ks \tag{1}$$

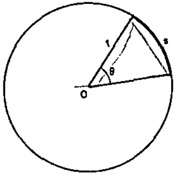


Fig 3.1

where k is a constant of proportionality If, for example, θ were measured in degrees,

$$360^\circ = k(2\pi)$$

since 360° encompasses the whole circle and 2π is its circumference Thus

$$k = \frac{360}{2\pi} = \frac{180}{\pi}$$

and

$$\theta = \frac{180}{\pi} s \tag{2}$$

Thus, an arc of length, say $\frac{\pi}{4}$ units will subtend an angle θ of

$$\theta = \frac{180}{\pi}\frac{\pi}{4} = 45^\circ$$

Now, would it not be more convenient to define a unit of angle so that the proportionality factor of (1) were unity, that is, so that θ were numerically equal to s? In this case, the angle subtended by an arc of length say $\pi/4$ would then simply be $\pi/4$ Such a measurement of angles is called *radian measure*,

$$2\pi \text{ radians} = 360 \text{ degrees} \tag{3}$$

Immediately we have

$$1 \text{ radian} = \frac{360}{2\pi} \text{ degrees} = \frac{180}{\pi} \text{ degrees} = 57\ 2958^\circ = 57^\circ\ 17'\ 44\ 8' \tag{4}$$

and

$$1^\circ = \frac{\pi}{180} \text{ radian} = 0\ 017453 \text{ radian *} \tag{5}$$

Radian measure is used exclusively in advanced theoretical mathematics, as we have mentioned before However, degree measure is frequently convenient in certain applications Thus the reader will find tables of the trigonometric functions expressed in radians as well as tables expressed in degrees (Table I, p 27) A brief table of sines, cosines, and tangents of angles measured in radians appears in Table III Since in a given problem, radian measure may be more convenient than degree measure and vice versa, we shall use whichever system seems most applicable Thus neither system will be exclusively adopted in this book The student will therefore learn to feel

* We now see that a *mil* is approximately a milliradian More exactly, 1 mil = 0 0009817 radian

TABLE III. Natural Trigonometric Functions in Radian Measure

Radians	sin	cos	tan
0	0.0000	1.0000	0.0000
0.10	0.0998	0.9950	0.1003
0.20	0.1987	0.9801	0.2027
0.30	0.2955	0.9553	0.3093
0.40	0.3894	0.9211	0.4228
0.50	0.4794	0.8776	0.5463
0.60	0.5646	0.8253	0.6841
0.70	0.6442	0.7648	0.8423
0.80	0.7174	0.6967	1.030
0.90	0.7833	0.6216	1.260
1.00	0.8415	0.5403	1.557
1.10	0.8912	0.4536	1.965
1.20	0.9320	0.3624	2.572
1.30	0.9636	0.2675	3.602
1.40	0.9854	0.1700	5.798
1.50	0.9975	0.0707	14.101
$\pi/2$	1.0000	0.0000	——
1.60	0.9996	−0.0292	−34.233

at home with both radians and degrees and at the same time gain familiarity with manipulating both. With a little experience and practice, he will quickly recognize $\frac{\pi}{6}$ as 30°, $\frac{\pi}{2}$ as 90°, π as 180°, etc.

Example 1. Express $7\pi/6$ radians in degrees.

Solution:

$$\frac{7\pi}{6}\text{ radians} = \frac{180}{\pi}\left(\frac{7\pi}{6}\right)\text{ degrees} = 210°$$

by (4).

Example 2. Express 0.85 radian in degrees.

Solution:

$$0.85\text{ radian} = \left(\frac{180}{\pi}\right)(0.85)\text{ degrees} \doteq (57.2958)(0.85)\text{ degrees}$$

$$= 48.70143° = 48°\,42'\,4''.$$

Example 3. Express 110° in radians.

Solution:

$$110° = \frac{\pi}{180}(110)\text{ radians} = (0.017453)(110)\text{ radians} = 1.920\text{ radians}$$

by (5). Sometimes it is convenient to leave it in the form $11\pi/18$.

Example 4. Find sin 0.4.

Solution: It is understood that 0.4 is 0.4 *radian*. That is, we rarely write sin (0.4 radian). The student has probably often omitted the degree sign and written, say, tan 45 rather than the more precise tan 45°. Of course, this is perfectly all right if it is understood that the 45 means 45 degrees. In practice, no confusion arises. It is generally clear from the context whether degrees or radians is intended Mathematical convention, though, usually dictates the degree symbol ° but omits the radian designation

Now from Table III,

$$\sin 0\ 4 = 0\ 3894$$

If Table III were not available, we could convert 0 4 radian into degrees,

$$0\ 4 \text{ radians} = \left(\frac{180}{\pi}\right)(0\ 4) \text{ degrees} = 22\ 92° = 22°\ 55\ \ 12'',$$

and then look up the sine in a table which expressed the angle in degrees

Example 5 An arc of 2 8 units length in a circle of radius 5 units subtends an angle θ What is this angle?

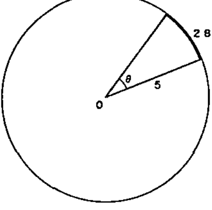


Fig 3 2

Solution Referring to Fig 3 2, we see that 2π radians corresponds to the circumference of the circle, which is $2\pi(5) = 10\pi$ Thus every unit of the circumference is $2\pi/10\pi = 0\ 2$ radians Hence

$$\theta = 2\ 8(0\ 2) = 0\ 56 \text{ radian}$$
$$= 32\ 0856° = 32°\ 5\ \ 8''$$

The analysis of Example 5 immediately leads to the formula for finding the angle θ measured in radians in terms of the arc subtended in any circle, namely,

$$\theta = \frac{\text{length of the subtending arc}}{\text{length of the radius}} \qquad (6)$$

TABLE IV Conversion of Degrees, Minutes Seconds to Radians

Degrees	Radians	Minutes	Radians	Seconds	Radians
1	0 01745 33	1	0 00029 09	1	0 00000 48
2	0 03490 66	2	0 00058 18	2	0 00000 97
3	0 05235 99	3	0 00087 27	3	0 00001 45
4	0 06981 32	4	0 00116 36	4	0 00001 94
5	0 08726 65	5	0 00145 44	5	0 00002 42
6	0 10471 98	6	0 00174 53	6	0 00002 91
7	0 12217 30	7	0 00203 62	7	0 00003 39
8	0 13962 63	8	0 00232 71	8	0 00003 88
9	0 15707 96	9	0 00261 80	9	0 00004 36
10	0 17453 29	10	0 00290 89	10	0 00004 85
20	0 34906 59	20	0 00581 78	20	0 00009 70
30	0 52359 88	30	0 00872 66	30	0 00014 54
40	0 69813 17	40	0 01163 55	40	0 00019 39
50	0 87266 46	50	0 01454 44	50	0 00024 24
60	1 04719 76	60	0 01745 33	60	0 00029 09
70	1 22173 05				
80	1 39626 34				
90	1 57079 63				

For convenience in converting degrees, minutes, and seconds to radians we include Table IV.

EXERCISE 3-1

Express the following (radian) angles in degrees and decimal fractions:

1. $\pi/6$.
2. $\pi/4$.
3. 0.4000.
4. 0.6472.
5. 1.2397.
6. 4.0593.
7. 1.9624.
8. 1.8901.
9. 3.1109.
10. 2.8752.

Write the following angles in degrees, minutes, and seconds:

11. 0.743251.
12. 5.726935.
13. 1.802375.
14. 2.566554.
15. 2.319641.
16. 3.739505.
17. 1.505044.
18. 0.814702.
19. 4.678887.
20. 6.154201.

Express these angles in radian measure:

21. 60°.
22. $22\frac{1}{2}$°.
23. 48°.
24. 17° 30′.
25. 97° 14′.
26. 54° 27′ 2″.
27. 27° 56′ 47″.
28. 76° 49′ 15″.
29. 92° 38′ 59″.
30. 64° 13′ 32″.

Determine the length of the arc subtended by the angles listed when the radius r of the circle is as shown:

31. $\theta = 45°$ $r = 10$.
32. $\theta = 60°$ $r = 1.7$.
33. $\theta = \pi/2$ $r = 11.3$.
34. $\theta = 78°$ $r = 0.98$.
35. $\theta = 16°$ $r = 448$.
36. $\theta = 43° 50'$. $r = 3.2$
37. $\theta = 19° 28'$. $r = 0.5$
38. $\theta = 147° 19' 10''$. $r = 117.0$
39. $\theta = 209° 47' 34''$. $r = 44.1$
40. $\theta = 190° 32' 45''$. $r = 0.064$

Evaluate the following functions:

41. $\sin \pi/4$.
42. $\sin 3\pi/2$.
43. $\sin 1.000$.
44. $\cos 5.932$.
45. $\sin 0.07$.
46. $\cos 1.34$.
47. $\tan 0.98$.
48. $\cot 0.32$.
49. $\cos 6.283$.
50. $\sin 1.59$.

3.2. Small Angles

A close examination of Table III reveals a curious fact, namely, that for small angles, say up to half a radian (about 30°), the sine of the angle is numerically very close to the value of the angle itself Thus it would seem that if θ were a small angle measured in radians, then

$$\sin\theta = \theta \tag{7}$$

(We shall use the symbol = to mean "approximately equal to") Let us see if we can establish this result purely analytically

Toward this end consider Fig 3 3, which represents a circle of unit radius The central angle subtended by the arc $\widehat{PP'}$ is 2θ At the points P and P' we have drawn tangents to the circle which intersect at Q (The line OQ is the bisector of the central angle) From the figure,

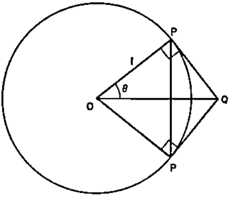


Fig 3 3

$$PP < \widehat{PP'} < PQ + QP' \tag{8}$$

But

$$\sin\theta = \tfrac{1}{2}PP,$$

$$\tan\theta = PQ = QP',$$

$$\theta = \tfrac{1}{2}\widehat{PP},$$

by definition of sine, tangent, and radian measure of an angle respectively Thus from (8),

$$2\sin\theta < 2\theta < \tan\theta + \tan\theta,$$

or, dividing by 2,

$$\sin\theta < \theta < \tan\theta \tag{9}$$

An acute angle, therefore, when measured in radians always lies between its sine and tangent This can be shown graphically by plotting the functions $y = \sin\theta$, $y = \theta$, and $y = \tan\theta$ from 0 to $\pi/2$ as in Fig 3 4

Now using reciprocal values of (9) leads to

$$\frac{1}{\sin\theta} > \frac{1}{\theta} > \frac{1}{\tan\theta}, \tag{10}$$

since the inequalities are reversed If we multiply by $\sin\theta$, (10) assumes the form

$$1 > \frac{\sin\theta}{\theta} > \cos\theta$$

But as θ approaches zero, $\cos\theta$ approaches one, that is $\lim_{\theta\to 0}\cos\theta = 1$ Since 1 is independent of θ, certainly $\lim_{\theta\to 0} 1 = 1$ is a true statement Thus

$$1 = \lim_{\theta\to 0} 1 \geq \lim_{\theta\to 0}\frac{\sin\theta}{\theta} \geq \lim_{\theta\to 0}\cos\theta = 1,$$

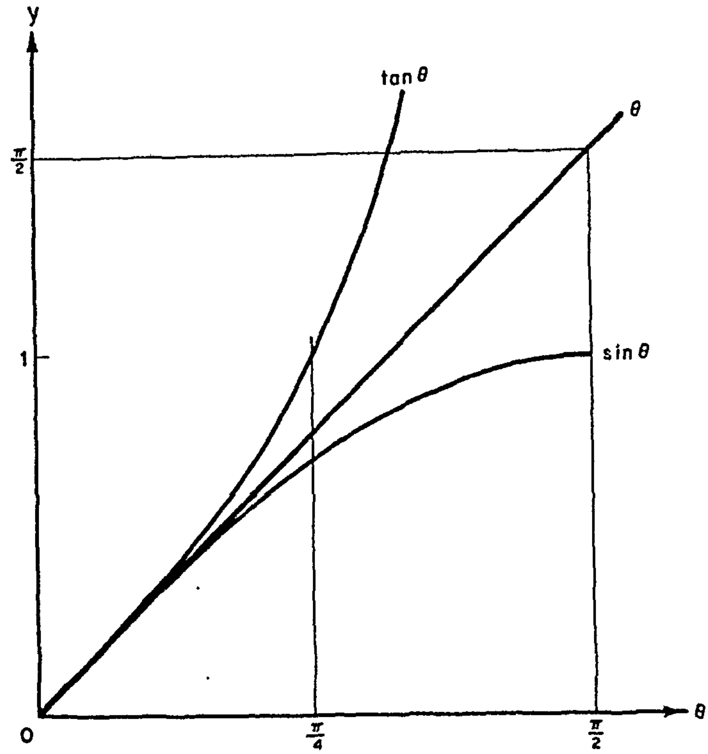


Fig. 3.4

which implies*

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = 1. \tag{11}$$

Thus for small θ,

$$\sin \theta \doteq \theta. \tag{12}$$

Also from (10), if we multiply by $\tan \theta$,

$$\frac{1}{\cos \theta} > \frac{\tan \theta}{\theta} > 1.$$

But $\lim_{\theta \to 0} \dfrac{1}{\cos \theta} = 1$ since $\cos \theta$ approaches one as θ approaches zero. Thus we also have the formulas

$$\lim_{\theta \to 0} \frac{\tan \theta}{\theta} = 1 \tag{13}$$

and

$$\tan \theta \doteq \theta \tag{14}$$

when θ is a small angle measured in *radians*.

* See Chapter 12 of Part II for an extended discussion of limits.

Note the simplicity of (12) and (14), which were first suggested by an inspection of Table III They probably would not have been as easily detected from Table I (p 27) However, if θ *is* measured in *degrees*, (11) and (12) become

$$\lim_{\theta \to 0} \frac{\sin \theta}{\theta} = \frac{\pi}{180}$$

and

$$\sin \theta = \frac{\pi}{180} \theta \tag{15}$$

respectively, with similar expressions for (13) and (14) The authors believe that (11) and (13) by themselves justify a strong case for the use of radians in theoretical mathematical work

One might also ask if a simple expression analogous to (12) exists for $\cos \theta$ An examination of Table III (p 51) gives little promise Although it appears that $\cos \theta$ is close to one for θ close to zero, the formula $\cos \theta = 1 - \theta$ is certainly not a very good approximation even at one-tenth of a radian However, let us approach this problem purely analytically We have (12) and we know a relation between sine and cosine This is sufficient information to deduce the analog of (12) for the cosine

First we note that in general

$$(1 - \tfrac{1}{2}x)^2 = 1 - x + \tfrac{1}{4}x^2$$

If $|x|$ is a small quantity compared with unity, then $\frac{1}{4}x^2$ will be small compared with $|x|$ Thus for small $|x|$,

$$(1 - \tfrac{1}{2}x)^2 = 1 - x,$$

or

$$1 - \tfrac{1}{2}x = \sqrt{1 - x} \tag{16}$$

From the familiar identity $\cos^2 \theta + \sin^2 \theta = 1$, we infer that

$$\cos \theta = \sqrt{1 - \sin^2 \theta} = \sqrt{1 - \theta^2} \tag{17}$$

by virtue of (12) for small angles θ measured in radians But from (16) with $x = \theta^2$,

$$\cos \theta = 1 - \tfrac{1}{2}\theta^2, \tag{18}$$

which is the desired result For example, even if θ is as large as one half a radian (28 65°),

$$1 - \tfrac{1}{2}\theta^2 = 1 - \tfrac{1}{2}(\tfrac{1}{2})^2 = \tfrac{7}{8} = 0\ 8750,$$

while from Table III

$$\cos 0\ 5 = 0\ 8776,$$

an error of only 0 3 percent!

EXERCISE 3-2

Determine the numerical values of the following functions without the use of tables:

1. $\sin 5''$.
2. $\tan 14' 12''$.
3. $\cos 4°$.
4. $\sec 0.001$.
5. $\csc 3° 28'$.
6. $\sin 10° 10'$.
7. $\sin 2° 17'$.
8. $\cos 1° 11' 15''$.
9. $\sin 2° 9' 11''$.
10. $\tan 4° 10'$.

3.3. Some Definitions and Terminology

As mathematics is developed on many frontiers and put to diverse uses, people introduce many different terms and definitions. Later it becomes apparent that two workers in unrelated fields have used different words to describe the same phenomenon. Such an occurrence is not unusual in fields of human endeavor and is by no means peculiar to the field of mathematics. In this section, we wish to expose the student to various definitions associated with trigonometric functions as well as describing synonymous expressions for essentially the same idea.

We first note that the trigonometric functions are *periodic*. A precise definition follows.

Definition. A function $f(x)$ is said to be *periodic* of *period* T if $f(x) = f(x + T)$ for all x for which both members are defined.

The sine of x is a periodic function of period 2π since the function repeats itself every 2π units. That is,

$$\sin x = \sin (x + 2\pi)$$

for all x. Similarly, cosine, secant, and cosecant are periodic of period 2π; tangent and cotangent are periodic of period π. The function $\cos x/2$ has period 4π, since

$$\cos \tfrac{1}{2}(x + 4\pi) = \cos (\tfrac{1}{2}x + 2\pi) = \cos \frac{x}{2},$$

and the function $\tan \pi x$ has period 1, since

$$\tan \pi(x + 1) = \tan (\pi x + \pi) = \tan \pi x.$$

Are there any other functions, besides the trigonometric functions, which are periodic? Certainly. We need only draw any curve on any finite interval, for example, those of Fig. 3.5. Then we merely repeat these functions as illustrated in Fig. 3.6 to obtain periodic functions. The function of Fig. 3.6a has period 2, that of Fig. 3.6b has period 3, and that of Fig. 3.6c has period $b - a$. The student might wonder if there is any connection between arbitrary

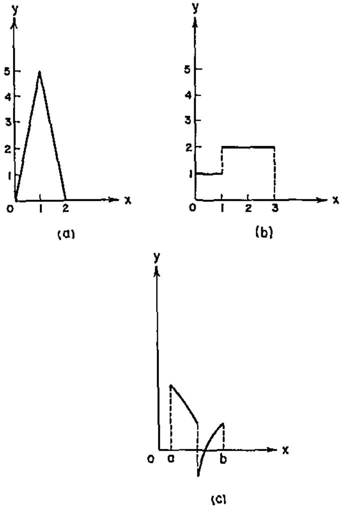


Fig 3 5

periodic functions and sines and cosines The answer is a very definite yes This quantitative relationship will be analyzed in Chapter 15 of Part II The process of expressing functions such as those illustrated in Fig 3 6 in terms of trigonometric functions is called *Fourier analysis*, and the resulting expansions are called *Fourier series*

Consider now the function

$$y = 2.5 \sin (3\theta + 15^\circ) \tag{19}$$

which we have plotted in Fig 3 7 If we ignore the position of the coordinate axes and their scales, (19) represents an ordinary sine curve A little reflection shows that the function $\sin (3\theta + 15^\circ)$ is just the function $\sin 3\theta$ displaced to

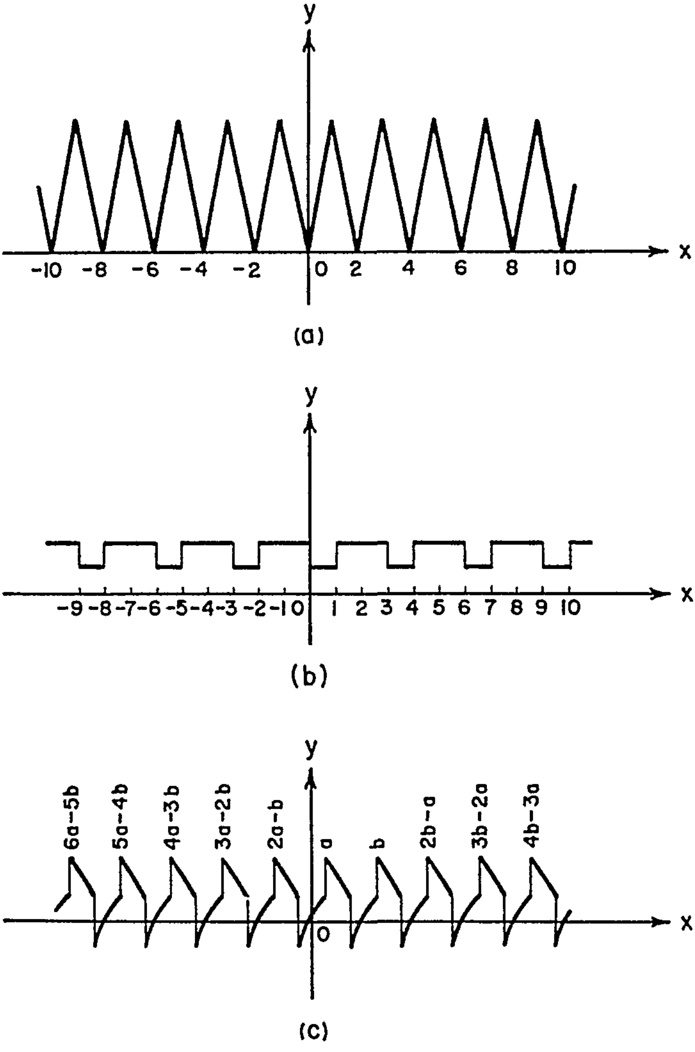


Fig. 3.6

the left by 5° with respect to the θ-axis. Also, $\sin 3\theta$ has period $2\pi/3$ since

$$\sin 3\left(\theta + \frac{2\pi}{3}\right) = \sin(3\theta + 2\pi) = \sin 3\theta.$$

The angle 15° is called a *phase angle*. The coefficient 2.5 before $\sin(3\theta + 15°)$ simply *stretches** the height of the sine curve by a factor of 2.5 without

---

* One can look on the coefficient 3 of 3θ which changes the period from 2π (for $\sin\theta$) to $2\pi/3$ (for $\sin 3\theta$) as *contracting* the width of the curve. A coefficient of less than one before θ, for example, $\sin \frac{1}{4}\theta$, would *stretch* the width from a period of 2π (for $\sin\theta$) to a period of 8π (for $\sin \frac{1}{4}\theta$).

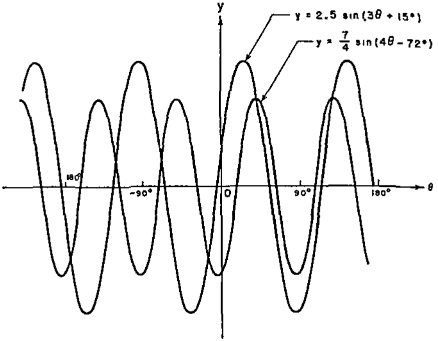


Fig 3 7

affecting the period or phase angle We call 2 5 the *amplitude* of the sine function This simply means that the maximum value assumed by 2 5 sin $(3\theta + 15°)$ for any value of θ is +2 5 For example,

$$y = \tfrac{7}{4}\sin(4\theta - 72°)$$

has period $\pi/2$, amplitude $+\frac{7}{4}$, and phase $-72°$ (see Fig 3 7), while

$$y = A\cos(a\theta + p)$$

has period $2\pi/a$, amplitude* A, and phase angle p

A phase angle simply shifts to the left the sine or cosine curve by the amount of the phase angle Thus since

$$y = \sin(\theta + 90°) = \cos\theta$$

one can say y is a cosine curve (with zero phase angle) or a sine curve (with 90° phase shift) If we do not specify the phase angle, we cannot distinguish between sine and cosine curves We therefore frequently use the term

* We are assuming that A is positive If A were negative, then we could write $y = -A\sin(a\theta - 90° + p)$ where now $-A$ is positive Thus the amplitude would be $-A$ and the phase $p - 90°$ *It is customary to assume a positive number for the amplitude*

sinusoidal function to describe a sine or cosine curve regardless of the phase angle (if any). In general, since

$$\sin(\theta + p) = \cos\left[\theta + \left(p - \frac{\pi}{2}\right)\right]$$

we can represent a sinusoidal function as either a sine or cosine curve with the appropriate phase angle. We have plotted some representative functions in Fig. 3.8.

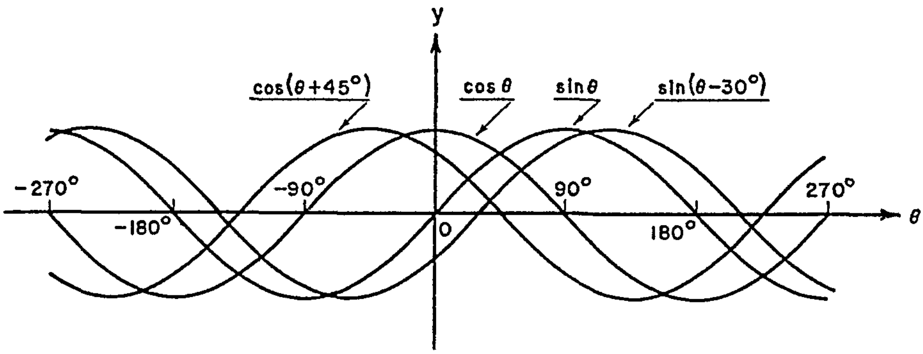


Fig. 3.8

In electrical engineering one frequently has occasion to deal with currents or voltages as functions of time t. A graph of a current or voltage plotted against time is often sinusoidal or perhaps even of the more general periodic form shown in Fig. 3.6. It is customary to call these curves *wave forms*. Thus, for example, Fig. 3.6a is referred to as a "triangular wave" or "saw tooth wave"; the sine function is referred to as a "sine wave."

More generally, we write a sinusoidal voltage, $e(t)$, as

$$e(t) = \sin \omega t. \tag{20}$$

Since time t is generally assumed to be measured in seconds, and since ωt has the dimensions of radians, we infer that ω must be measured in radians/second. We call ω the *angular frequency* of the sine curve of (20). The *frequency* f is defined as

$$f = \frac{\omega}{2\pi} \tag{21}$$

and is measured in cycles/second. The *period* T of the sine wave is simply $1/f$ since

$$e(t + T) = \sin \omega(t + T) = \sin \omega\left(t + \frac{1}{f}\right) = \sin\left(\omega t + \frac{\omega}{f}\right)$$

$$= \sin(\omega t + 2\pi) = \sin \omega t = e(t) \tag{22}$$

by (21) Thus we write

$$T = \frac{1}{f}\text{seconds} \tag{23}$$

Some representative functions are plotted in Fig 3 9

One also refers to the "period" as a complete *cycle* Thus the term "cycles/second" used in describing f Actually, since f has dimensions of cycles/second, the period T should have dimensions of seconds/cycle However, it is understood that the period T is the time to complete one cycle,

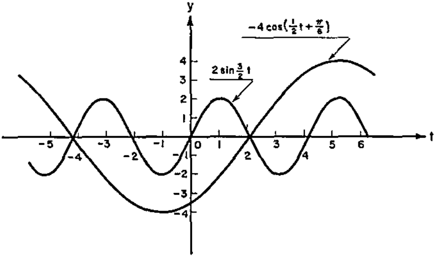


Fig 3 9

by definition of period, and it is customary to express T in seconds If the student wishes to be very precise, he may write $T = T \cdot 1$ where T has the dimensions of seconds/cycle, and the "1" of the product $T \cdot 1$ carries the dimensions of cycles

It is often convenient to classify functions as *even* or *odd* We say a function $f(x)$ is an *even function* if

$$f(x) = f(-x) \tag{24}$$

Examples of even functions are $\cos x$, $3\sec 4x$, 1, x^2, x^4 We say a function $f(x)$ is an *odd function* if

$$f(x) = -f(-x) \tag{25}$$

Examples of odd functions are $2\sin x$, $-\tan \pi x$, x, x^3 Most functions are neither even nor odd For example,

$$f(x) = x^2 - 3x$$

is neither even nor odd since

$$f(-x) = (-x)^2 - 3(-x) = x^2 + 3x$$

is equal to neither $f(x)$ nor $-f(x)$. However, we do have the remarkable result that *any* function may be written as an even function plus an odd function. For, clearly,

$$f(x) = \frac{f(x) + f(-x)}{2} + \frac{f(x) - f(-x)}{2}$$

identically. Now let

$$g(x) = \tfrac{1}{2}[f(x) + f(-x)]$$

and

$$h(x) = \tfrac{1}{2}[f(x) - f(-x)].$$

We assert that $g(x)$ is an even function and $h(x)$ an odd function; for,

$$g(-x) = \tfrac{1}{2}[f(-x) + f(x)] = \tfrac{1}{2}[f(x) + f(-x)] = g(x)$$

and

$$h(-x) = \tfrac{1}{2}[f(-x) - f(x)] = -\tfrac{1}{2}[f(x) - f(-x)] = -h(x).$$

Finally we would like to make one last definition. The trigonometric functions sine and cosine are sometimes called *circular functions* because of their intimate connection with a circle. Later, in Chapter 14 of Part II, we shall introduce *hyperbolic functions*, which bear the same relation to a

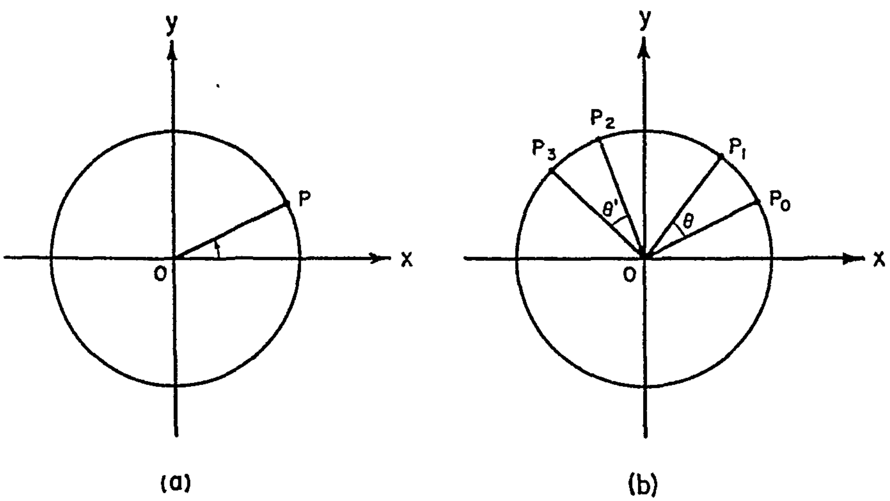


Fig. 3.10

curve called the hyperbola as the circular functions do to the circle. Other interpretations of the relation of sine and cosine to the circle may be given. One argument is to consider a circle, Fig. 3.10a, where we assume the radius OP is rotating about O at a constant speed or, as we say in engineering parlance, moving with a constant *angular velocity*. This means that in equal intervals of time equal angles are swept out. For example, suppose at times t_0, t_1, t_2, t_3 the point P (the tip of the radius) is at the points P_0, P_1, P_2, P_3

respectively Then if the time differences $t_1 - t_0$ and $t_3 - t_2$ are equal, the angles θ and θ' will be equal (Fig 3 10b) If the tip P of the radius makes a complete revolution in T seconds, then we say that it is rotating at an angular velocity of $\omega = 2\pi/T$ radians per second Thus, for example, $\theta = \omega(t_1 - t_0)$ Now if we plot the length of the projection of P on the y-axis versus t (Fig 3 11), the point P is said to move in *simple harmonic motion* If we assume that

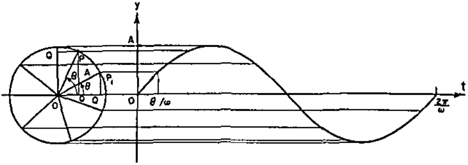


Fig 3 11

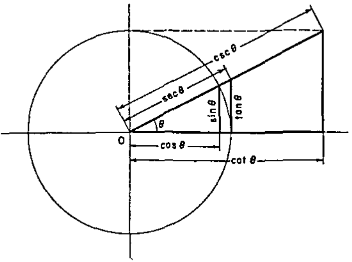


Fig 3 12

the radius starts to rotate with constant angular velocity ω at time $t = 0$ when it is on the positive direction of the x axis, then the angle swept out in the time t_1 seconds will be simply $\theta_1 = \omega t_1$ Thus

$$y = Q_1P_1 = A \sin \theta_1 = A \sin \omega t_1,$$

where A is the radius of the circle A plot of the ordinate versus time will yield a sine wave of angular frequency ω and amplitude A Similarly, the point Q, the projection of P on the x-axis, also moves in simple harmonic motion, and

$$x = OQ = Q'P = A \cos \theta = A \cos \omega t.$$

Thus simple harmonic motion is nothing more than sinusoidal motion. This fact is used in elementary physics and vibration theory. For example, the motion of a simple pendulum in the absence of damping approaches simple harmonic motion for small angles.

Another interpretation of the trigonometric functions in terms of a circle may be made by considering Fig. 3.12, known as a *line diagram*. If the circle has unit radius, then the numerical lengths of the various semichords, tangents, and secants have precisely the numerical values indicated on the diagram. This again emphasizes the term "circular functions."

EXERCISE 3-3

Express the following functions as sinusoids with phase angles whose magnitude is less than 45° (e.g., $\sin(\theta + 70°) = \cos(\theta - 20°)$):

1. $\sin(\theta + 100°)$.
2. $\cos(\theta - 90°)$.
3. $\cos(\theta - 50°)$.
4. $\sin(\theta + 50°)$.
5. $\sin(\theta + 134°)$.

Express the following functions as sinusoids having positive amplitudes and phase angles whose magnitude is equal to or less than 90°:

6. $17 \sin(\theta + 130°)$.
7. $10 \sin(\omega t - 217°)$.
8. $23 \cos(x - 197°)$.
9. $-4 \sin(\alpha + 100°)$.
10. $11 \cos(\beta + 350°)$.
11. $49 \cos(x - 160°)$.
12. $-37 \sin(\theta + 120°)$.
13. $2 \cos(t + 250°)$.
14. $-3.7 \cos(x - 200°)$.
15. $-5 \sin(y + 290°)$.

What is the period of each of the following functions?

16. $\sin 5x$.
17. $\cos 14x$.
18. $\sin(11x + 40°)$.
19. $\cos(x + \pi/2)$.
20. $\sin(\omega t + 14°)$.

PROBLEMS

1. Let $f(x)$ be any function of x. Then

$$f(x) = g(x) + h(x)$$

where $g(x)$ is an odd function and $h(x)$ is an even function Show that $g(x)$ and $h(x)$ are unique That is, show there is only one way of decomposing $f(x)$ into even and odd functions

2 Show that the only function which is both even and odd is $f(x) \equiv 0$

3. Show that, when a is much less than one, then

$$\frac{1}{1 + a} = 1 - a$$

4. The sun subtends an angle at the earth of 32′ 4″ The distance of the sun from the earth is 92,000,000 miles What is the diameter of the sun?

5. If the sun subtends an angle of 32′ 4″ at the earth, how far from the eye must a penny be held to just hide the sun? The diameter of a penny is $\frac{3}{4}$ inch

6. If the moon subtends an angle at the earth of 30 49″ when it is 236,500 miles away, what is the diameter of the moon?

7. A railroad is inclined 53′ above the horizontal How many feet does it rise in a mile?

8. Draw line diagrams for an angle θ in the second, third, and fourth quadrants

4

ADDITION THEOREMS

In earlier chapters we developed numerous relations among the trigonometric functions. Most of them were deducible almost immediately from the appropriate triangle. There are, however, two very important formulas which are not as obvious. These are known as *addition formulas*. They are

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi \tag{1}$$

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi \tag{2}$$

and simply express the sine and cosine of the *sum* of any two angles in terms of the sines and cosines of the individual angles. Our main purpose in this chapter will be to derive these formulas and consider some of their numerous consequences. Other applications will appear in future chapters.

Before proving (1) and (2), let us whet the student's appetite with a simple application. Suppose we wish to compute exactly the cosine of 15°. No technique we have developed until now is applicable. However, let $\theta = \phi = 15°$ in (1). Then

$$\cos(15° + 15°) = \cos^2 15° - \sin^2 15°.$$

But $\sin^2 15° = 1 - \cos^2 15°$. Hence

$$\cos 30° = 2\cos^2 15° - 1$$

and

$$\cos 15° = \sqrt{\frac{1 + \cos 30°}{2}}.$$

Since $\cos 30° = \frac{1}{2}\sqrt{3}$, the above formula yields

$$\cos 15° = \tfrac{1}{2}\sqrt{2 + \sqrt{3}}. \tag{3}$$

4.1. The Addition Formulas

One finds many different proofs of (1) and (2) in various books Most of the proofs readily establish (1) or (2) when θ, ϕ, and $\theta + \phi$ are all acute For general angles, the proofs tend to become long-winded in their consideration of the various cases for example, θ in the first quadrant, ϕ in the second quadrant, and $\theta + \phi$ in the third quadrant, or θ in the first quadrant, ϕ and $\theta + \phi$ in the second quadrant, etc We shall give a proof which avoids this annoying treatment of a multiplicity of special cases *

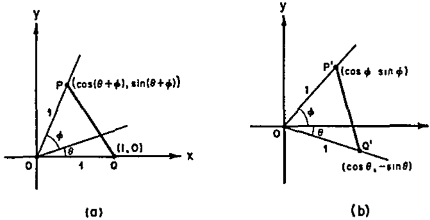


Fig 4 1

Suppose we consider two angles θ and ϕ as in Fig 4 1 The lines OP and OQ of Fig 4 1a as well as the lines OP and OQ' of Fig 4 1b are of unit length Hence triangles OPQ and $OP'Q'$ are congruent, and in particular $PQ = P\,Q'$ In this illustration, θ and ϕ are both acute angles, as is their sum In Fig 4 2 we portray the situation in which θ is acute and ϕ is in the third quadrant Now whether we refer to Fig 4 1 or Fig 4 2, the coordinates of P are always $(\cos(\theta + \phi), \sin(\theta + \phi))$, the coordinates of Q are (1,0), the coordinates of P' are $(\cos\phi, \sin\phi)$, and the coordinates of Q' are $(\cos(-\theta), \sin(-\theta)) = (\cos\theta, -\sin\theta)$ Similarly, no matter in which quadrants θ, ϕ, and $\theta + \phi$ lie, the coordinates of the corresponding points P, Q, P', Q' will always be as described above

Now let us apply the distance formula [(13) of Section 1 4 of Chapter 1] to write

$$\begin{aligned} PQ &= \sqrt{[\cos(\theta + \phi) - 1]^2 + [\sin(\theta + \phi) - 0]^2} \\ &= \sqrt{\cos^2(\theta + \phi) - 2\cos(\theta + \phi) + 1 + \sin^2(\theta + \phi)} \\ &= \sqrt{2 - 2\cos(\theta + \phi)} \end{aligned} \tag{4}$$

* In Chapter 8, we shall indicate another (essentially similar) proof using the law of cosines, and in Section 13 6 of Chapter 13, we shall give an elegant analytic proof using complex numbers

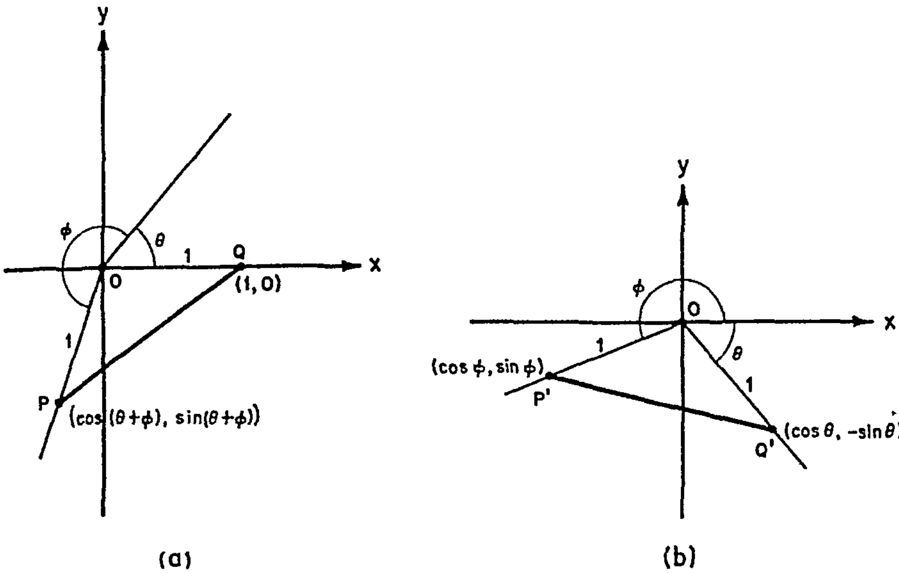


Fig. 4.2

[since $\sin^2(\theta + \phi) + \cos^2(\theta + \phi) = 1$]. The distance formula applied to Fig. 4.1b or Fig. 4.2b yields

$$
\begin{aligned}
P'Q' &= \sqrt{[\cos\phi - \cos\theta]^2 + [\sin\phi + \sin\theta]^2} \\
&= \sqrt{[\cos^2\phi - 2\cos\theta\cos\phi + \cos^2\theta] + [\sin^2\phi + 2\sin\theta\sin\phi + \sin^2\theta]} \\
&= \sqrt{2 - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi}. \qquad (5)
\end{aligned}
$$

But $PQ = P'Q'$, since they are corresponding parts of congruent triangles. Hence $(PQ)^2 = (P'Q')^2$. Thus from (4) and (5)

$$2 - 2\cos(\theta + \phi) = 2 - 2\cos\theta\cos\phi + 2\sin\theta\sin\phi$$

which immediately reduces to (1):

$$\cos(\theta + \phi) = \cos\theta\cos\phi - \sin\theta\sin\phi.$$

If we replace ϕ by $-\phi$ in (1), we obtain

$$
\begin{aligned}
\cos(\theta - \phi) &= \cos\theta\cos(-\phi) - \sin\theta\sin(-\phi) \\
&= \cos\theta\cos\phi + \sin\theta\sin\phi, \qquad (6)
\end{aligned}
$$

since $\sin\phi$ is an odd function and $\cos\phi$ is an even function. To deduce (2), we recall that

$$\sin x = \cos\left(\frac{\pi}{2} - x\right).$$

Thus if we let $x = \theta + \phi$, and apply (6), we get

$$\sin(\theta + \phi) = \cos\left(\frac{\pi}{2} - \theta - \phi\right) = \cos\left[\left(\frac{\pi}{2} - \theta\right) - \phi\right]$$

$$= \cos\left(\frac{\pi}{2} - \theta\right)\cos\phi + \sin\left(\frac{\pi}{2} - \theta\right)\sin\phi \qquad (7)$$

But $\cos\left(\frac{\pi}{2} - \theta\right) = \sin\theta$ and $\sin\left(\frac{\pi}{2} - \theta\right) = \cos\theta$ Equation (7) then becomes (2).

$$\sin(\theta + \phi) = \sin\theta\cos\phi + \cos\theta\sin\phi$$

The analog of (6) for the sine is obtained by replacing ϕ by $-\phi$ in (2)

$$\sin(\theta - \phi) = \sin\theta\cos(-\phi) + \cos\theta\sin(-\phi)$$

$$= \sin\theta\cos\phi - \cos\theta\sin\phi \qquad (8)$$

One immediate consequence of (1) and (2) is that we may obtain $\tan(\theta + \phi)$ in terms of $\tan\theta$ and $\tan\phi$ For, by definition of tangent,

$$\tan(\theta + \phi) = \frac{\sin(\theta + \phi)}{\cos(\theta + \phi)} = \frac{\sin\theta\cos\phi + \cos\theta\sin\phi}{\cos\theta\cos\phi - \sin\theta\sin\phi}$$

by (1) and (2) Now divide numerator and denominator of this expression by $\cos\theta\cos\phi$ We obtain

$$\tan(\theta + \phi) = \frac{\dfrac{\sin\theta}{\cos\theta} + \dfrac{\sin\phi}{\cos\phi}}{1 - \dfrac{\sin\theta}{\cos\theta}\dfrac{\sin\phi}{\cos\phi}} = \frac{\tan\theta + \tan\phi}{1 - \tan\theta\tan\phi} \qquad (9)$$

Since tangent is an odd function we may write immediately that

$$\tan(\theta - \phi) = \frac{\tan\theta - \tan\phi}{1 + \tan\theta\tan\phi} \qquad (10)$$

We illustrate the use of the above formulas with some simple examples

Example 1. Find $\cos 75°$

Solution We first write $75° = 45° + 30°$ Then

$$\cos 75° = \cos(45° + 30°) = \cos 45°\cos 30° - \sin 45°\sin 30°$$

by (1) But $\cos 45° = \sin 45° = \frac{\sqrt{2}}{2}$ and $\cos 30° = \frac{\sqrt{3}}{2}$, $\sin 30° = \frac{1}{2}$

Hence

$$\cos 75° = \frac{\sqrt{6} - \sqrt{2}}{4}$$

Example 2. If $\cos x = \frac{3}{5}$ and $\cos y = \frac{5}{13}$ where x and y are both in the first quadrant, find $\sin (x + y)$.

Solution: From the formula $\sin x = \sqrt{1 - \cos^2 x}$ we conclude that

$$\sin x = \sqrt{1 - (\tfrac{3}{5})^2} = \tfrac{4}{5}$$

and similarly that $\sin y = \frac{12}{13}$. By (2)

$$\begin{aligned}\sin (x + y) &= \sin x \cos y + \cos x \sin y \\ &= \tfrac{4}{5}\tfrac{5}{13} + \tfrac{3}{5}\tfrac{12}{13} = \tfrac{56}{65}.\end{aligned}$$

Example 3. Express $\sin (x + y + z)$ in terms of $\sin x$, $\sin y$, $\sin z$, $\cos x$, $\cos y$, $\cos z$.

Solution: Let $u = y + z$. Then from (2)

$$\sin (x + y + z) = \sin (x + u) = \sin x \cos u + \cos x \sin u. \tag{11}$$

But

$$\cos u = \cos (y + z) = \cos y \cos z - \sin y \sin z \tag{12}$$

by (1) and again by (2)

$$\sin u = \sin (y + z) = \sin y \cos z + \cos y \sin z. \tag{13}$$

Substituting (12) and (13) in (11), we obtain

$$\begin{aligned}\sin (x + y + z) &= \sin x[\cos y \cos z - \sin y \sin z] \\ &\quad + \cos x[\sin y \cos z + \cos y \sin z] \\ &= -\sin x \sin y \sin z + \sin x \cos y \cos z \\ &\quad + \cos x \sin y \cos z + \cos x \cos y \sin z.\end{aligned}$$

Example 4. Express $\tan (x + 60°)$ in terms of $\tan x$.

Solution: From (9)

$$\tan (x + 60°) = \frac{\tan x + \tan 60°}{1 - \tan x \tan 60°} = \frac{\tan x + \sqrt{3}}{1 - \sqrt{3} \tan x},$$

since $\tan 60° = \sqrt{3}$.

Equations (1) and (2) should be committed to memory. Let us mention a very practical application of these important formulas. We recall from Chapter 2 that we derived a multiplicity of formulas for the sine and cosine (and other trigonometric functions as well) of $\pm 90° \pm \theta$, $\pm 180° \pm \theta$, and similar angles. However, instead of memorizing this imposing array, we find it more convenient to use (1) or (2). For example,

$$\cos (180° - \theta) = \cos 180° \cos \theta + \sin 180° \sin \theta = -\cos \theta,$$

since $\cos 180° = -1$ and $\sin 180° = 0$. (We have used the slight variation, (6), of (1) in the above illustration.) The values of $\cos 180°$ and $\sin 180°$

should be recalled from a visual picture of the graphs of the trigonometric functions With a little practice, the above type of calculation can be done mentally

EXERCISE 4-1

Determine the values of the following functions Do not reduce to decimal form

1 $\cos 22\frac{1}{2}°$
2 $\cos 105°$
3 $\tan 75°$
4 $\tan 15°$
5 $\sin 195°$
6 $\cos 165°$
7 $\sin 7\frac{1}{2}°$
8 $\sin 255°$

Find $\sin (x + y)$ for the following cases

9 $\sin x = 0.6$, $\cos y = 0.8$, x and y in I quadrant
10 $\sin y = 0.4$, $\cos x = 0.4$, x and y in I quadrant
11 $\sin x = 0.5$, $\sin y = 0.7$, x in I, y in II quadrant
12 $\cos x = 0.8$, $\cos y = 0.8$, x in I, y in IV quadrant
13 $\cos y = -0.9$ $\sin x = 0.6$ x in II y in III quadrant
14 $\sin x = -0.1$, $\sin y = 0.3$, x in IV, y in I quadrant

Find $\cos (x - y)$ when

15 $\sin x = 0.6$, $\cos y = 0.9$, x and y in I quadrant
16 $\sin x = 0.8$, $\sin y = 0.7$ x in IV, y in II quadrant
17 $\cos x = 0.1$, $\sin y = 0.5$, x in II, y in IV quadrant
18 $\sin x = 0.4$, $\cos y = 0.4$, x in III, y in I quadrant
19 $\cos x = 0.2$ $\cos y = 0.3$, x in I y in III quadrant
20 $\cos x = 0.3$, $\sin y = 0.1$, x in II y in I quadrant

Reduce each expression to a single term

21 $\sin 100° \sin 35° + \cos 100° \cos 35°$
22 $\cos 95° \sin 15° - \sin 95° \cos 15°$
23 $\cos 22° \cos 17° - \sin 22° \sin 17°$
24 $\sin 43° \cos 18° + \sin 18° \cos 43°$
25 $\sin^2 50° + \cos^2 50°$

4·2. The Double-Angle and Half-Angle Formulas

If we let $\theta = \phi$ in (1) and (2), we obtain the *double angle formulas*

$$\cos 2\theta = \cos^2 \theta - \sin^2 \theta \tag{14a}$$

$$= 2 \cos^2 \theta - 1 \tag{14b}$$

$$= 1 - 2 \sin^2 \theta, \tag{14c}$$

and

$$\sin 2\theta = 2 \sin \theta \cos \theta \tag{15}$$

From (9) we also get

$$\tan 2\theta = \frac{2 \tan \theta}{1 - \tan^2 \theta}. \tag{16}$$

We can also deduce the *half-angle formulas* from (14) by solving (14b) for $\cos \theta$:

$$\cos \theta = \pm\sqrt{\frac{1 + \cos 2\theta}{2}}$$

and then letting $\theta = \dfrac{\phi}{2}$

$$\cos \frac{\phi}{2} = \pm\sqrt{\frac{1 + \cos \phi}{2}}. \tag{17}$$

We thus have a formula for the cosine of half the angle in terms of the cosine of the angle. Hence the name "half-angle formula." Similarly, solving (14c) for $\sin \theta$ and then letting $\theta = \frac{1}{2}\phi$ leads to the half-angle formula

$$\sin \frac{\phi}{2} = \pm\sqrt{\frac{1 - \cos \phi}{2}}. \tag{18}$$

The square roots appearing in (17) and (18) could be plus or minus. Clearly we use a plus sign with (17) if $\frac{1}{2}\phi$ is in the first or fourth quadrant and a minus sign if $\frac{1}{2}\phi$ is in the second or third quadrant. Similarly, (18) carries a positive sign if $\frac{1}{2}\phi$ is in the first or second quadrant and a negative sign otherwise. For example, to find the exact value of $\cos 112° 30'$, we write

$$\cos 112° 30' = -\sqrt{\frac{1 + \cos 225°}{2}}.$$

But $\cos 225° = -\cos 45° = -\dfrac{\sqrt{2}}{2}$ and

$$\cos 112° 30' = -\sqrt{\frac{1}{2}\left(1 - \frac{\sqrt{2}}{2}\right)} = -\tfrac{1}{2}\sqrt{2 - \sqrt{2}}.$$

The half-angle formula for the tangent is given by

$$\tan \frac{\phi}{2} = \frac{\sin \dfrac{\phi}{2}}{\cos \dfrac{\phi}{2}} = \pm\sqrt{\frac{1 - \cos \phi}{1 + \cos \phi}}. \tag{19}$$

Now by simple algebra we may eliminate the square root sign in (19). For suppose we multiply numerator and denominator, under the radical sign, by $1 - \cos\phi$ Then

$$\tan\frac{\phi}{2} = \pm\sqrt{\frac{(1-\cos\phi)(1-\cos\phi)}{(1+\cos\phi)(1-\cos\phi)}} = \pm\sqrt{\frac{(1-\cos\phi)^2}{1-\cos^2\phi}}$$

$$= \pm\sqrt{\frac{(1-\cos\phi)^2}{\sin^2\phi}} = \frac{1-\cos\phi}{\sin\phi} \tag{20}$$

An alternate form for $\tan\frac{\phi}{2}$ may be obtained by multiplying by $1 + \cos\phi$

$$\tan\frac{\phi}{2} = \pm\sqrt{\frac{(1-\cos\phi)(1+\cos\phi)}{(1+\cos\phi)(1+\cos\phi)}} = \pm\sqrt{\frac{1-\cos^2\phi}{(1+\cos\phi)^2}}$$

$$= \frac{\sin\phi}{1+\cos\phi} \tag{21}$$

Again one must be careful of the sign If $\frac{1}{2}\phi$ is in the first or third quadrant we use the plus sign before the radical of (19) If $\frac{1}{2}\phi$ is in the second or fourth quadrant, we use the negative sign As far as (20) and (21) are concerned, the student may readily verify that $\tan\frac{1}{2}\phi$ and $\sin\phi$ *always* have the same sign regardless of the quadrant in which the angle lies For instance, if $180° < \frac{1}{2}\phi < 270°$, then $360° < \phi < 450°$ or $0 < \phi < 180°$ Clearly both $\tan\frac{1}{2}\phi$ and $\sin\phi$ are positive in this case (Of course, $1 \pm \cos\phi$ is never negative)

One can also use the above basic formulas to express products of trigonometric functions in terms of sums of trigonometric functions and vice versa For example, if we write

$$\sin(x+y) = \sin x\cos y + \cos x\sin y,$$

$$\sin(x-y) = \sin x\cos y - \cos x\sin y,$$

and then add and divide by 2, we get

$$\sin x\cos y = \tfrac{1}{2}[\sin(x+y) + \sin(x-y)] \tag{22}$$

Similarly, if we write

$$\cos(x+y) = \cos x\cos y - \sin x\sin y,$$

$$\cos(x-y) = \cos x\cos y + \sin x\sin y,$$

and then add and divide by 2, we get

$$\cos x\cos y = \tfrac{1}{2}[\cos(x+y) + \cos(x-y)] \tag{23}$$

Subtraction and division by 2 yields

$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]. \tag{24}$$

Now if we let $u = x + y$, $v = x - y$ in (22), we obtain

$$\sin u + \sin v = 2 \sin \tfrac{1}{2}(u + v) \cos \tfrac{1}{2}(u - v), \tag{25}$$

since $x = \frac{1}{2}(u + v)$ and $y = \frac{1}{2}(u - v)$. Similar changes of variable reduce (23) and (24) to

$$\cos u + \cos v = 2 \cos \tfrac{1}{2}(u + v) \cos \tfrac{1}{2}(u - v), \tag{26}$$

and

$$\cos u - \cos v = -2 \sin \tfrac{1}{2}(u + v) \sin \tfrac{1}{2}(u - v), \tag{27}$$

respectively.

EXERCISE 4-2

Plot the following pairs of graphs, each to the same scale:

1. $\sin \theta$, $\sin 2\theta$.
2. $\cos \theta$, $\cos 2\theta$.
3. $\sin \alpha$, $2 \sin \alpha$.
4. $\cos \theta$, $2 \cos \theta$.
5. $\sin^2 \theta$, $-\cos 2\theta$.
6. $\cos^2 \theta$, $\cos 2\theta$.
7. $\sin t$, $\sin t/2$.
8. $\cos \theta$, $\cos \theta/2$.
9. $\tan \theta$, $\tan 2\theta$.
10. $\tan \theta$, $2 \tan \theta$.

Find $\sin 2A$, $\sin A/2$, $\cos 2A$, $\cos A/2$ for angles having the following properties:

11. $\cos A = \frac{12}{13}$, A in I quadrant.
12. $\sin A = \frac{4}{5}$, A in II quadrant.
13. $\tan A = \frac{3}{2}$, A in I quadrant.
14. $\sin A = -\frac{5}{8}$, A in III quadrant.
15. $\cos A = \frac{2}{5}$, A in IV quadrant.

4.3. The Axiom of Induction*

Let us consider a familiar formula of algebra in order to illustrate a fundamental principle of mathematics. The student recalls that

$$\begin{aligned} (a + b) &= a + b, \\ (a + b)^2 &= a^2 + 2ab + b^2, \\ (a + b)^3 &= a^3 + 3a^2b + 3ab^2 + b^3, \\ (a + b)^4 &= a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4, \end{aligned} \tag{28}$$

* This section may be omitted on a first reading or in a short course.

and that in general it is stated that*

$$(a+b)^n = \sum_{j=0}^{n} \binom{n}{j} a^j b^{n-j}, \tag{29}$$

where n is a positive integer The symbol $\binom{n}{j}$ is called the *binomial coefficient* and is defined as

$$\binom{n}{j} = \frac{n!}{j!\,(n-j)!}, \tag{30}$$

where $n!$ (read "n factorial") is the product of the positive integers $1, 2, \ldots, n$ For example, $4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$ and

$$\binom{7}{5} = \frac{7!}{5!\,2!} = \frac{7\ 6 \cdot 5!}{5!\ 2} = \frac{42}{2} = 21.$$

The symbol $0!$ is defined as 1. Thus

$$\binom{n}{0} = \frac{n!}{0!\,n!} = 1,$$

for any nonnegative integer n

We recall that (29) is called the *binomial theorem*. For instance, if we let $n = 3$ in (29), we have

$$\begin{aligned}(a+b)^3 &= \sum_{j=0}^{3} \binom{3}{j} a^j b^{3-j} \\ &= \binom{3}{0} a^0 b^3 + \binom{3}{1} a^1 b^{3-1} + \binom{3}{2} a^2 b^{3-2} + \binom{3}{3} a^3 b^{3-3}. \end{aligned} \tag{31}$$

Since

$$\binom{3}{0} = \frac{3!}{0!\,3!} = 1 = \binom{3}{3},$$

$$\binom{3}{1} = \frac{3!}{1!\,2!} = 3 = \binom{3}{2},$$

* A sum of n terms, say

$$a_1 + a_2 + \quad + a_n,$$

may be compactly written by using the "Σ" (capital Greek letter "sigma") notation Thus

$$\sum_{k=1}^{n} a_k = a_1 + a_2 + \ldots + a_n$$

The index k is called a *dummy index*, since, for example,

$$\sum_{k=1}^{n} a_k = \sum_{i=1}^{n} a_i = \sum_{j=1}^{n} a_j,$$

independent of the index

equation (31) becomes

$$(a + b)^3 = b^3 + 3ab^2 + 3a^2b + a^3$$

[cf. (28)].

However, how did we arrive at (29)? It certainly seems plausible from an inspection of (28)—yet, as we know, this does not constitute a mathematical proof. One way to *prove* (29) is to use the *Axiom of Induction.*

Axiom of Induction. Suppose that a property $\mathscr{P}$ can be shown to hold for $n = 1$. If it can be proved for $n = k + 1$, *assuming* it is true for $n = 1, 2, \ldots, k$, then it is true for all positive integers n.

This is an *axiom* and hence not subject to proof. However, it certainly looks reasonable. The Axiom of Induction is one of the fundamental axioms of mathematics. It is used, for example, in developing the number system where such theorems as: "If m and n are integers, then $n + m = m + n$" are proved. Has the reader ever questioned or tried to prove this result?

Let us apply the Axiom of Induction to the proof of a few simple theorems before attempting the proof of the binomial theorem and more complicated trigonometric propositions.

Example 1. Prove that

$$\sum_{j=1}^{n} j = \frac{n(n+1)}{2}$$

by induction.

Solution: Certainly the formula is true if $n = 1$ since

$$\sum_{j=1}^{1} j = 1,$$

and

$$\frac{1 \cdot (1+1)}{2} = 1.$$

Now assume it true for $n = k$:

$$\sum_{j=1}^{k} j = \frac{k(k+1)}{2}. \tag{32}$$

We must prove that

$$\sum_{j=1}^{k+1} j = \frac{(k+1)(k+2)}{2}. \tag{33}$$

To do this write

$$\sum_{j=1}^{k+1} j = \sum_{j=1}^{k} j + (k+1) = \frac{k(k+1)}{2} + (k+1),$$

by our induction hypothesis. But

$$\frac{k(k+1)}{2} + (k+1) = \frac{k(k+1) + 2(k+1)}{2} = \frac{(k+1)(k+2)}{2},$$

which establishes (33).

The student, of course, recognizes this example as an arithmetic series.

Example 2. Find

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} \tag{34}$$

by using the Axiom of Induction

Solution Here the problem is more complicated We must first make an educated guess as to what the sum of (34) is, and then try to prove it by induction To do this, let us consider some special cases

$$\sum_{j=1}^{1} \frac{1}{j(j+1)} = \frac{1}{1\ 2} = \frac{1}{2}, \tag{35}$$

$$\sum_{j=1}^{2} \frac{1}{j(j+1)} = \frac{1}{1\ 2} + \frac{1}{2\ 3} = \frac{1}{2} + \frac{1}{6} = \frac{2}{3},$$

$$\sum_{j=1}^{3} \frac{1}{j(j+1)} = \frac{1}{1\ 2} + \frac{1}{2\ 3} + \frac{1}{3\ 4} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} = \frac{3}{4},$$

$$\sum_{j=1}^{4} \frac{1}{j(j+1)} = \frac{1}{1\ 2} + \frac{1}{2\ 3} + \frac{1}{3\ 4} + \frac{1}{4\ 5} = \frac{1}{2} + \frac{1}{6} + \frac{1}{12} + \frac{1}{20} = \frac{4}{5}$$

Hence it seems plausible that

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{n}{n+1} \tag{36}$$

Let us see if we can rigorously prove this by the use of induction Certainly it is true if $n = 1$ by (35) Now *assume* it true for $n = k$

$$\sum_{j=1}^{k} \frac{1}{j(j+1)} = \frac{k}{k+1}$$

Then

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} = \frac{k}{k+1} + \frac{1}{(k+1)(k+2)},$$

by our induction hypothesis But

$$\frac{k}{k+1} + \frac{1}{(k+1)(k+2)} = \frac{k(k+2)+1}{(k+1)(k+2)} = \frac{k^2+2k+1}{(k+1)(k+2)}$$

$$= \frac{(k+1)^2}{(k+1)(k+2)} = \frac{k+1}{k+2} = \frac{k+1}{(k+1)+1}.$$

Accordingly, it is true for $n = k + 1$, which proves (36)

If we had been so unfortunate as to choose the wrong law, we would not have been able to prove our theorem For example, suppose we thought

$$\sum_{j=1}^{n} \frac{1}{j(j+1)} = \frac{1}{2n}$$

Now the above formula is certainly true if $n = 1$. However,

$$\sum_{j=1}^{k+1} \frac{1}{j(j+1)} = \sum_{j=1}^{k} \frac{1}{j(j+1)} + \frac{1}{(k+1)(k+2)} = \frac{1}{2k} + \frac{1}{(k+1)(k+2)}$$
$$= \frac{k^2 + 5k + 2}{2k(k+1)(k+2)},$$

which is not the same as

$$\frac{1}{2(k+1)}.$$

Let us now use the Axiom of Induction to prove the binomial theorem (29). After we have done this, we shall apply it to a nontrivial trigonometric problem.

Since (29) is true for $n = 1$ (and $n = 2, 3, 4$), let us assume it true for $n = k$:

$$(a + b)^k = \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}, \tag{38}$$

and *prove* it true for $n = k + 1$. Now from (38)

$$(a + b)^{k+1} = (a + b)(a + b)^k = a(a + b)^k + b(a + b)^k$$
$$= a \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j} + b \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j}$$
$$= \sum_{j=0}^{k} \binom{k}{j} a^{j+1} b^{k-j} + \sum_{j=0}^{k} \binom{k}{j} a^j b^{k-j+1}.$$

If we make the change of dummy index $j = r - 1$ in the first sum and $j = r$ in the second sum,

$$(a + b)^{k+1} = \sum_{r=1}^{k+1} \binom{k}{r-1} a^r b^{k-r+1} + \sum_{r=0}^{k} \binom{k}{r} a^r b^{k-r+1}$$
$$= \sum_{r=1}^{k} \binom{k}{r-1} a^r b^{k-r+1} + \binom{k}{k} a^{k+1} b^{k-(k+1)+1}$$
$$+ \binom{k}{0} a^0 b^{k-0+1} + \sum_{r=1}^{k} \binom{k}{r} a^r b^{k-r+1},$$

where we have explicitly written out the last term of the first sum and the first term of the last sum. Combining the two summation terms under one sigma sign, the above formula becomes

$$(a + b)^{k+1} = b^{k+1} + \sum_{r=1}^{k} \left[\binom{k}{r-1} + \binom{k}{r} \right] a^r b^{k-r+1} + a^{k+1}. \tag{39}$$

But

$$\binom{k}{r-1} + \binom{k}{r} = \frac{k!}{(r-1)!\,(k-r+1)!} + \frac{k!}{r!\,(k-r)!}$$

$$= \frac{k!}{(r-1)!\,(k-r)!}\left[\frac{1}{k-r+1} + \frac{1}{r}\right]$$

$$= \frac{k!}{(r-1)!\,(k-r)!}\left[\frac{k+1}{r(k-r+1)}\right] = \frac{(k+1)!}{r!\,(k-r+1)!}$$

$$= \binom{k+1}{r}$$

Equation (39) then becomes

$$(a+b)^{k+1} = b^{k+1} + \sum_{r=1}^{k}\binom{k+1}{r}a^r b^{k-r+1} + a^{k+1}$$

We may write

$$b^{k+1} = \binom{k+1}{0}a^0 b^{k+1}$$

$$a^{k+1} = \binom{k+1}{k+1}a^{k+1}b^0$$

Thus

$$(a+b)^{k+1} = \binom{k+1}{0}a^0 b^{k+1} + \sum_{r=1}^{k}\binom{k+1}{r}a^r b^{k-r+1} + \binom{k+1}{k+1}a^{k+1}b^0$$

$$= \sum_{r=0}^{k+1}\binom{k+1}{r}a^r b^{k+1-r},$$

which proves our contention

Now let us consider an application to trigonometry We recall that

$$\cos 2\theta = 2\cos^2\theta - 1,$$

or

$$\cos^2\theta = \tfrac{1}{2}[\cos 2\theta + 1] \tag{40}$$

We are going to derive a formula for $\cos^{2n}\theta$ in terms of the cosine of multiple angles, where n is a positive integer If we square (40),

$$\cos^4\theta = \tfrac{1}{4}[\cos^2 2\theta + 2\cos 2\theta + 1], \tag{41}$$

and since $\cos^2 2\theta = \frac{1}{2}[\cos 4\theta + 1]$, we may write (41) as

$$\cos^4\theta = \tfrac{1}{8}[\cos 4\theta + 4\cos 2\theta + 3] \tag{42}$$

Thus we have expressed the fourth power of $\cos\theta$ in terms of multiple angles In order to write a general formula for $\cos^{2n}\theta$ in terms of the cosines of multiples of θ, we shall appeal to the Axiom of Induction However, what is the law we are trying to prove? It is certainly not obvious from (40) and (42) just what we should assume for $\cos^{2k}\theta$ and then attempt to prove for $\cos^{2(k+1)}\theta$. This is not a new problem (see Example 2). In fact, it is a common situation preliminary to most inductive proofs. What we do, of course, is to compute $\cos^6\theta$, $\cos^8\theta$, etc. (by directly evaluating the coefficients as we did to obtain $\cos^4\theta$), until some rule of formation of the coefficients of the cosines of the multiple angles becomes apparent. Then we try to prove this rule by induction. If we have unfortunately chosen the wrong rule, we shall not be able to prove our theorem, as we have seen above.

Toward this end let us compute, for example, $\cos^8\theta$. From (41)

$$\cos^8\theta = (\cos^4\theta)^2 = \tfrac{1}{64}[\cos^2 4\theta + 16\cos^2 2\theta + 9 + 8\cos 4\theta\cos 2\theta + 6\cos 4\theta + 24\cos 2\theta]. \quad (43)$$

But

$$\cos^2 4\theta = \tfrac{1}{2}(\cos 8\theta + 1),$$

$$\cos^2 2\theta = \tfrac{1}{2}(\cos 4\theta + 1),$$

and

$$\cos 4\theta\cos 2\theta = \tfrac{1}{2}[\cos 6\theta + \cos 2\theta]$$

by (23). Substituting all these in (43) and collecting terms yields

$$\cos^8\theta = \frac{1}{2^7}[\cos 8\theta + 8\cos 6\theta + 28\cos 4\theta + 56\cos 2\theta + 35]. \quad (44)$$

The student may now compute $\cos^{10}\theta$, $\cos^{12}\theta, \ldots$ (or, more conveniently, $\cos^{16}\theta$, $\cos^{32}\theta, \ldots$) if necessary. However, let us assume that

$$\begin{aligned}\cos^{2n}\theta &= \frac{1}{2^{2n-1}}\left[\cos 2n\theta + 2n\cos 2(n-1)\theta + \frac{(2n)(2n-1)}{2!}\cos 2(n-2)\theta \right.\\ &\qquad \left. + \cdots + \frac{(2n)!}{(n-1)!\,(n+1)!}\cos 2\theta + \frac{(2n)!}{2n!\,n!}\right] \qquad (45)\\ &= \frac{1}{2^{2n-1}}\left[\sum_{j=0}^{n}\binom{2n}{j}\cos 2(n-j)\theta - \frac{(2n)!}{2(n!)^2}\right]\end{aligned}$$

is valid for $n = k$, and attempt to prove it true for $n = k + 1$. Certainly it is true for $n = 1, 2, 4$ by our above direct calculations. If we can show it to be true for $n = k + 1$, this will, of course, by the Axiom of Induction, establish (45) for all positive integers n.

Now

$$\begin{aligned}\cos^{2(k+1)}\theta &= \cos^{2k}\theta\cos^2\theta = \cos^{2k}\theta[\tfrac{1}{2}(\cos 2\theta + 1)]\\ &= \frac{1}{2^{2k}}\left[\sum_{j=0}^{k}\binom{2k}{j}\cos 2(k-j)\theta\cos 2\theta - \frac{(2k)!}{2(k!)^2}\cos 2\theta\right] + \tfrac{1}{2}\cos^{2k}\theta \quad (46)\end{aligned}$$

by our induction hypothesis But

$$\cos 2(k-j)\theta \cos 2\theta = \tfrac{1}{2}[\cos 2(k-j+1)\theta + \cos 2(k-j-1)\theta],$$

and, substituting in (46), we get

$$\cos^{2(k+1)}\theta = \frac{1}{2^{2k+1}}\left\{\sum_{j=0}^{k}\binom{2k}{j}[\cos 2(k-j+1)\theta + \cos 2(k-j-1)\theta]\right.$$

$$\left. - \frac{(2k)!}{k!\,k!}\cos 2\theta\right\} + \frac{1}{2^{2k}}\left[\sum_{j=0}^{k}\binom{2k}{j}\cos 2(k-j)\theta - \frac{(2k)!}{2k!\,k!}\right] \quad (47)$$

But

$$\sum_{j=0}^{k}\binom{2k}{j}\cos 2(k-j+1)\,\theta = \sum_{i=-1}^{k-1}\binom{2k}{i+1}\cos 2(k-i)\theta$$

$$= \binom{2k}{0}\cos 2(k+1)\theta + \binom{2k}{1}\cos 2k\theta + \sum_{i=1}^{k-1}\binom{2k}{i+1}\cos 2(k-i)\theta,$$

on letting $i = j - 1$, and

$$\sum_{j=0}^{k}\binom{2k}{j}\cos 2(k-j-1)\theta = \sum_{i=1}^{k+1}\binom{2k}{i-1}\cos 2(k-i)\theta$$

$$= \sum_{i=1}^{k-1}\binom{2k}{i-1}\cos 2(k-i)\theta + \binom{2k}{k-1}\cos 0 + \binom{2k}{k}\cos 2\theta,$$

on changing the dummy index to $i = j + 1$

Using these results in (47) leads to

$$\cos^{2(k+1)}\theta = \frac{1}{2^{2k+1}}\left\{\sum_{i=1}^{k-1}\left[\binom{2k}{i+1} + 2\binom{2k}{i} + \binom{2k}{i-1}\right]\cos 2(k-i)\theta\right.$$

$$\left. + \cos 2(k+1)\theta + 2(k+1)\cos 2k\theta + \frac{(2k)!}{k!\,k!} + \frac{(2k)!}{(k-1)!\,(k+1)!}\right\}$$

By a straightforward manipulation of the binomial coefficients (as in the proof of the binomial theorem), we show that

$$\binom{2k}{i+1} + 2\binom{2k}{i} + \binom{2k}{i-1} = \binom{2k+2}{i+1}$$

and

$$\frac{(2k)!}{k!\,k!} + \frac{(2k)!}{(k-1)!\,(k+1)!} = \frac{1}{2}\binom{2k+2}{k+1} = \binom{2k+2}{k+1} - \frac{1}{2}\binom{2k+2}{k+1}$$

Hence

$$\cos^{2(k+1)}\theta = \frac{1}{2^{2(k+1)-1}}\left[\cos 2(k+1)\theta + 2(k+1)\cos 2k\theta + \sum_{i=1}^{k-1}\binom{2(k+1)}{i+1}\cos 2(k-i)\theta + \binom{2(k+1)}{k+1} - \frac{1}{2}\binom{2(k+1)}{k+1}\right]$$

$$= \frac{1}{2^{2(k+1)-1}}\left[\sum_{i=-1}^{k}\binom{2(k+1)}{i+1}\cos 2(k-i)\theta - \frac{1}{2}\binom{2(k+1)}{k+1}\right].$$

On letting $j = i + 1$, we have our result.

EXERCISE 4-3

Using the Axiom of Induction, prove the following:

1. $\sum_{i=1}^{n} i^2 = \dfrac{n(n+1)(2n+1)}{6}$,
2. $\sum_{i=1}^{n} (2i)^2 = \dfrac{2n(n+1)(2n+1)}{3}$.
3. $\sum_{i=1}^{n} (2i-1)^2 = \dfrac{n(2n+1)(2n-1)}{3}$.
4. $\sum_{i=1}^{n} i^3 = \dfrac{n^2(n+1)^2}{4}$.
5. $\sum_{i=1}^{n} i(i+1) = \dfrac{n(n+1)(n+2)}{3}$.
6. $\sum_{i=1}^{n} (3i-2) = \dfrac{n(3n-1)}{2}$.
7. The number of permutations of n different things is $n!$
8. $1\cdot 2\cdot 3 + 2\cdot 3\cdot 4 + \ldots + n(n+1)(n+2) = \dfrac{n(n+1)(n+2)(n+3)}{4}$.
9. The sum of the interior angles of an n-sided polygon is $(n-2)180°$.
10. $(1+x)^n > 1 + nx \qquad$ for $x > 0$ and $n = 2, 3, 4, \ldots$.

PROBLEMS

1. Develop an expression for $\cos(\theta/4)$ in terms of $\sin\theta$ and $\cos\theta$.

2. Express $\sin(\theta/4)$ in terms of $\sin\theta$ and $\cos\theta$.

3 Prove that

$$\cos(x + y + z) = \cos x \cos y \cos z - \sin x \sin y \cos z - \sin x \cos y \sin z - \cos x \sin y \sin z$$

4 By using the functions of special angles and small angles developed in Chapter 3, along with the additional formulas, it is now possible to construct tables of trigonometric functions Compute $\sin\theta$, $\cos\theta$, and $\tan\theta$ for $\theta = 0°, 5°, 10°, \ldots, 45°$

5 Show that the angle θ formed by two intersecting lines of slopes m_1 and m_2 respectively is given by

$$\tan\theta = \frac{m_2 - m_1}{1 + m_1 m_2}$$

6 Demonstrate that when two lines intersect at right angles their slopes must be related by

$$m_1 = \frac{-1}{m_2}$$

7 Show that

$$\cot(x \pm y) = \frac{\cot x \cot y \pm 1}{\cot y \pm \cot x}$$

8 Prove by induction that

$$\sin^{2n}\theta = \frac{(-1)^n}{2^{2n-1}}\left[\sum_{j=0}^{n}(-1)^j\binom{2n}{j}\cos 2(n-j)\theta - \frac{(-1)^n}{2}\frac{(2n)!}{n!\,n!}\right]$$

where n is a positive integer

9 Develop by induction, a formula for $\sin^{2n+1}\theta$ in terms of $\sin(2n+1)\theta$, $\sin(2n-1)\theta, \ldots, \sin\theta$

10 Develop, by induction, a formula for $\cos^{2n+1}\theta$ in terms of $\cos(2n+1)\theta$, $\cos(2n-1)\theta, \ldots, \cos\theta$

5

THE INVERSE TRIGONOMETRIC FUNCTIONS

If we are given an angle, say x_0, then it is a trivial problem to find $y_0 = \sin x_0$. We simply reduce x_0 to an angle in the first quadrant (if it is not already an acute angle) and look up its sine in the tables. For example, if $x_0 = -101° 10'$,

$$y_0 = \sin(-101° 10') = -\sin 78° 50' = -0.9811.$$

If the angle were not an integral multiple of ten minutes, for example $x_0 = -101° 14'$, then we could still find y_0 by the use of interpolation (see Section 7.2 of Chapter 7).

So far, we have said nothing new. Let us now consider the *inverse* problem. Namely, if we are *given* y_0, can we find an x_0 such that

$$y_0 = \sin x_0?$$

For example, can we find an x_0 such that

$$\sin x_0 = 0.5?$$

We know that 30° is an angle whose sine is $\frac{1}{2}$. Thus we may say "30° is an angle whose sine is $\frac{1}{2}$." If we do not recognize the angle, for example, if we had to find an x_0 such that

$$\sin x_0 = 0.6240,$$

then we would have to resort to our tables. A little reflection indicates that we need only look under the column of *sines* until we find 0.6240. In our case, we find

$$\sin 38° 30' = 0.6225$$
$$\sin 38° 40' = 0.6248.$$

Thus the angle whose sine is 0.6240 must be found by interpolation (see Chapter 7, Section 7.2). It is 38° 37' (to the nearest minute).

It is easily seen that such an argument applies to any of the trigonometric functions There are, of course, limitations For example, the problem "Find an angle whose sine is 2 5" is impossible* since the sine of an angle is always between +1 and −1

5.1. The Inverse Sine and Cosine

The equation that expresses the statement "The sine of y is x" is written

$$x = \sin y$$

We introduce the notation

$$y = \arcsin x \tag{1}$$

to mean "y is the angle whose sine is x" Equation (1) is read "y equals the arcsine of x" Similarly, we write

$$y = \arccos x, \tag{2}$$

to indicate that y is the angle whose cosine is x The right-hand sides of (1) and (2) are called "inverse trigonometric functions" In particular, $y = \arcsin x$ is the *inverse sine* and $y = \arccos x$ the *inverse cosine* The present section is devoted to a discussion of these two functions

Our first important observation is that the inverse functions are *not* single valued (see Section 1 3 of Chapter 1) That is, for example,

$$\frac{\pi}{6} = \arcsin \tfrac{1}{2},$$

$$\frac{5\pi}{6} = \arcsin \tfrac{1}{2},$$

$$\frac{13\pi}{6} = \arcsin \tfrac{1}{2}, \text{ etc },$$

are all true equations since $\pi/6$, $5\pi/6$, $13\pi/6$, etc , are all angles whose sine is $\frac{1}{2}$ The graph of $y = \arcsin x$ is shown in Fig 5 1 It is clear that some lines parallel to the y-axis cut the curve in more than one place, and we see, geometrically, that arcsin x is not single valued

How can we remove this ambiguity? We do so by defining a *principal angle* This principal angle lies between $-\pi/2$ and $+\pi/2$ and is indicated by a heavy line on Fig 5 1 Thus when we write

$$y = \arcsin x$$

* See however, Section 14 4 of Chapter 14

we mean (unless the contrary is explicitly stated) "find the angle y between $-\pi/2$ and $+\pi/2$ which satisfies the above equation." This solution is called the *principal value*. For example, if we write

$$y = \arcsin \frac{\sqrt{2}}{2}$$

we mean $y = 45°$. Note that the curve $y = \arcsin x$ (see Fig. 5.1) considered between $y = -\pi/2$ and $y = +\pi/2$ *is* single valued. Thus there is only *one* value of y between $-\pi/2$ and $+\pi/2$ which corresponds to a given x (between -1 and $+1$). (Query: Would it have been possible to define the principal value of arcsin from 0 to π rather than from $-\pi/2$ to $+\pi/2$?)

Sometimes to emphasize that we are dealing with principal values we capitalize the "a" of arcsin x, viz.:

$$y = \text{Arcsin } x. \tag{3}$$

In all advanced mathematics, however, it is always understood that we are dealing with principal values—unless something explicitly is said to the contrary—and hence we frequently omit the capitalization. For emphasis in the present chapter, we shall strictly adhere to the notation of (3). Returning for a moment to Fig. 5.1, we see that if a sine curve is rotated about a 45° line through the origin the resulting curve is the arcsine curve (see Fig. 5.2). Geometrically stated, the curves are congruent.

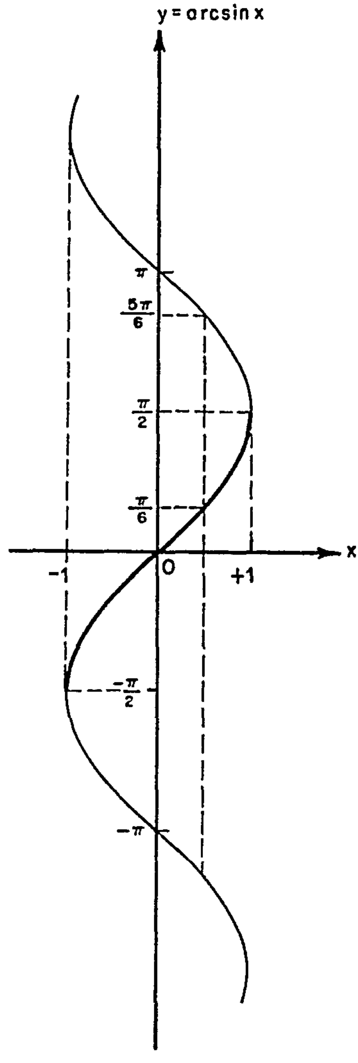


Fig. 5.1

Let us consider now the possible multivaluedness of the arcsine. Suppose

$$y = \arcsin x_0. \tag{4}$$

Then one solution of this equation is

$$y_0 = \text{Arcsin } x_0. \tag{5}$$

However, $y_0 + 2\pi$ is also a solution since

$$\sin (y_0 + 2\pi) = \sin y_0 = x_0.$$

Similarly, $\pi - y_0$ is also a solution since

$$\sin (\pi - y_0) = \sin y_0 = x_0.$$

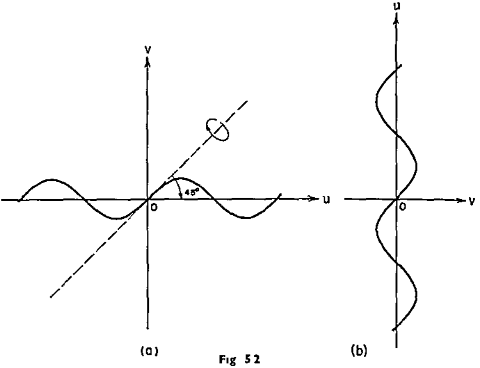


Fig 5 2

In general we see, therefore that if

$$y = y_0 + 2n\pi \tag{6}$$

or if

$$y = \pi - y_0 + 2n\pi \equiv -y_0 + (2n + 1)\pi \tag{7}$$

where n is an integer, positive, negative, or zero, then this value of y satisfies (4) We can express (6) and (7) by the single relation

$$y = (-1)^m y_0 + m\pi, \qquad m = 0, \pm 1, \pm 2, \tag{8}$$

If m is an even integer, (8) reduces to (6) If m is an odd integer, (8) reduces to (7) Thus we may succinctly write all solutions of (4) as

$$\arcsin x_0 = (-1)^m \operatorname{Arcsin} x_0 + m\pi \tag{9}$$

where m is an integer, positive, negative, or zero

Equipped with this information, let us see if we can simplify the following expressions

$$\begin{gathered} \sin(\operatorname{Arcsin} x), \\ \sin(\arcsin x), \\ \operatorname{Arcsin}(\sin x), \\ \arcsin(\sin x) \end{gathered} \tag{10}$$

The first one is trivial since by definition

$$\sin(\text{Arcsin } x) = x.$$

To simplify the second, we use (9) to write

$$\begin{aligned}\sin(\arcsin x) &= \sin[(-1)^m \text{Arcsin } x + m\pi] \\ &= \sin[(-1)^m \text{Arcsin } x] \cos m\pi \\ &+ \cos[(-1)^m \text{Arcsin } x] \sin m\pi,\end{aligned}$$

by the addition formula (2) of Chapter 4. But $\sin m\pi = 0$, $\cos m\pi = (-1)^m$. Thus

$$\begin{aligned}\sin(\arcsin x) &= (-1)^m \sin[(-1)^m \text{Arcsin } x] \\ &= (-1)^{m+m} \sin(\text{Arcsin } x) \\ &= \sin(\text{Arcsin } x) = x.\end{aligned}$$

[Note that $(-1)^{m+m} = +1$ and $\sin(-1)^m\theta = (-1)^m \sin\theta$, since $\sin\theta$ is an odd function.]

The remaining equations of (10) do not have simple answers. For example, if $x = 5\pi/6$, $\sin x = \frac{1}{2}$ and Arcsin $\frac{1}{2} = \pi/6$. Thus

$$\text{Arcsin}\left(\sin\frac{5\pi}{6}\right) = \frac{\pi}{6}$$

while

$$\arcsin\left(\sin\frac{5\pi}{6}\right) = (-1)^n\frac{\pi}{6} + n\pi,$$

$$n = 0, \pm 1, \pm 2, \ldots.$$

The inverse cosine function is treated just as the arc sine was. We write

$$y = \arccos x$$

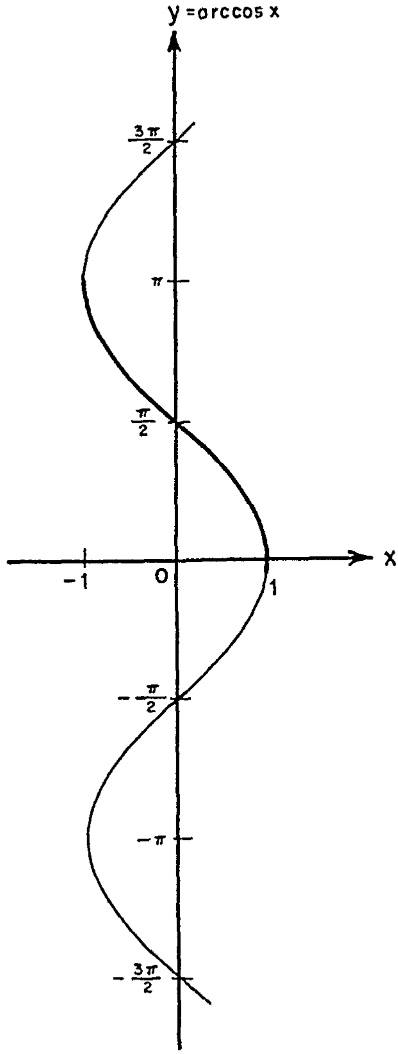


Fig. 5.3

to mean the angle whose cosine is x. Its graph is plotted in Fig. 5.3 and of course is multivalued. The principal value is between 0 and π,

$$y = \text{Arccos } x, \qquad 0 \leqq y \leqq \pi.$$

(Query: Could we have chosen $-\pi/2$ to $+\pi/2$ as the range of the principal values of arccosine?)

If

$$y_0 = \arccos x_0,$$

then we also see that

$$y_0 + 2n\pi = \arccos x_0$$

as well as

$$-y_0 + 2n\pi = \arccos x_0$$

Hence $\cos(y_0 + 2n\pi) = \cos y_0$ and $\cos(-y_0 + 2n\pi) = \cos(-y_0) = \cos y_0$ These solutions may be subsumed in the single formula

$$y = 2n\pi \pm y_0, \qquad n = 0, \pm 1, \pm 2, \quad .,$$

or, analogously to (9),

$$\arccos x_0 = \pm \operatorname{Arccos} x_0 + 2n\pi \tag{11}$$

where n is an integer, positive, negative, or zero

EXERCISE 5-1

Find every value of x less than 360° which satisfies the following equations

1. $\sin x = \sqrt{2}/2$.
2. $\cos x = \sqrt{3}/2$
3. $\sin x = 1$
4. $\cos x = 1$
5. $\cos x = -\frac{1}{2}$
6. $\sin x = 0\ 3338$
7. $\sin x = 0\ 5373$
8. $\sin x = -0\ 8258$
9. $\cos x = 0\ 9652$
10. $\sin x = 0\ 2447$
11. $x = \arccos -0\ 7490$
12. $x = \arccos 0\ 9969$
13. $x = \arcsin 0\ 6993$
14. $x = \arccos -0\ 8339$
15. $x = \arcsin 0\ 3746$
16. $x = \arcsin -0\ 6988$
17. $x = \arccos -0\ 2618$
18. $x = \arccos -0\ 8307$.
19. $x = \arcsin 0\ 9659$
20. $x = \arccos 0\ 7050$

Simplify the following expressions

21. $\cos(\operatorname{Arccos} x)$
22. $\cos(\arccos x)$
23. $\operatorname{Arcsin}(\sin x) \quad 0 \leq x \leq \frac{\pi}{2}$
24. $\operatorname{Arccos}(\cos x) \quad 0 \leq x \leq \pi$
25. $\operatorname{Arcsin}(\sin x)$

Find the value of each of the following expressions

26. $\cos(\operatorname{Arcsin} \frac{1}{2})$
27. $\sin(\operatorname{Arccos} \sqrt{2}/2)$
28. $\cos(\operatorname{Arcsin} \sqrt{3}/2)$
29. $\sin(\operatorname{Arccos} 1)$
30. $\tan(\operatorname{Arcsin} \frac{1}{2})$
31. $\tan(\operatorname{Arccos} -\frac{3}{5})$
32. $\sin(\operatorname{Arccos} -\frac{1}{3})$
33. $\tan(\operatorname{Arcsin} \frac{1}{4})$
34. $\tan(\operatorname{Arcsin} \frac{12}{13})$
35. $\tan(\operatorname{Arccos} -\sqrt{2}/2)$

5.2. Addition Formulas for the Inverse Functions

Certain interesting identities analogous to those derived in Chapter 4 can be deduced for the inverse functions. In particular, we can express the sum of Arcsin x and Arcsin y in terms of the arcsine of a certain function of x and y. We shall call this an "addition formula."

Let

$$u = \text{Arcsin}\, x, \qquad v = \text{Arcsin}\, y.$$

Then

$$u + v = \text{Arcsin}\, x + \text{Arcsin}\, y, \tag{12}$$

and

$$\sin (u + v) = \sin u \cos v + \cos u \sin v. \tag{13}$$

But $\sin u = x$ and $\sin v = y$. Furthermore, from the relation $x = \sin u$ we deduce $\cos u = \sqrt{1 - x^2}$ and from the relation $y = \sin v$ we deduce $\cos v = \sqrt{1 - y^2}$. Since u and v are in either the first or fourth quadrants by definition of the principal value of arcsine, we *always* use the plus sign before the square roots $\sqrt{1 - x^2}$ and $\sqrt{1 - y^2}$, since cosine is positive in the first and fourth quadrants. Thus (13) becomes

$$\sin (u + v) = x\sqrt{1 - y^2} + \sqrt{1 - x^2}\, y,$$

or

$$u + v = \arcsin (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}).$$

Note that since $u + v$ may no longer be between $-\pi/2$ and $+\pi/2$ we must use arcsine rather than Arcsine. Equation (12) then yields the desired addition formula:

$$\text{Arcsin}\, x + \text{Arcsin}\, y = \arcsin (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}). \tag{14}$$

If $|u + v| < \dfrac{\pi}{2}$, then

$$\arcsin (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}) = \text{Arcsin}\, (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}).$$

If $u + v < -\pi/2$,

$$\arcsin (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}) = -\pi - \text{Arcsin}\, (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}),$$

and, if $u + v > \pi/2$,

$$\arcsin (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}) = \pi - \text{Arcsin}\, (x\sqrt{1 - y^2} + y\sqrt{1 - x^2}).$$

For example, if $x = \sqrt{2}/2$ and $y = \sqrt{3}/2$, then

$$\text{Arcsin}\,\frac{\sqrt{2}}{2} + \text{Arcsin}\,\frac{\sqrt{3}}{2} = \arcsin\left(\frac{\sqrt{2}}{2}\sqrt{1-\tfrac{3}{4}} + \frac{\sqrt{3}}{2}\sqrt{1-\tfrac{1}{2}}\right)$$

$$= \arcsin\left(\frac{\sqrt{2}(1+\sqrt{3})}{4}\right) = \arcsin 0.9659$$

Since $u + v$ is in the second quadrant, we must have

$$\arcsin 0.9659 = \pi - \text{Arcsin}\, 0.9659 = 180° - 75° = 105°$$

The corresponding formula for the cosine may be similarly deduced We write

$$u = \text{Arccos}\, x, \qquad v = \text{Arccos}\, y$$

Then

$$\cos(u + v) = \cos u \cos v - \sin u \sin v$$

$$= xy - \sqrt{1-x^2}\sqrt{1-y^2},$$

and both square roots are positive since the principal value of u and v lie in the first two quadrants where the sine is positive Thus

$$u + v = \arccos(xy - \sqrt{1-x^2}\,\sqrt{1-y^2}),$$

and again we must use arccosine rather than Arccosine since $u + v$ may not be a principal value Hence by definition of u and v,

$$\text{Arccos}\, x + \text{Arccos}\, y = \arccos(xy - \sqrt{1-x^2}\,\sqrt{1-y^2}) \qquad (15)$$

where the determination of $\arccos(xy - \sqrt{1-x^2}\,\sqrt{1-y^2})$ depends on the magnitude of $u + v$ If $u + v < \pi$

$$\arccos(xy - \sqrt{1-x^2}\,\sqrt{1-y^2} = \text{Arccos}(xy - \sqrt{1-x^2}\sqrt{1-y^2})$$

If $u + v > \pi$,

$$\arccos(xy - \sqrt{1-x^2}\,\sqrt{1-y^2}) = 2\pi - \text{Arccos}(xy - \sqrt{1-x^2}\,\sqrt{1-y^2})$$

EXERCISE 5-2

Determine the values of the following functions Leave the results in radical form

1. $\sin(\text{Arcsin}\,\frac{4}{5} + \text{Arcsin}\,\frac{12}{13})$
2. $\cos(\text{Arccos}\,\frac{3}{4} + \text{Arccos}\,\frac{4}{5})$
3. $\sin(\text{Arcsin}\,\frac{1}{3} + \text{Arcsin}\,\frac{1}{2})$
4. $\sin(\text{Arccos}\,\frac{7}{8} + \text{Arccos}\,\frac{3}{5})$
5. $\cos(\text{Arcsin}\,\frac{3}{5} + \text{Arcsin}\,\frac{5}{13})$

Prove the following:

6. $\text{Arcsin } u = \frac{1}{2} \text{Arcsin } (2u\sqrt{1 - u^2})$.
7. $2 \text{ Arccos } u = \text{Arccos } (2u^2 - 1)$.
8. $\cos (\frac{1}{2} \text{Arcsin } x) = \sqrt{\dfrac{1 + \sqrt{1 - x^2}}{2}}$.
9. $\sin (\frac{1}{2} \text{Arcsin } y) = \sqrt{\dfrac{1 - \sqrt{1 - y^2}}{2}} \qquad 0 \leqq y < 1.$

Evaluate:

10. $\sin (2 \text{ Arcsin } \frac{2}{3})$.
11. $\cos (2 \text{ Arccos } \frac{4}{5})$.
12. $\sin (3 \text{ Arcsin } \frac{2}{5})$.
13. $\tan (2 \text{ Arcsin } \frac{12}{13})$.
14. $\tan (2 \text{ Arccos } \frac{5}{6})$.
15. $\cot (2 \text{ Arcsin } \frac{5}{8})$.

5.3. The Inverse Tangent Functions

The functions

$$y = \text{arctan } x \tag{16}$$
$$y = \text{arccot } x$$

are the inverse tangent and cotangent functions respectively. Their graphs are plotted in Figs. 5.4 and Fig. 5.5 respectively. The principal values are indicated by heavy lines. Thus

$$y = \text{Arctan } x, \qquad -\frac{\pi}{2} < y < \frac{\pi}{2},$$

$$y = \text{Arccot } x, \qquad 0 < y < \pi.$$

From Fig. 5.4 we see that if $y_0 = \text{arctan } x_0$, then

$$y = y_0 + 2n\pi$$

and

$$y = (y_0 + \pi) + 2n\pi = y_0 + (2n + 1)\pi$$

where n is an integer are also solutions of $y = \text{arctan } x_0$. Hence we may write

$$\text{arctan } x_0 = \text{Arctan } x_0 + n\pi \tag{17}$$

where n is an integer, positive, negative, or zero. Similarly, we can deduce

$$\text{arcot } x_0 = \text{Arccot } x_0 + n\pi \tag{18}$$

where n is an integer, positive, negative, or zero.

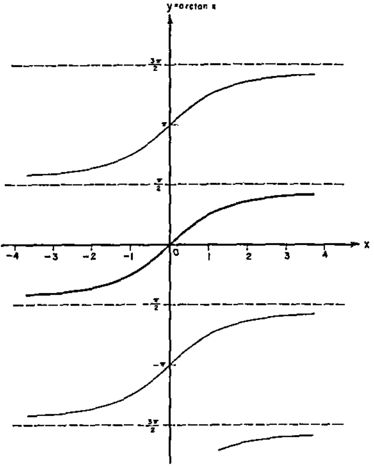


Fig 5 4

Certain simple relations between arctangent and arccotangent may be derived Consider, for instance, the identity

$$x = \tan y = \frac{1}{\cot y}$$

From $x = \tan y$ follows $y = \text{Arctan}\, x$ and from $x = 1/\cot y$ follows $1/x = \cot y$ or $y = \text{Arccot}\,(1/x)$ Since arctangent and arccotangent do not have the same range of principal values, we cannot infer that

$$\text{Arctan}\, x = \text{Arccot}\, \frac{1}{x} \tag{19}$$

in general However, (19) is true for angles in the first quadrant, that is, for $x > 0$

Another relation is derivable from the identity

$$\tan\left(\frac{\pi}{2} - y\right) = \cot y.$$

If we call the common value of the above expression x, then

$$y = \operatorname{arccot} x$$

and

$$\frac{\pi}{2} - y = \arctan x.$$

If y is restricted to the range $0 < y < \pi/2$, then we may write

$$y = \operatorname{Arccot} x \tag{20}$$

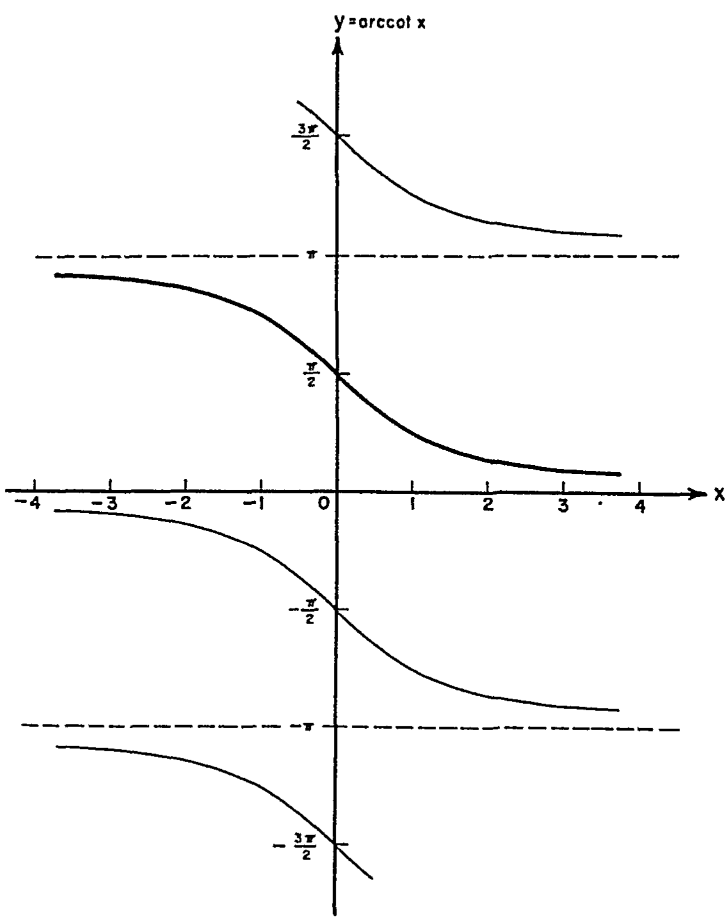


Fig. 5.5

in place of arccot x since the interval $(0,\pi)$ is the range of the principal values of arccotangent Similarly, if $0 < y < \pi$, we infer $-\frac{\pi}{2} < \frac{\pi}{2} - y < \frac{\pi}{2}$, and hence $(\pi/2) - y$ lies in the range of the principal values of arctangent Thus we may write

$$\frac{\pi}{2} - y = \text{Arctan}\, x \tag{21}$$

From (20) and (21) we easily conclude that

$$\text{Arccot}\, x = \frac{\pi}{2} - \text{Arctan}\, x \tag{22}$$

Equation (22) is sometimes taken as the defining equation for Arccot x when we are given the definition of Arctan x

The addition formula for arctangent analogous to (14) and (15) for the arcsine and arccosine is readily deduced Let

$$u = \text{Arctan}\, x, \qquad v = \text{Arctan}\, y \tag{23}$$

Then

$$\tan(u + v) = \frac{\tan u + \tan v}{1 - \tan u \tan v} = \frac{x + y}{1 - xy},$$

or

$$u + v = \arctan\left(\frac{x + y}{1 - xy}\right)$$

Equation (23) then implies

$$\text{Arctan}\, x + \text{Arctan}\, y = \arctan\left(\frac{x + y}{1 - xy}\right) \tag{24}$$

While both u and v lie between $-\pi/2$ and $+\pi/2$ by definition of the principal value of arctangent, $u + v$ need *not* lie in this range Hence as in the case of arcsine and arccosine, we must use $\arctan\left(\frac{x + y}{1 - xy}\right)$ rather than $\text{Arctan}\left(\frac{x + y}{1 - xy}\right)$ More explicitly, if $|u + v| < \pi/2$,

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \text{Arctan}\left(\frac{x + y}{1 - xy}\right),$$

while, if $u + v > \pi/2$,

$$\arctan\left(\frac{x + y}{1 - xy}\right) = \pi + \text{Arctan}\left(\frac{x + y}{1 - xy}\right),$$

and, if $u + v < -\pi/2$,

$$\arctan\left(\frac{x+y}{1-xy}\right) = -\pi + \text{Arctan}\left(\frac{x+y}{1-xy}\right).$$

EXERCISE 5-3

Find every value of x less than 360° which satisfies the following equations:

1. arctan 0.2309 $= x$.
2. Arctan 0.6371 $= x$.
3. arctan $-1.5399 = x$.
4. Arctan $-0.4452 = x$.
5. Arctan 1.0599 $= x$.
6. Arctan $-0.7400 = x$.
7. Arctan 0.4245 $= x$.
8. Arctan $-0.0875 = x$.
9. Arctan 3.7321 $= x$.
10. Arctan 6.0844 $= x$.

Find explicit expressions for y:

11. $y = \tan(\text{Arctan}\, x)$.
12. $y = \tan(\arctan x)$.
13. $y = \arctan(\tan x)$.
14. $y = \text{Arctan}(\tan x)$.

Find the value of each of the following expressions:

15. $\cos(\text{Arctan}\, \frac{5}{6})$.
16. $\sin(\text{Arctan}\, -\frac{4}{5})$.
17. $\text{Arctan}(\sec \pi)$.
18. $\sin(\arctan \frac{1}{2} + \arctan \frac{1}{3})$.
19. $\cos(2 \arctan \theta)$.
20. $\tan(2 \arctan x)$.

Prove the following:

21. $\text{Arccot}\, 1 + \text{Arccot}\, \frac{1}{2} + \text{Arccot}\, \frac{1}{3} = \pi$.
22. $2\, \text{Arctan}\, 1 + 2\, \text{Arctan}\, \frac{1}{2} + 2\, \text{Arctan}\, \frac{1}{3} = \pi$.
23. $8\, \text{Arccot}\, 3 + 4\, \text{Arccot}\, 7 = \pi$.

5.4. The Inverse Secant and Cosecant

The inverse secant and cosecant functions are represented by

$$y = \text{arcsec}\, x,$$
$$y = \text{arccsc}\, x$$

and are plotted in Fig. 5.6 and Fig. 5.7 respectively. These functions are rarely used, but we include them here for completeness.

There is some ambiguity in defining the principal values. For example, some authors define

$$y = \text{Arcsec}\, x$$

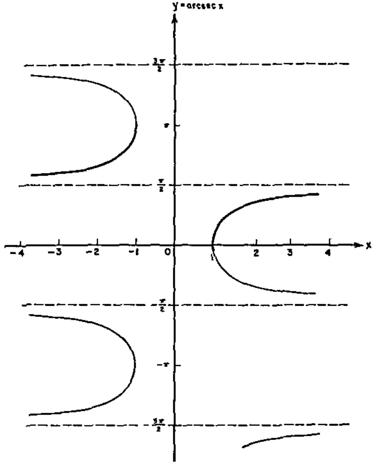


Fig 5 6

for $0 < y < \pi/2$ and $-\pi < y < -\pi/2$, and others define the principal values for $0 < y < \pi$ Both of these definitions can be supported by appealing to their use in higher mathematics In this book, we shall adopt the second point of view To lend credence to this choice, let us *define* Arcsec x as

$$\text{Arcsec } x = \text{Arccos } \frac{1}{x} \tag{25}$$

Since the cosine of an angle is never greater than one and the secant never less than one, the x of (25) must always be greater than or equal to one in absolute value, that is, $|x| \geqq 1$ From this definition of Arcsecant, we see that the range of principal values of arcsecant must be the same as that of arccosine, namely,

$$y = \text{Arcsec } x, \qquad 0 < y < \pi$$

Similarly, we *define*

$$\text{Arccsc}\, x = \text{Arcsin}\, \frac{1}{x}, \qquad |x| \geqq 1 \tag{26}$$

and note that the range of principal values is the same as that of Arcsine, namely, $-\pi/2 < y < \pi/2$. (Consistent with the definition of $y = \text{Arcsec}\, x$ for $0 < y < \pi/2$, $-\pi < y < -\pi/2$, some authors also define $y = \text{Arccsc}\, x$ for this same range.)

From the definitions of Arcsecant and Arccosecant, we deduce from (11) and (9) that

$$\begin{aligned} \text{arcsec}\, x &= \pm \text{Arcsec}\, x + 2n\pi, \\ \text{arccsc}\, x &= (-1)^n\, \text{Arccsc}\, x + n\pi, \end{aligned} \tag{27}$$

where n is an integer, positive, negative, or zero.

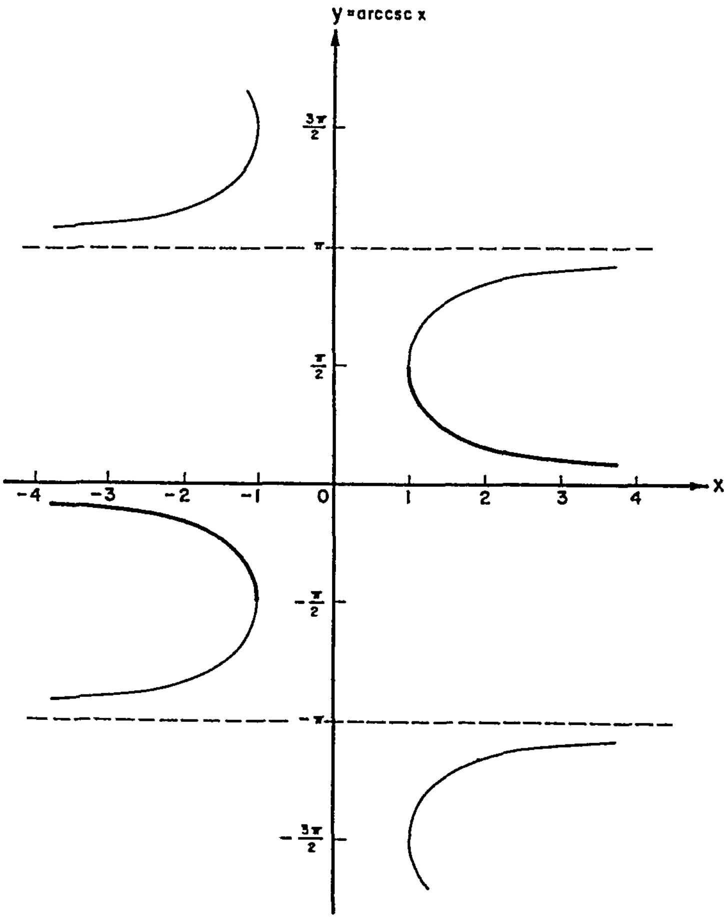


Fig. 5.7

EXERCISE 5-4

Evaluate each of the following expressions

1 Arcsec 1

2 Arcsec $2/\sqrt{3}$

3 Arccsc 2

4 Arcsec 1 278

5 Arccsc 1 056

Prove the following

6 Arcsin x = Arccsc $1/x$

7 Arccos x = Arcsec $1/x$ $\quad 0 < x \leq 1$

8 Arcsec x + Arccsc $x = \pi/2$ $\quad x \geqq 1$

9 Arcsec $10/\sqrt{3}$ + Arcsec $\sqrt{5}/2 = \pi/4$

10 Arccsc $\frac{25}{7}$ = 2 Arccsc $\sqrt{50}$

5 5. A Remark on Notation

In Chapter 1, Section 1 3, we talked briefly about *functions* If we write

$$y = f(x), \tag{28}$$

then under certain conditions it may be possible to solve for x in terms of y We may then write

$$x = g(y) \tag{29}$$

and of course $y = f(g(y))$ For example, if x and y are both positive, then $y = \sqrt{x}$ can be solved for x in terms of y, as $x = y^2$ Similarly, if $-\pi/2 < x < \pi/2$, then $y = \tan x$ may be solved for x in terms of y, as $x = \text{Arctan}\, y$ Now a standard notation is to call the function g of (29) the *inverse function* (with respect to f) and write

$$x = f^{-1}(y)$$

Thus the symbol f^{-1} is identical with g The superscript "-1" on f is a matter of notation and is *not* to be interpreted as an exponent That is, $f^{-1}(y)$ is in general *unequal* to $\dfrac{1}{f(y)}$ Notice how elegant this notation makes expressions such as $y = f(g(y))$ become, viz $y = f(f^{-1}(y))$

With this new notation we may write

$$y = \arcsin x$$

as

$$y = \sin^{-1} x$$

and

$$y = \text{Arcsin}\, x$$

as

$$y = \text{Sin}^{-1} x$$

Of course similar expressions are used for the other five trigonometric functions.

The "-1" notation is probably almost as widely used as the "arc" notation to indicate inverse trigonometric functions. In this book, however, we shall consistently adhere to the "arc" notation. We mention the "-1" notation in passing for the benefit of the student who intends to continue his study of mathematics and hence will undoubtedly, at some time or another, come across the "-1" notation. We then want him to interpret $\cos^{-1} x$ as $\arccos x$ and not as $\sec x$!

EXERCISE 5-5

Determine $f^{-1}(y)$ for each of the functions listed:

1. $y = x + 3$.
2. $y = 2x + 7$.
3. $y = 5x + 11$.
4. $y = x^2 + 3$.
5. $y = x^2 + 2x + 8$.

5.6. Polar Coordinates

Returning to Section 1.2 (Chapter 1), we recall that we defined a *coordinate system*. On a set of axes (see Fig. 5.8), the *coordinates* (x,y) uniquely determine the point P. There are other ways of describing P. Perhaps the next most common way is by the use of *polar coordinates*. If we call r the length of the line from the origin to the point P and θ the indicated angle, then the pair of numbers (r,θ) is spoken of as the *polar coordinates* of the point P.*

The relation between polar and cartesian coordinates is easy to derive. From Fig. 5.8,

$$x = r\cos\theta,\qquad y = r\sin\theta. \tag{30}$$

Thus if r and θ are assigned, x and y are uniquely determined by (30). These equations can also be solved for r and θ. First, if we square both equations and add,

$$x^2 + y^2 = r^2\cos^2\theta + r^2\sin^2\theta = r^2(\cos^2\theta + \sin^2\theta) = r^2$$

and

$$r = \sqrt{x^2 + y^2}. \tag{31}$$

* The coordinates (x,y) of a point, which we discussed in Chapter 1, are called *cartesian coordinates*. The adjective "cartesian" is now necessary. For if we are given a point numerically, for example (2,0.7), it would not be clear whether we were referring to a point whose abscissa were 2 and ordinate 0.7, or to a point whose distance from the origin were 2 and whose angle measured from the positive direction of the x-axis were 0.7 radians.

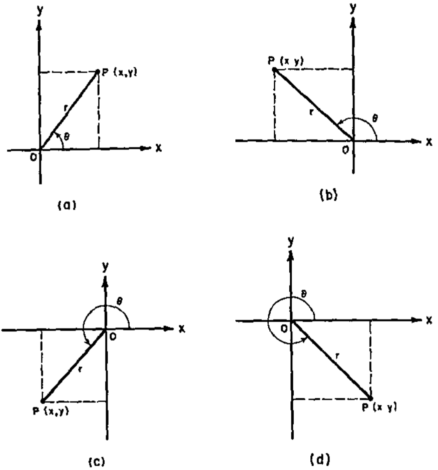


Fig 5.8

We always take the positive square root Thus r is always nonnegative If we divide the two equations of (30), we get

$$\frac{y}{x} = \frac{r \sin \theta}{r \cos \theta} = \tan \theta,$$

or

$$\theta = \arctan \frac{y}{x} \tag{32}$$

The choice of a quadrant for θ is determined by the signs of x and y For example, if $x > 0, y > 0$, θ is in the first quadrant, if $x > 0, y < 0$, θ is in the fourth quadrant, if $x < 0, y > 0$ θ is in the second quadrant, and, if $x < 0$, $y < 0$, θ is in the third quadrant

Thus we see that given any point described by the cartesian coordinates (x,y), we can represent it uniquely in terms of polar coordinates (r,θ) by

(31) and (32). Conversely, given any point (r,θ) expressed in polar coordinates, we can uniquely represent it in terms of the cartesian coordinates (x,y) by means of (30).

Example 1. Express $(-2,3)$ in polar coordinates.

Solution: From (31),

$$r = \sqrt{(-2)^2 + (3)^2} = \sqrt{13} = 3.606$$

and

$$\theta = \arctan \frac{3}{-2}.$$

Since $x < 0$, $y > 0$, θ must lie in the second quadrant. Hence

$$\theta = \text{Arctan}\,(-\tfrac{3}{2}) + \pi = -56^\circ\, 20' + 180^\circ = 123^\circ\, 40'.$$

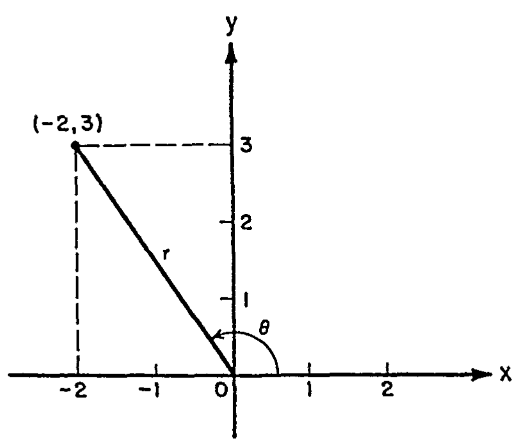


Fig. 5.9

A rough check on the answer is to plot the point $(-2,3)$ as in Fig. 5.9. The sketch also yields a convenient check on the correctness of the quadrant in which θ lies. We would recommend that the student always draw a sketch.

Example 2. If $\left(5, -\frac{\pi}{6}\right)$ are the polar coordinates of a point P, what are its cartesian coordinates?

Solution: From (30),

$$x = 5 \cos\left(-\frac{\pi}{6}\right) = \frac{5\sqrt{3}}{2} = 4.330,$$

$$y = 5 \sin\left(-\frac{\pi}{6}\right) = -\frac{5}{2} = -2.500.$$

These results are sketched in Fig. 5.10.

Polar coordinates are extensively studied in that branch of mathematics known as *analytic geometry*. However, the only nontrivial use we shall

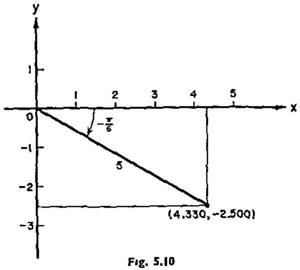


Fig. 5.10

make of this subject is in Chapter 11 of Part II where complex numbers are treated

EXERCISE 5-6

In the following problems provide the missing numbers:

	x	y	r	θ
1.	3	3	—	—.
2.	—	—	10	30°
3.	—	5	—	135°.
4.	−12	—	12	—
5.	—	−5	$5\sqrt{2}$	—.
6.	2	—	—	60°
7.	—	—	17	45°
8.	−15	15	—	—
9.	—	17	—	90°.
10.	$4\sqrt{3}$	—	8	—.
11.	—	−5	5	—.
12.	—	—	12	300°.
13.	8	—	—	315°.
14.	14	−14	—	—
15.	$\sqrt{2}$	—	2	—.
16.	—	—	1 5	58°
17.	—	—	16	63° 10′.
18.	$\frac{1}{2}$	—	$\sqrt{3}/2$	—.
19.	—	—	13 9	122° 40′.
20.	—	—	42 97	79° 30′.

Plot each of the following curves in polar coordinates:

21. $r = \sin\theta$.
22. $r = \cos\theta$.
23. $r = 2 - \sin\theta$.
24. $r = 1 - \cos\theta$.
25. $r = 1 - \sin 3\theta$.
26. $r = \csc\theta$.
27. $r^2 = 2a^2\cos 2\theta$.
28. $r\theta = a$.
29. $r = a\theta$.
30. $r = a\theta^2$.

PROBLEMS

1. Discuss the possibility of defining the principal value of the inverse sine function as lying in the range 0 to π, rather than the range $-\pi/2$ to $\pi/2$.

2. Discuss the possibility of defining the principal value of the inverse cosine function as lying in the range $-\pi/2$ to $\pi/2$, rather than the range 0 to π.

3. Develop an expression for

$$\text{Arcsin}\, x + \text{Arccos}\, y.$$

4. Is this a true statement:

$$\arctan x = \frac{\arcsin x}{\arccos x}?$$

5. Show that the equation of a straight line in polar coordinates is

$$p = r\cos(\theta - \alpha)$$

where p is the distance of the line from the origin. (Hint: See Problem 1 of Chapter 2.)

6
TRIGONOMETRIC IDENTITIES

In Chapter 1 (footnote p 4), we mentioned the distinction between an equation and an identity Let $f(x)$ and $g(x)$ be two functions of the variable x If

$$f(x) = g(x) \tag{1}$$

for *all* values of x for which both members are defined,* then we call (1) an *identity* Thus

$$x = x \tag{2}$$

is a trivial identity,

$$(x + 3)^2 = x^2 + 6x + 9 \tag{3}$$

is also an identity Sometimes, to underscore the fact that a particular relation *is* an identity, the symbol "$\equiv$" meaning "equals identically" is used instead of "$=$" We may then write (1) (2), and (3) as

$$f(x) \equiv g(x),$$
$$x \equiv x,$$
$$(x + 3)^2 \equiv x^2 + 6x + 9$$

respectively

An equation, on the other hand, is true only for certain values of the variables, known as *roots* of the equation Accordingly,

$$x + 3 = 7,$$

which is true only for $x = 4$ is an equation—not an identity (Sometimes equations are termed *conditional equations* when it is necessary specifically to distinguish them from identities)

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* For example $\frac{1}{\tan x}$ is not defined at $x = 0$ Thus while $\cot x = \frac{1}{\tan x}$ is an identity it is not defined for $x = 0 \ \pm\frac{\pi}{2}, \pm\pi$

Identities involving trigonometric functions form the subject matter of this chapter. These studies serve several purposes. For one, many of the relations are important in their own right and are frequently used even in elementary trigonometry. Of even greater importance is that trigonometric identities play an important role in higher mathematics; for example, in reducing certain expressions (known as "integrals") to tractable forms. Finally, of course, the manipulation of identities increases the user's familiarity with trigonometric functions and their properties.

6.1. Elementary Identities

From earlier chapters the student is familiar with the fundamental identities

$$\cot \theta \equiv \frac{1}{\tan \theta},$$

$$\sec \theta \equiv \frac{1}{\cos \theta},$$

$$\csc \theta \equiv \frac{1}{\sin \theta},$$

$$\tan \theta \equiv \frac{\sin \theta}{\cos \theta},$$

$$\cot \theta \equiv \frac{\cos \theta}{\sin \theta},$$

$$\sin^2 \theta + \cos^2 \theta \equiv 1,$$

$$1 + \tan^2 \theta \equiv \sec^2 \theta,$$

$$1 + \cot^2 \theta \equiv \csc^2 \theta.$$

Except for the last three, which are variations on the Pythagorean theorem, these are of the nature of definitions of the various functions involved. Many of the relationships developed in Chapters 2 and 4 are also identities. A partial list includes (10), (15), (16), (17), (18), (19), (20), (21), (22), (29), (30), (31), (32) of Chapter 2 and (1), (2), (6), (8), (9), (10), (14), (15), (16), (17), (18), (19), (20), (21), (22), (23), (24), (25), (26), (27), (28), (29), (30) of Chapter 4.

6.2. The Proof of Identities

In order to prove an asserted identity, we must show that it follows from a known identity.

Example 1. Prove that

$$1 + \tan^2 \theta \equiv \sec^2 \theta.$$

Solution We start with the known identity

$$\sin^2\theta + \cos^2\theta \equiv 1$$

and divide by $\cos^2\theta$ to obtain

$$\frac{\sin^2\theta}{\cos^2\theta} + \frac{\cos^2\theta}{\cos^2\theta} \equiv \frac{1}{\cos^2\theta}$$

But this is the same as

$$\tan^2\theta + 1 \equiv \sec^2\theta, \tag{4}$$

which is the desired result

We observe in the foregoing example that $\cos\theta$ is zero when $\theta = (n + \frac{1}{2})\pi$ and n is an integer Hence, strictly speaking, we should write

$$1 + \tan^2\theta \equiv \sec^2\theta, \qquad \theta \neq (n + \tfrac{1}{2})\pi, \qquad n = 0, \pm 1, \pm 2, \tag{5}$$

in order to avoid situations which would involve division by zero However, it is customary to write (4) rather than (5) [see the footnote on p 106]

The method of proving identities by starting with a known identity and performing permissible operations is logically rigorous However, except for cases of the type treated in the next section, it is often more convenient to develop the proof by starting with the asserted identity and working back to some known identity, recognizing that the steps in the proof may be performed in inverse order to provide a formal proof

Example 2 Prove that

$$\frac{\tan\theta - 1}{1 - \cot\theta} \equiv \tan\theta$$

Solution Multiply numerator and denominator of the left hand member by $\tan\theta$

$$\frac{\tan\theta - 1}{1 - \cot\theta} = \frac{\tan\theta(\tan\theta - 1)}{\tan\theta - \tan\theta\cot\theta}$$

Since $\tan\theta\cot\theta \equiv 1$, we have

$$\frac{\tan\theta - 1}{1 - \cot\theta} = \frac{\tan\theta(\tan\theta - 1)}{\tan\theta - 1} = \tan\theta$$

The formal proof would require performing the steps in reverse order

It is sometimes convenient in proving complicated identities "to work from both ends to the middle," that is, to transform both sides of the identity to a common third expression

Example 3 Prove

$$\frac{\sin\theta}{1 - \cos\theta} + \frac{1 - \cos\theta}{\sin\theta} \equiv \frac{2\tan\theta}{\sec\theta - \cos\theta}$$

Solution: The left-hand side of this identity will be considered first. Add the fractions after finding the common denominator, namely, $\sin\theta(1 - \cos\theta)$:

$$\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta} = \frac{\sin^2\theta + (1-\cos\theta)^2}{\sin\theta(1-\cos\theta)}$$

$$= \frac{\sin^2\theta + (1 - 2\cos\theta + \cos^2\theta)}{\sin\theta(1-\cos\theta)}.$$

Since $\sin^2\theta + \cos^2\theta = 1$, the above expression becomes

$$\frac{2 - 2\cos\theta}{\sin\theta(1-\cos\theta)}$$

and on canceling $(1 - \cos\theta)$:

$$\frac{\sin\theta}{1-\cos\theta} + \frac{1-\cos\theta}{\sin\theta} = \frac{2}{\sin\theta} = 2\csc\theta.$$

The problem is now to show that the right-hand side of the original assertion is equal to $2\csc\theta$. This can conveniently be done by expressing $\tan\theta$ and $\sec\theta$ in terms of $\sin\theta$ and $\cos\theta$:

$$\frac{2\tan\theta}{\sec\theta - \cos\theta} = \frac{2\dfrac{\sin\theta}{\cos\theta}}{\dfrac{1}{\cos\theta} - \cos\theta}.$$

Multiply numerator and denominator of this expression by $\cos\theta$ to obtain

$$\frac{2\sin\theta}{1-\cos^2\theta}.$$

Since $1 - \cos^2\theta = \sin^2\theta$, this becomes

$$\frac{2\sin\theta}{1-\cos^2\theta} = \frac{2\sin\theta}{\sin^2\theta} = \frac{2}{\sin\theta} = 2\csc\theta,$$

which is the desired result. In a formal proof, the starting point would be

$$2\csc\theta \equiv 2\csc\theta.$$

Before beginning to prove a questionable identity, it is advisable to substitute one or two numerical values of θ to determine if the assertion is actually an identity. (Remember, if even *one* value of θ exists for which the relation is not true, it cannot be an identity.) However, resist the temptation to use simple values of θ such as $0, \pi/2, \pi$, for such values are often roots of the *equation* represented by the relation.

Example 4. Is the following statement an identity:

$$(\sin\theta + \cos\theta)^2 = \sin^6\theta + 2\sin^5\theta\cos\theta + \cos^4\theta?$$

Solution If we substitute $\theta = 0$, we obtain

$$(\sin 0 + \cos 0)^2 = (0 + 1)^2 = 1,$$

and

$$\sin^6 0 + 2\sin^5 0 \cos 0 + \cos^4 0 = 0 + 0 + 1 = 1$$

Similarly, if we let $\theta = \pi/2$,

$$\left(\sin\frac{\pi}{2} + \cos\frac{\pi}{2}\right)^2 = (1 + 0)^2 = 1,$$

and

$$\sin^6\frac{\pi}{2} + 2\sin^5\frac{\pi}{2}\cos\frac{\pi}{2} + \cos^4\frac{\pi}{2} = 1 + 0 + 0 = 1$$

However, if we take for example, $\theta = \pi/6$,

$$\left(\sin\frac{\pi}{6} + \cos\frac{\pi}{6}\right)^2 = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}\right)^2 = 1 + \frac{1}{2}\sqrt{3},$$

while

$$\sin^6\frac{\pi}{6} + 2\sin^5\frac{\pi}{6}\cos\frac{\pi}{6} + \cos^4\frac{\pi}{6} = \frac{1}{64} + \frac{\sqrt{3}}{32} + \frac{9}{16} = \frac{37}{64} + \frac{1}{32}\sqrt{3},$$

which is unequal to $1 + \frac{1}{2}\sqrt{3}$ The "identity" is therefore not an identity

Even if a questionable identity is satisfied for a number of numerical values of the variable, this does not guarantee its truth for *all* values However, it would seem plausible, and thus give the student sufficient confidence to attempt to prove it in general

Several general principles for verifying identities can be inferred from the foregoing examples

1 Unless the identity is known to be true, check its plausibility with numerical values
2 If one side involves only one function of the angle, write the other side in terms only of this function (If the function is a secant or cosecant, it may be preferable to use the corresponding more familiar cosine or sine function)
3 Where possible, factor expressions, treating trigonometric functions as algebraic variables
4 Fractions with several terms in the numerator may be broken into several parts, on the other hand, if one side represents the sum of several fractions, determine the least common denominator and add
5 When the best method of proceeding is not clear, write all the functions in terms of sines and cosines (An extension of this method, to be developed in Section 13 6 of Chapter 13, is to express all functions in terms of exponentials)
6 It sometimes helps to multiply the numerator and denominator of a fraction by the same factor

EXERCISE 6-2

Prove the following identities:

1. $1 + \tan 2x \tan x \equiv \sec 2x.$
2. $\left(\cos\theta + 2\cos\frac{\theta}{2} + 1\right)^2 + \left(\sin\theta + 2\sin\frac{\theta}{2}\right)^2 \equiv 16\cos^4\frac{\theta}{4}.$
3. $1 - \sin\theta \equiv (1 + \sin\theta)(\sec\theta - \tan\theta)^2.$
4. $\sin(x + y + z) + 4\sin x \sin y \sin z \equiv \sin(x + y - z)$ $+ \sin(x - y + z) + \sin(-x + y + z).$
5. $\dfrac{\sin(x + y)}{\sin(x - y)} \equiv \dfrac{\tan x + \tan y}{\tan x - \tan y}.$
6. $3\cos^4\theta + 6\sin^2\theta \equiv 3(1 + \sin^4\theta).$
7. $\sin\alpha\cos\alpha\tan\alpha + \sin\alpha\cos\alpha\cot\alpha \equiv 1.$
8. $1 + 8\sin^2\frac{\theta}{2} \equiv 5 - 4\cos\theta.$
9. $\tan x - \tan y \equiv \dfrac{\sin(x - y)}{\cos x \cos y}.$
10. $\tan x \equiv \cot x - 2\cot 2x.$
11. $\cos(x + y)\cos(x - y) \equiv \cos^2 x - \sin^2 y.$
12. $(1 + \cos y)\cos x + \sin y \sin x \equiv 2\cos\frac{y}{2}\cos\left(x - \frac{y}{2}\right).$
13. $\dfrac{1 - \cos x}{x^2} \equiv \dfrac{1}{2}\left(\dfrac{\sin\frac{1}{2}x}{\frac{1}{2}x}\right)^2.$
14. $\dfrac{\sin\theta + 5\csc\theta}{\cot\theta + 3\csc\theta} \equiv \dfrac{\sin^2\theta + 5}{\cos\theta + 3}.$
15. $\dfrac{\sin 3\theta - \sin\theta}{\cos 3\theta + \cos\theta} \equiv \tan\theta.$
16. $\dfrac{1 + \tan^2\theta}{1 - \tan^2\theta} \equiv \dfrac{1}{\cos 2\theta}.$
17. $(\sin\alpha - \cos\alpha)^2 \equiv 1 - \sin 2\alpha.$
18. $\dfrac{\cot\theta + 4\sec\theta}{\cot\theta\csc\theta} \equiv \sin\theta + 4\tan^2\theta.$
19. $\tan^2 x - \sin^2 x \equiv \tan^2 x \sin^2 x.$
20. $(\sec x - \tan x)(1 + \sin x) \equiv \cos x.$
21. $\tan x - \tan y \equiv \dfrac{\sin(x - y)}{\cos x \cos y}.$
22. $\dfrac{\sin A + \sin B}{\csc A + \csc B} \equiv \sin A \sin B.$
23. $\tan\alpha \equiv \dfrac{1 - \cos^2\alpha + \sin^2\alpha}{2\sin\alpha\cos\alpha}.$

24. $\sin^2 x \sec^2 x + \cos^2 x \csc^2 x \sec^2 y + \tan^2 y \equiv \sec^2 x + \csc^2 x \sec^2 y - 2.$

25. $\tan\theta \sin\theta \equiv \sec\theta - \cos\theta.$

26. $\dfrac{\tan\theta}{\sec\theta - \cos\theta} \equiv \dfrac{1}{\sin\theta}$

27. $\dfrac{\csc x}{1 + \cot x} \equiv \dfrac{\sec x}{1 + \tan x}.$

28. $\dfrac{3\sin\theta + \cos\theta}{3\tan\theta + 1} \equiv \cos\theta$

29. $\cot\theta \equiv \dfrac{1 + \cos\theta + \cos 2\theta}{\sin\theta + \sin 2\theta}.$

30. $\cos^2\theta\cos^2\phi + \sin^2\phi\cos^2\theta + \sin^2\theta \equiv 1$

31. $\tan A + \cot A \equiv 2\csc 2A$

32. $\dfrac{\cos 5\theta - \cos 2\theta}{\sin 2\theta - \sin 5\theta} \equiv \tan\dfrac{7\theta}{2}$

33. $\dfrac{\sin 2x}{\sin 4x} \equiv \frac{1}{2}\sec 2x$

34. $\cos 2\alpha \equiv \cos^4\alpha - \sin^4\alpha$

35. $\tan x(1 - \cot^2 x) + \cot x(1 - \tan^2 x) \equiv 0.$

36. $\dfrac{\tan x - \tan y}{1 + \tan x \tan y} \equiv \dfrac{\cot y - \cot x}{1 + \cot x \cot y}.$

37. $\dfrac{\sec\theta + \sin\theta + \cot\theta}{\cos\theta} \equiv \sec^2\theta + \tan\theta + \csc\theta.$

38. $\dfrac{\cot^2\alpha}{\csc^2\alpha - 2\csc\alpha - 3} \equiv \dfrac{\csc\alpha - 1}{\csc\alpha - 3}.$

39. $3\cos^4 x + 6\sin^2 x \equiv 3(1 + \sin^4 x)$

40. $\dfrac{\cos^2\alpha\cot^2\alpha}{\cot\alpha - \cos\alpha} \equiv \cos\alpha + \cot\alpha$

41. $\tan^4\theta + \tan^2\theta \equiv \sin^2\theta\sec^4\theta$

42. $\dfrac{3 - 4\sin^2 x}{\cos^2 x} \equiv 3 - \tan^2 x$

43. $\dfrac{\sec\theta}{\tan\theta + \cot\theta} \equiv \sin\theta$

44. $\dfrac{\sec^2\theta}{\sec^2\theta - 1} \equiv \csc^2\theta$

45. $\cos^6\theta - \sin^6\theta \equiv (2\cos^2\theta - 1)(1 - \sin^2\theta\cos^2\theta).$

46. $\dfrac{\sin^3\theta - \cos^3\theta}{\sin\theta - \cos\theta} \equiv 1 + \sin\theta\cos\theta.$

47. $\dfrac{1}{\csc\theta + \cot\theta} - \dfrac{1}{\csc\theta - \cot\theta} \equiv -2\cot\theta.$

48. $\dfrac{1}{\sec \theta + \tan \theta} \equiv \dfrac{1 - \sin \theta}{\cos \theta}$.

49. $\dfrac{\sin x + \sin y}{\sin x - \sin y} \equiv \dfrac{\csc y + \csc x}{\csc y - \csc x}$.

50. $\dfrac{\cos \theta}{1 - \sin \theta} \equiv \dfrac{1 + \sin \theta}{\cos \theta}$.

6.3. A Logical Difficulty

There is one difficulty associated with the method wherein a questioned identity is "proved" by reducing it to an established identity. This difficulty may most conveniently be expressed symbolically. Let Q represent the questioned statement and I the established identity. For example, Q may be the proposition (as in Example 2)

$$\frac{\tan \theta - 1}{1 - \cot \theta} \equiv \tan \theta$$

and I the identity

$$\tan \theta \equiv \tan \theta.$$

We then prove the proposition "Q implies I" (or, "if Q, then I"). We interpret this to be equivalent to the converse proposition "I implies Q," and, since I is known to be true, we assert that Q is true.

In general, of course, the truth of a theorem carries no implication about the truth of its converse. However, in the particular cases treated, the proof of the converse was obvious from the proof of the theorem and was indicated in each case considered. This was so because each step in the proof was reversible. (That is, if the operation and its inverse are performed, the net result is the original expression.) There are, though, some operations that are not reversible; for example, multiplication by zero and the operation designated by $\sqrt{\ }$, which means "take the *positive* square root of." If we multiply a number by zero and then divide by zero, the result is not the original number; in fact, the result is not defined. Likewise, $\sqrt{(-5)^2}$ is not -5 but $+5$. If we suspect either multiplication by a number identically zero or see a radical sign, it is well to check an alleged proof by starting with the known identity and proceeding to the desired relation.

Example 5. "Prove" that $3 = 2$.

Solution: Let $3 = 2$. Multiply by zero:

$$3 \cdot 0 = 2 \cdot 0$$

or

$$0 = 0.$$

Thus the statement "three equals two" implies the statement "zero equals zero" We recognize that the statement "zero equals zero" is true We then (falsely) assert that this implies "three equals two" is true

In formal terms, "A implies B" is a proposition or theorem, and "B implies A" is its converse A proposition may be true or false without requiring that the component parts be true or false For example, " 'Martians are taller than earthmen' implies 'Martians are taller than I (an earthman)' " is a true proposition, whether or not there are Martians " 'John loves Mary' implies 'All that glitters is not gold' " is a false proposition, regardless of the state of the amours of John and Mary Finally, even if a proposition is true and the statements true, the converse is not necessarily true " 'All cows eat grass' implies 'Elsie, the cow, eats grass' " is a true proposition whose component parts are true, but the converse " 'Elsie, the cow, eats grass' implies 'All cows eat grass' " is not a true theorem, even though both statements are true, because the fact that one cow eats grass, or drinks beer, or whatever, does not imply that all cows do, even though they, in fact, may

EXERCISE 6-3

Determine whether or not the following are identities

1 $\dfrac{\sin x \cos x}{1 - \cos x} = \sqrt{2}(\csc x + \cot x)$

2 $\tan\theta(\sin\theta + \cos\theta)^2 \cot\theta - 2\sin\theta\cos\theta = 1$

3 $\tan\theta - \cot\theta = (1 - 2\cos^2\theta)$

4 $(1 - \cos x)(1 + \sec x) = \sin x$

5 $\dfrac{\sec^2\alpha - 5}{\sec^2\alpha - 6\tan\alpha + 7} = \dfrac{\tan\alpha + 2}{\tan\alpha - 4}$

6 $\dfrac{\tan x}{1 - \tan^2 x} + \dfrac{\cot x}{\cot^2 x - 1} = \dfrac{1}{\cot x - \tan x}$

7 $(\sin x \cos y)^2 + (\sin x \sin y)^2 + \cos^2 x = 1$

8 $\tan^2 x \sin^2 x - \cos^2 x = \sec^2 x - 2$

9 $\tan x = \tan^2 x \sec^2 x - \tan^2 x$

10 $\cos^2 6x \sin 12x = (1 + \cos 12x)\sin 6x(\sin^2 3x - \cos^2 3x)$

6 4. Identities Involving Inverse Functions

Identities involving inverse functions are less frequently encountered than those involving the direct functions However, they do exist and demonstrating them sometimes involves techniques not encountered with the identities already studied The following is an example

Example 6. Prove that

$$\sin(2 \text{ Arcsin } x) = 2x\sqrt{1 - x^2}.$$

Solution: Let

$$\phi = \text{Arcsin } x.$$

Then

$$\sin(2 \text{ Arcsin } x) = \sin 2\phi = 2 \sin \phi \cos \phi \tag{6}$$

by the double angle formula [(15) of Chapter 4]. Now

$$\sin \phi = \sin(\text{Arcsin } x) = x \tag{7}$$

and

$$\cos \phi = \pm\sqrt{1 - \sin^2 \phi} = \pm\sqrt{1 - x^2}. \tag{8}$$

But ϕ is a principal value. Hence

$$-\frac{\pi}{2} \leqq \phi \leqq \frac{\pi}{2}.$$

If ϕ is in the first or fourth quadrant, $\cos \phi$ is always nonnegative and hence we always choose the plus sign in (8). Substituting (7) and (8) in (6) gives us the desired proposition.

EXERCISE 6-4

Establish the following identities:

1. $\cos(\frac{1}{2} \text{ Arccos } y) = \sqrt{\frac{1 + y}{2}}.$
2. $\sin(\frac{1}{2} \text{ Arccos } 2x) = \sqrt{\frac{1}{2} - x}.$
3. $3 \text{ Arccos } u = \text{Arccos}(4u^3 - 3u) \qquad \frac{1}{2} \leqq u \leqq 1.$
4. $\text{Arcsin } u = \frac{1}{3} \text{ Arcsin}(3u - 4u^3) \qquad -\frac{1}{2} \leqq u \leqq \frac{1}{2}.$
5. $2 \text{ Arctan } u = \text{Arctan } \frac{2u}{1 + u^2} \qquad -1 < u < 1.$

PROBLEMS

1. Simplify the following:

(a) $\frac{\tan x - \sec x}{\tan^3 x - \sec^3 x}.$ (b) $\frac{\sin^2 \phi - \cos^2 \phi}{\sin^4 \phi - \cos^4 \phi}.$

(c) $\frac{\cot \alpha \cos \alpha}{\csc \alpha - \sin \alpha}.$ (d) $\frac{2 \cos \phi + 3 \sin \phi \cot \phi}{\sin \phi}.$

(e) $\frac{\tan 4\alpha}{\sec 4\alpha}.$

2. Prove

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$$

3. Prove

$$\sin 4\alpha \cos 3\alpha = \cos 4\alpha \sin 3\alpha + \sin\alpha$$

4. Prove

$$\sin 4x = 8\cos^3 x \sin x - 4\cos x \sin x$$

5. Prove

$$\sin^2\theta = 4(\cos^2\theta - \cos^4\theta)$$

6. Prove

$$2\sin\frac{\theta}{2}\cos\frac{3\theta}{2} + 4\sin^2\frac{\theta}{2}\sin\theta = \sin\theta$$

7. Prove

$$\sin\frac{\pi}{6}(n+2)\ \sin\frac{\pi}{6}(n+1) - \sin\frac{\pi}{6}(n+1)\ \sin\frac{\pi}{6}n + \sin\frac{\pi}{6}(n-1) = \frac{\sqrt{3}}{4} \qquad (n \text{ an integer})$$

8. Prove

$$\sin(n+1)\theta + 2\sin\frac{\theta}{2}\cos\left(n+\frac{3}{2}\right)\theta + 4\sin^2\frac{\theta}{2}\sin(n+1)\theta = 2\sin(n+1)\theta - \sin n\theta \qquad (n \text{ an integer})$$

9. Prove that

$$\sin\frac{\pi}{5}\sin\frac{2\pi}{5}\sin\frac{3\pi}{5}\sin\frac{4\pi}{5} = \frac{5}{16}.$$

10. Prove that

$$\sin\frac{\pi}{12}\sin\frac{\pi}{4}\sin\frac{5\pi}{12}\sin\frac{7\pi}{12}\sin\frac{3\pi}{4}\sin\frac{11\pi}{12} = \frac{1}{32}$$

7

TRIGONOMETRIC EQUATIONS

An equation is a relationship that holds only for certain values of the variables, known as *roots*. Indeed, an equation generally does not state a fact but rather poses a question. For example,

$$x + 3 = 7$$

does not state "There is a value of x such that when three is added the result is seven," but instead asks "Is there a value of x which when added to three yields seven?"

In terms of the equation presented, this discussion seems quibbling. On the other hand, consider

$$x^2 = -1.$$

The nature of this equation as a question is more clearly seen. "Is there a value of x whose square is minus one?" When this problem was first encountered, the answer appeared to be "No." However, the introduction of imaginary numbers* enabled us to answer "Yes, $x = \pm i$." It may be, however, that we have a more specific question in mind. "Is there a real value of x whose square is minus one?" The answer to this remains "No."

Similarly, when we write

$$\sin \theta = \tfrac{1}{2}$$

we mean "Is there a real value of θ whose sine is one-half?" Specifying "real value" may seem pedantic, but suppose we write

$$\sin \theta = 2.$$

This means "Is there a real value of θ whose sine is two?" Of course not, but it is shown in Chapter 14 that there are imaginary values of θ which satisfy this relation.

* See Chapter 11 of Part II for a discussion of complex numbers.

In the following discussion, unless specifically stated to the contrary, it is assumed that we are only interested in *real* roots of equations

7.1. Trigonometric Equations

Solving a trigonometric equation means to find all values of the angle that satisfy the equation, that is, that are roots of the equation In general (except in cases of the type to be treated in Section 7 4), there is an infinite number of roots of a trigonometric equation if angles larger than 2π are considered For example, the roots of the equation $\sin\theta = 0$ are $\theta = n\pi$, where n is any integer, positive, negative, or zero However, we shall only seek those roots θ such that $0 \leq \theta < 2\pi$

In solving trigonometric equations involving a single function, we treat that function as an ordinary algebraic variable, solving by any of the standard algebraic procedures, such as factoring If the equation involves more than one function of a variable, all the functions should be reduced to a single function using, if necessary, various identities, such as those considered in previous chapters It is to be noted that if functions of various multiples of the angle appear they must be reduced to functions of the angle itself (or of the same multiple) These techniques are illustrated in the following examples

Example 1 Find the roots of the equation

$$2\sin^2 x + 1 = 3\sin x$$

Solution This equation is merely a quadratic equation in $\sin x$, and is recognized when it is written in the form

$$2\sin^2 x - 3\sin x + 1 = 0$$

By use of the quadratic formula, the roots of this equation are found to be

$$\sin x = \frac{3 \pm \sqrt{9-8}}{4} = 1, \tfrac{1}{2}$$

[With a little thought one could have factored directly the original equation as $(\sin x - 1)(2\sin x - 1) = 0$] Corresponding to these values of $\sin x$, we have

$$x = 30^\circ, 150^\circ \qquad (\text{for } \sin x = \tfrac{1}{2})$$
$$x = 90^\circ \qquad (\text{for } \sin x = 1)$$

These are the only roots of the equation that lie between 0 and 360°

Observe in the above example that even though the equation is a quadratic equation, there are more than two roots This is because the equation is a quadratic in the *sine* of x, and there are indeed only *two* values of $\sin x$ that satisfy the equation However, there are many values of x corresponding to each of these values of $\sin x$

Example 2. Find the roots of the equation

$$\sin\theta - 2\sin\theta\cos\theta = 0.$$

Solution: We might express the cosine in terms of the sine. However, we can recognize that $\sin\theta$ is a common factor. Hence

$$\sin\theta(1 - 2\cos\theta) = 0,$$

and

$$\sin\theta = 0,$$

or

$$1 - 2\cos\theta = 0.$$

Thus

$$\sin\theta = 0,$$

$$\cos\theta = \tfrac{1}{2},$$

to which correspond the values of θ,

$$\theta = 0, 180°,$$

and

$$\theta = 60°, 300°$$

respectively.

Example 3. Solve the following equation for ϕ:

$$2\cos^2\phi + 7\sin\phi = 5.$$

Solution: Here it is simplest to express the equation in terms of $\sin\phi$ by writing $(1 - \sin^2\phi)$ for $\cos^2\phi$:

$$2(1 - \sin^2\phi) + 7\sin\phi = 5,$$

or

$$2\sin^2\phi - 7\sin\phi + 3 = 0.$$

The solution of this equation is

$$\sin\phi = \frac{7 \pm \sqrt{49 - 4\cdot 2\cdot 3}}{2\cdot 2} = \frac{7 \pm 5}{4} = 3, \tfrac{1}{2}.$$

Clearly, there is no (real) value of ϕ for which $\sin\phi = 3$, so we need only concern ourselves with the case $\sin\phi = \frac{1}{2}$. This occurs for $\phi = 30°, 150°$.

Example 4. Find the roots of the equation

$$\tfrac{1}{2}\cos 4\theta + \cos^2 2\theta = 1.$$

Solution: It might appear at first that it would be appropriate to express each term as a function of θ. However, in this case it will be seen that it is simpler to express each term as a function of 4θ. This is done by using the relation

$$\cos^2 x = \tfrac{1}{2}(1 + \cos 2x)$$

with $x = 2\theta$, so that the original equation becomes

$$\tfrac{1}{2}\cos 4\theta + \tfrac{1}{2}(1 + \cos 4\theta) = 1,$$

or

$$2\cos 4\theta = 1$$

whence

$$\cos 4\theta = \tfrac{1}{2}$$

and

$$4\theta = 60°\ 300°, 420\ \ 660°\ 780°, 1020°\ 1140°, 1380°,$$

so that

$$\theta = 15°, 75°, 105°, 165°, 195°, 255°, 285°, 345°$$

Observe how we had to determine values of 4θ greater than 360° in order to determine all values of θ less than 360

EXERCISE 7 1

Solve for all values of the angle less than 360°

1 $4\cos^2\theta = 3$
2 $\tan\theta\sin\theta - \sin\theta = 0$
3 $\cos 2x = \cos x$
4 $\sin 3x + \sin x = \cos x$
5 $\sin 4x - \cos 3x - \sin 2x = 0$
6 $\cos 2\theta\csc\theta - 2\cos 2\theta = 0$
7 $\sin 2x = \sqrt{2}\sin x$
8 $\cos 2\theta = 3\sin\theta + 2$
9 $2\cos^2 x = 1$
10 $\sqrt{3}\sin x + \cos x = 1$
11 $\sin^4 x + \cos^4 x = \frac{1}{2}$
12 $\cos 2x\sec x + \sec x + 1 = 0$
13 $3\cos x - \cos 2x = 2$
14 $\csc x\cot x = 2\sqrt{3}$
15 $\tan^2 x + 3\csc^2 x = 7$
16 $\sin^2 2x - \sin 2x = 2$
17 $\csc x + \cot x = \sqrt{3}$
18 $\sin x + \cos x = 0$
19 $2\tan^2 x + 3\sec x = 0$
20 $4\sec^2 x - 7\tan^2 x = 3$

7 2 Interpolation and the Use of Tables

In the examples previously considered the values of the various numbers were carefully selected so the answers would be those special angles whose trigonometric functions were simple fractions In actuality, however, such fortuitous combinations of values occur rarely, and tables must be used to yield correct results Unless the angle to be determined happens to correspond directly to a table entry, a process known as *interpolation* will have to be employed This process will first be illustrated by an example and then explained in detail

Example 5 Find the roots of the equation

$$3\sin\theta - 1 = 0$$

Solution Clearly

$$\sin\theta = 0\ 3333$$

A portion of the appropriate table is reproduced as Fig 7 1 It is seen that $\sin\theta$ lies between the tabular values 0 3311 and 0 3338 and hence that θ must lie between

Degrees	sin	csc	tan	cot	sec	cos	
18° 0′	.3090	3.236	.3249	3.0777	1.051	.9511	72° 0′
10′	118	207	281	0475	052	502	50′
20′	145	179	314	3.0178	053	492	40′
30′	173	152	346	2.9887	054	483	30′
40′	201	124	378	9600	056	474	20′
50′	228	098	411	9319	057	465	10′
19° 0′	.3256	3.072	.3443	2.9042	1.058	.9455	71° 0′
10′	283	046	476	8770	059	446	50′
20′	311	3.021	508	8502	060	436	40′
30′	338	2.996	541	8239	061	426	30′
40′	365	971	574	7980	062	417	20′
50′	393	947	607	7725	063	407	10′
20° 0′	.3420	2.924	.3640	2.7475	1.064	.9397	70° 0′
10′	[illegible]	901	673	[illegible]	065	387	50′
[illegible]	[illegible]	[illegible]	[illegible]	[illegible]	111	.9001	10′
26° 0′	.4384	2.281	.4877	2.0503	1.113	.8988	64° 0′
10′	410	268	913	0353	114	975	50′
20′	436	254	950	0204	116	962	40′
30′	462	241	.4986	2.0057	117	949	30′
40′	488	228	.5022	1.9912	119	936	20′
50′	514	215	059	9768	121	923	10′
27° 0′	.4540	2.203	.5095	1.9626	1.122	.8910	63° 0′
	cos	sec	cot	tan	csc	sin	Degrees

Fig. 7.1

19° 20′ and 19° 30′. Between these two values, sin θ varies by 0.3338 − 0.3311 or 0.0027, while θ varies by 10′. The sine of the desired value of θ is 0.3333 − 0.3311 or 0.0022 greater than sin 19° 20′. Hence the desired value of θ is about $\frac{0.0022}{0.0027}$ of 10′ greater than 19° 20′. This is about 8′ greater than 19° 20′. Therefore the solution is $\theta = 19° 28'$.

In general, interpolation is required when using tables because a table gives only discrete values of the variables, rather than the continuous values given by graphs. (On the other hand, it is impossible to read very precise values from most graphs.) If we were to plot the points given by a table, we would have the dotted curve shown in Fig. 7.2a, rather than the smooth curve of Fig. 7.2b. However, for precision in reading values we can magnify the curve of Fig. 7.2a greatly. A portion of such a curve is shown in Fig. 7.3. The heavy dots indicate tabular values; the broken line is a plot of the actual function. We encounter no difficulty in determining sin θ when θ is a multiple of 10′; it can be read directly from the tables. Suppose we wish to determine, say, 73° 18′. This value may be read directly from the broken curve on the graph as 0.9578. Conversely, Arcsin 0.9583 is 73° 24′. The only difficulty here is that it is very time-consuming to draw a magnified portion of a graph every time we wish to determine some value that does not appear directly in the

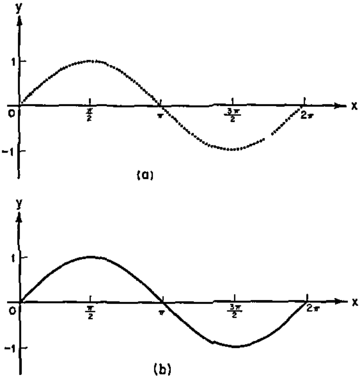


Fig 7 2

tables However, the graphical approach does give us a clue as to how to find intermediate values of functions, without actually constructing the graph This process is known as *interpolation*

The interpolation process we use, which strictly speaking, is known as *linear interpolation* is based on the observation that between two tabulated points the actual curve representing the function is almost a straight line (see Fig 7 3) We therefore assume that to a reasonable approximation the curve is made up of straight line segments connecting the tabulated points (A careful scrutiny of Fig 7 3 shows that the function in fact exhibits a slight curvature) We therefore can use simple proportion to determine intermediate values of the function

Consider Fig 7 4 which is merely Fig 7 3 redrawn without the grid lines and marked to illustrate the determination of sin 72° 44 (point *A*) The point *O* is one tabulated point (sin 72° 40) and *A* is the next (sin 72° 50) The line *OB'* is drawn parallel to the abscissa and *AB* is drawn parallel to the ordinate, along the line corresponding to 72° 44 Clearly the triangles *AOB* and *A'OB'* are similar Accordingly,

$$\frac{AB}{A B} = \frac{OB}{OB} \tag{1}$$

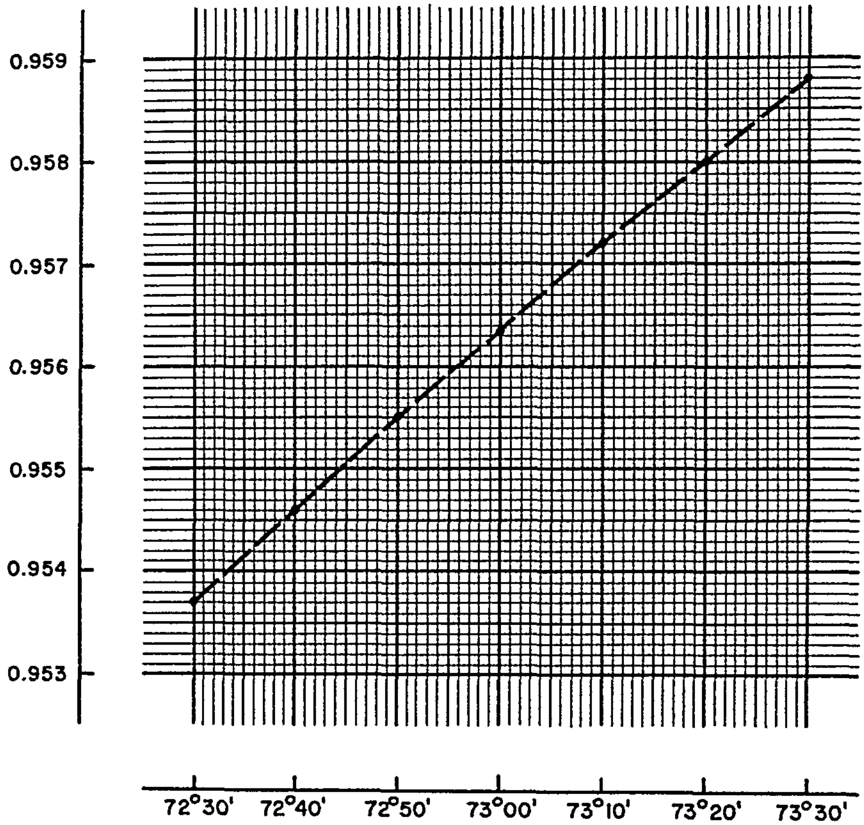

0.959
0.958
0.957
0.956
0.955
0.954
0.953
72°30'
72°40'
72°50'
73°00'
73°10'
73°20'
73°30'

Fig. 7.3

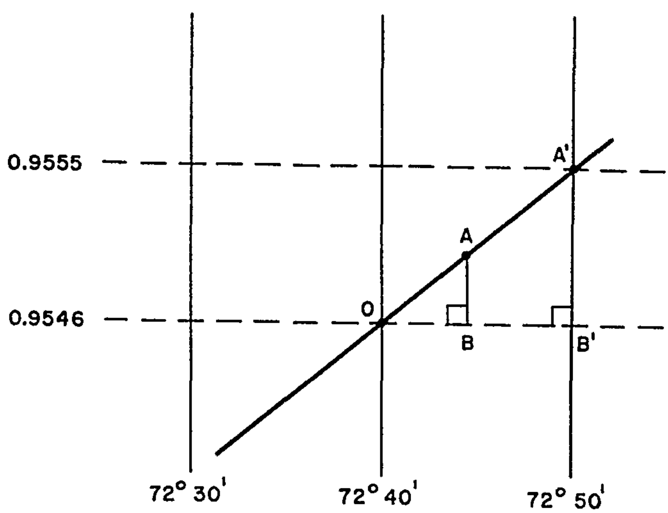

0.9555
0.9546
A'
A
O
B
B'
72° 30'
72° 40'
72° 50'

Fig. 7.4

However, $A'B'$ is known to be of length 0 0009 (obtained by subtracting 0 9546 from 0 9555), OB' is 10' and OB is 4' (the difference between 72° 44' and 72° 40') Hence

$$AB = (A'B')\frac{OB}{OB'} = 0\,0009 \times \frac{4}{10} = 0\,00036 = 0\,0004$$

(We must round off* the value to four decimal places because otherwise we would be pretending that we could obtain five figures from a four-place table) The value of sin 72° 44' is therefore

$$\begin{aligned} \sin 72^\circ 44' &= \sin 72^\circ 40' + 0\,0004 \\ &= 0\,9546 + 0\,0004 \\ &= 0\,9550 \end{aligned}$$

The foregoing procedure may be generalized as follows Let $f(x)$ be a function of x (It does not even have to be a trigonometric function, as we shall see later when we study logarithms) Assume $f(x)$ is tabulated and we wish to find the value $f(x_0)$ corresponding to x_0 Let the nearest values to x_0 in the table be x_1 and x_2, x_1 being the value just below x_0 and x_2 being the value just above x_0 Corresponding to these we have $f(x_1)$ and $f(x_2)$ respectively Then

$$f(x_0) = f(x_1) + \frac{x_0 - x_1}{x_2 - x_1}[f(x_2) - f(x_1)] \tag{2}$$

This rule is merely a more explicit statement of (1) The quantity $x_0 - x_1$ is the length OB, $x_2 - x_1$ is OB', and $f(x_2) - f(x_1)$ is $A'B'$ Observe that this expression gives correct results regardless of whether the function is increasing or decreasing

There do exist more general types of interpolation formulas that take into account the curvature of the function, but a study of these is beyond the scope of a first course in trigonometry

EXERCISE 7 2

Determine the values of the following functions

1	sin 14° 27	9	sin 298° 17
2	sin 38° 14	10	tan 5° 11
3	cos 93° 18	11	cos 128° 11
4	cos 197° 53	12	sin 165° 14
5	tan 300° 5	13	sin 205° 48
6	cot 32° 17	14	cos 49° 44
7	sec 44° 39	15	tan 70° 19
8	csc 127° 15	16	cot 31° 57

* If the last digit is 5, it is customary to round off to the nearest *even* number That is 0 00035 becomes 0 0004 but 0 00025 becomes 0 0002

17. $\sin 110^\circ 1'$.
18. $\cos 70^\circ 38'$.
19. $\sin 285^\circ 19'$.
20. $\sin 314^\circ 16'$.
21. $\tan 10^\circ 28'$.
22. $\tan 152^\circ 12'$.
23. $\cot 17^\circ 32'$.
24. $\sec 231^\circ 16'$.
25. $\csc 44^\circ 58'$.
26. $\sec 15^\circ 40'$.
27. $\sec 28^\circ 2'$.
28. $\tan 97^\circ 14'$.
29. $\sin 49^\circ 22'$.
30. $\sin 110^\circ 5'$.
31. $\cos 215^\circ 51'$.
32. $\sin 357^\circ 6'$.
33. $\cos 216^\circ 46'$.
34. $\sin 131^\circ 33'$.
35. $\cos 63^\circ 17'$.
36. $\sec 350^\circ 22'$.
37. $\tan 292^\circ 33'$.
38. $\sin 57^\circ 54'$.
39. $\sin 129^\circ 41'$.
40. $\sec 84^\circ 39'$.

Find the smallest value of each angle which satisfies each of the following equations:

41. $\sin x = 0.7172$.
42. $\cos \theta = 0.7940$.
43. $\tan \phi = 0.3650$.
44. $\sin \alpha = 0.5085$.
45. $\tan x = 4.0593$.
46. $\cos x = -0.9860$.
47. $\tan \beta = 0.0970$.
48. $\sin \phi = 0.3345$.
49. $\sec \theta = -1.154$.
50. $\csc x = 1.210$.
51. $\sin y = 0.8520$.
52. $\sin x = 0.2431$.
53. $\cos \theta = -0.8650$.
54. $\cot \psi = 0.7146$.
55. $\sec \phi = -2.210$.
56. $\sin \theta = 0.9880$.
57. $\tan \theta = 3.2000$.
58. $\cos x = -0.5860$.
59. $\sec x = -1.350$.
60. $\sin \phi = 0.6675$.
61. $\csc x = 1.403$.
62. $\cos \phi = 0.4530$.
63. $\sin \alpha = -0.1225$.
64. $\sin x = 0.9970$.
65. $\cos \beta = 0.7482$.
66. $\csc \alpha = -7.182$.
67. $\tan \theta = 0.2871$.
68. $\sin x = 0.6311$.
69. $\sec \phi = -3.371$.
70. $\sin \theta = 0.8143$.
71. $\csc \alpha = 1.120$.
72. $\cos x = -0.2610$.
73. $\cos y = 0.9657$.
74. $\cot \psi = -0.5065$.
75. $\sin \theta = -0.9540$.
76. $\sec x = 6.950$.
77. $\sin \theta = 0.1540$.
78. $\cot \beta = 4.0750$.
79. $\tan \alpha = -1.3800$.
80. $\sin \phi = 0.5163$.

Determine all values of the angle less than 360° which satisfy each of the following equations:

81. $\cos \frac{x}{2} + \cos x = 1$.
82. $\sin^2 \theta - 5 \sin \theta = 3$.
83. $2 \sin 3\theta - 3 \cos^2 3\theta = -2$.
84. $3 \sin 2x + 2 \cos^2 x = 2$.
85. $3 \tan x - 1 = 0$.
86. $\sin x - 2 \cos x - 1 = 0$.
87. $\cos^2 x + 2 \sin x = 0$.
88. $2 - \cos x = 3 \sin x$.
89. $4 \sin^2 x - 7 \sin x + 3 = 0$.
90. $5 \cos x - 4 \sin x = 4$.
91. $\sin 5x + \sin 3x + 2 \cos x = 0$.
92. $\sin x = 3 \cos x$.
93. $5 \cos^2 \theta + 9 \cos \theta = -2$.
94. $2 \csc^2 \theta - \csc \theta = 6$.
95. $4 \cos^2 x + \cos x = 3$.
96. $3 \cot^2 x - 5 \cot x = 1$.
97. $\csc \theta \cot \theta = 1000 \sec \theta \tan \theta$.
98. $3 \sin x \tan x - 5 \sec x = 7$.
99. $7 \cos \theta - 4 = \sin \theta \tan \theta$.
100. $\tan x - \sec x = 2$.

7.3. Equations Involving $A \sin x + B \cos x$

A particular equation that occurs quite frequently in practical problems involving vibration and wave motion is the following Find the values of C and θ that make the following relation an identity for all x

$$A \sin x + B \cos x = C \sin (x + \theta) \tag{3}$$

(A and B are known values)

The procedure here is somewhat different from those already demonstrated Expand the right-hand side of the equation using the rule for the sine of the sum of two angles [see (2) of Chapter 4]

$$A \sin x + B \cos x = C \sin x \cos \theta + C \cos x \sin \theta \tag{4}$$

If this relation holds for all x, it must in particular hold for $x = \pi/2$ Thus

$$A \sin \frac{\pi}{2} + B \cos \frac{\pi}{2} = C \sin \frac{\pi}{2} \cos \theta + C \cos \frac{\pi}{2} \sin \theta,$$

or

$$A + 0 = C \cos \theta + 0,$$

and

$$A = C \cos \theta \tag{5}$$

It must hold also for $\theta = 0$ Thus

$$A \sin 0 + B \cos 0 = C \sin 0 \cos \theta + C \cos 0 \sin \theta,$$

or

$$0 + B = 0 + C \sin \theta,$$

and

$$B = C \sin \theta \tag{6}$$

The problem has now been reduced to solving (5) and (6) for C and θ Writing these equations in slightly different form furnishes the clue to their solution

$$\cos \theta = \frac{A}{C}, \tag{5a}$$

$$\sin \theta = \frac{B}{C} \tag{6a}$$

These relations, however, are clearly seen to hold for the right triangle of Fig 7 5 Thus

$$C = \sqrt{A^2 + B^2} \tag{7}$$

and

$$\theta = \arctan \frac{B}{A} \tag{8}$$

Our results may be summarized by writing

$$A \sin x + B \cos x \equiv \sqrt{A^2 + B^2} \sin \left(x + \arctan \frac{B}{A}\right). \tag{9}$$

Care must be taken, however, to select the correct quadrant for θ. This may be inferred from (5) and (6).

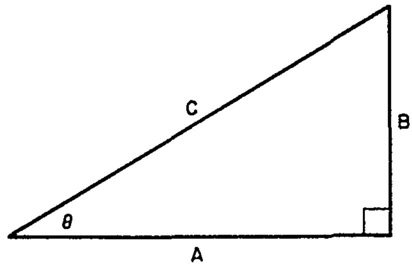


Fig. 7.5

A similar relation, the proof of which is left to the student, is

$$A \sin x + B \cos x \equiv \sqrt{A^2 + B^2} \cos \left(x - \arctan \frac{A}{B}\right). \tag{10}$$

Example 6. Express $3 \sin x + 4 \cos x$ in the form $C \sin (x + \phi)$.

Solution: $C = \sqrt{3^2 + 4^2} = 5$ and

$$\tan \phi = \tfrac{4}{3}.$$

Since $\sin \phi = \frac{4}{5} > 0$ and $\cos \phi = \frac{3}{5} > 0$, ϕ must be in the first quadrant. From the tables, using interpolation,

$$\phi = 53^\circ\, 8'.$$

Example 7. Express $-12 \sin \omega t + 5 \cos \omega t$ in the form $C \sin (\omega t + \phi)$.

Solution: $C = \sqrt{12^2 + 5^2} = 13$ and

$$\tan \phi = \frac{5}{-12} = -0.4167.$$

Since $\sin \phi = \frac{5}{13} > 0$ and $\cos \phi = -\frac{12}{13} < 0$, we infer that ϕ must be in the second quadrant. From the tables, using interpolation,

$$\text{Arctan } 0.4167 = 22^\circ\, 37'.$$

Thus

$$\phi = 180^\circ - 22^\circ\, 37' = 157^\circ\, 23'.$$

Example 8. Find the roots of the equation

$$5 \sin \theta + 6 \cos \theta = 7.$$

Solution First we use (9) to rewrite the left hand side of the equation as

$$5 \sin \theta + 6 \cos \theta = \sqrt{5^2 + 6^2} \sin (\theta + \arctan \tfrac{6}{5})$$
$$= \sqrt{61} \sin (\theta + \arctan 1\ 200)$$
$$= 7\ 810 \sin (\theta + 50^\circ\ 11')$$

Thus the original equation becomes

$$7\ 810 \sin (\theta + 50^\circ\ 11') = 7,$$

or

$$\sin (\theta + 50^\circ\ 11\ ) = \frac{7}{7\ 810} = 0\ 8963,$$

whence

$$\theta + 50^\circ\ 11\ = 63^\circ\ 41\ , \qquad 116^\circ\ 19$$

Hence θ may take on the values

$$63^\circ\ 41\ - 50^\circ\ 11\ = 13^\circ\ 30\ ,$$
$$116^\circ\ 19\ - 50^\circ\ 11\ = 66^\circ\ 8$$

EXERCISE 7 3

Find numerical values of the constants in the following equations

1 $4 \sin \theta + 5 \cos \theta = C \sin (\theta + \phi)$
2 $5 \sin x + 6 \cos\ x = C \cos (x - \phi)$
3 $5 \sin \omega t + 12 \cos \omega t = C \cos (\omega t + \phi)$
4 $A \sin \phi + 7 \cos \phi = 10 \cos (\phi + \theta)$
5 $A \cos \alpha + B \sin \alpha = 5 \cos (\alpha + 30^\circ)$
6 $1\ 313 \cos \omega t + B \sin \omega t = 1\ 800 \cos (\omega t + 43^\circ\ 10\ )$
7 $1\ 2 \sin \omega t + 14 \cos \omega t = C \sin (\omega t + \phi)$
8 $-5\ 9 \sin \omega t + 2\ 3 \cos \omega t = C \cos (\omega t + \phi)$
9 $\sin \alpha + B \cos \alpha = C \cos (\alpha - 57^\circ\ 20')$
10 $-4 \sin \omega t + 6 \cos (\omega t - 30^\circ) = C \cos (\omega t + \phi)$

Find the roots of the following equations

11 $3 \sin x + 4 \cos x = 4$
12 $24 \cos 3x - 7 \sin 3x = 19$
13 $2 \cos x + \sin x = -1$
14 $8 \sin \theta + 1\ 2 \cos \theta = 3$
15 $\sqrt{3} \sin \theta + \cos \theta = \sqrt{2}$

7 4 Equations Involving Inverse Functions

Trigonometric equations involving inverse functions are, of course, also encountered One method of solving such equations is to reduce them to algebraic equations in the unknown by using appropriate identities to determine trigonometric functions of inverse trigonometric functions.

Example 9. Solve for x the equation

$$\text{Arcsin } x = 2 \text{ Arctan } x.$$

Solution: It is first noted that the angle must lie in the first or fourth quadrants, because only in these quadrants do the sine and tangent have the same sign [and the interval $(-\pi/2, \pi/2)$ corresponds to the principal values of both sine and tangent].

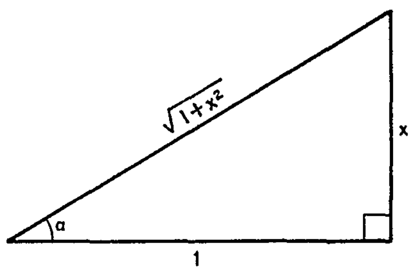


Fig. 7.6

Take the sine of each side:

$$x = \sin 2(\text{Arctan } x).$$

Now Arctan x may be considered to be an angle θ,

$$\begin{aligned} x &= \sin 2\theta = 2 \sin \theta \cos \theta \\ &= 2 \sin (\text{Arctan } x) \cos (\text{Arctan } x). \end{aligned}$$

With the aid of Fig. 7.6 (since $\tan \theta = x$), it is seen that

$$\sin \theta = \frac{x}{\sqrt{1 + x^2}},$$

$$\cos \theta = \frac{1}{\sqrt{1 + x^2}}.$$

Hence

$$x = 2 \frac{x}{\sqrt{1 + x^2}} \frac{1}{\sqrt{1 + x^2}} = \frac{2x}{1 + x^2},$$

or

$$x(1 + x^2) = 2x.$$

Therefore $x = \pm 1, 0$, which is the desired result. As a check, we may determine the actual angles involved.

$$\text{Arcsin } (+1) = \frac{\pi}{2}$$

and

$$2 \text{ Arctan } (+1) = 2\left(\frac{\pi}{4}\right) = \frac{\pi}{2},$$

while

$$\text{Arcsin } (-1) = -\frac{\pi}{2}$$

and

$$2 \text{ Arctan } (-1) = 2\left(-\frac{\pi}{4}\right) = -\frac{\pi}{2}$$

clearly

$$\text{Arcsin } 0 = 2 \text{ Arctan } 0$$

Other types of equations which may be solved with the aid of tables can also often be solved by using appropriate identities

Example 10. Solve for x

$$\text{Arcsin } x = \text{Arcsin } \tfrac{5}{13} + \text{Arcsin } \tfrac{3}{5}$$

Solution This can clearly be solved with the aid of tables As an alternative, we may use (14) of Chapter 5 to write

$$\begin{aligned} \text{Arcsin } x &= \arcsin\left(\tfrac{5}{13}\sqrt{1 - \tfrac{9}{25}} + \tfrac{3}{5}\sqrt{1 - \tfrac{25}{169}}\right) \\ &= \arcsin\left(\tfrac{20}{65} + \tfrac{36}{65}\right) = \arcsin \tfrac{56}{65} \end{aligned}$$

or

$$x = \tfrac{56}{65}$$

EXERCISE 7-4

Solve the following equations

1. $\arctan \dfrac{\theta - 2}{\theta + 4} + \arctan \dfrac{\theta + 2}{\theta + 4} = \dfrac{\pi}{4}$
2. $\arctan (1 - 2x) + \arctan 2x = \arctan \frac{4}{3}$
3. $\text{Arccos } x + 2 \text{ Arcsin } x = 1$
4. $\text{Arcsin } y = 2 \text{ Arctan } 3y$
5. $\text{Arctan } x = \text{Arcsin } \frac{7}{25} + \text{Arccos } \frac{4}{5}$

7.5. Equations Involving Both Trigonometric and Algebraic Functions

In his experience in solving equations, the student has undoubtedly learned that a real root of an equation is the point at which the graph of the function represented crosses the x-axis Thus the solutions of the equation

$$f(x) = x^2 - x - 2 = 0 \tag{11}$$

are seen, from Fig 7 7, to be $x = -1$ and $x = +2$

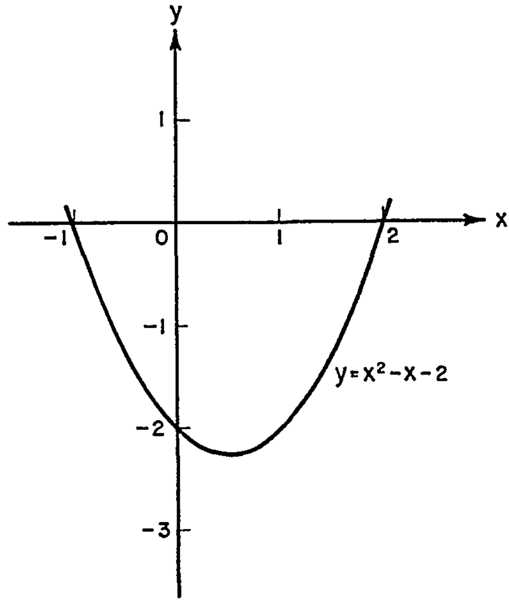


Fig. 7.7

This is indeed the most natural way of considering roots of an equation; however, another way of interpreting the solution of an equation is not to equate the function to zero, but to break it into two functions. The solution of the equation is then the value of the variable at which the corresponding curves intersect. For example, (11) may be written

$$x^2 = x + 2. \tag{12}$$

The curves $y_1 = x^2$ and $y_2 = x + 2$ are shown in Fig. 7.8. These curves intersect at $x = -1$ and $x = +2$. That is, for these values of x, (12) is satisfied.

What is the value of this method of solution? It is particularly useful for equations involving powers of x greater than the second or for equations which contain other than simple algebraic functions. In such cases, it is frequently easier to draw or sketch the graphs of each separate function, rather than to plot the original expression. The method, of course, is limited to the determination of real roots, just as is the method of finding the intersection of the graph and the x-axis.

Example 11. Solve for real values of x, using the method just explained:

$$x^4 - 5x - 6 = 0.$$

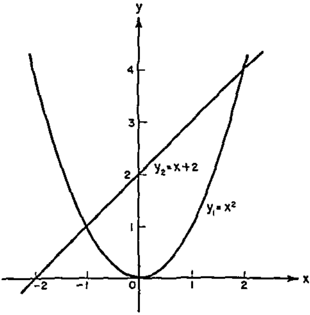

y
4
3
2
1
-2
-1
0
1
2
x
$y_2 = x + 2$
$y_1 = x^2$

Fig. 7.8

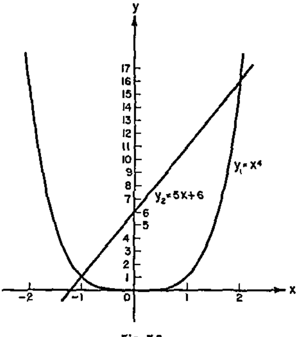

y
17
16
15
14
13
12
11
10
9
8
7
6
5
4
3
2
1
-2
-1
0
1
2
x
$y_2 = 5x + 6$
$y_1 = x^4$

Fig. 7.9

Solution: First write the equation in the form

$$x^4 = 5x + 6.$$

The graphs of the equations $y_1 = x^4$ and $y_2 = 5x + 6$ are shown in Fig. 7.9. It is seen that *these intersect at* $x = -1$ *and* $x = +2$. *These are the desired real roots.* (The remaining roots may be found by solving the reduced equation which results when the original equation is divided by the factors $x + 1$ and $x - 2$,

$$\frac{x^4 - 5x - 6}{(x + 1)(x - 2)} = x^2 + x + 3 = 0.$$

The roots of this equation are

$$x = \frac{-1 \pm \sqrt{1 - 12}}{2} = -\tfrac{1}{2} \pm \frac{\sqrt{-11}}{2}$$

which are complex.)

In general, of course, the roots of equations are not simply integers, as they were in the foregoing examples. In such cases, a process akin to interpolation may be used. This will be illustrated by an example so simple that a direct attack on it will yield the correct result (which can then be used as a check on the method).

Example 12. Find the roots of the equation

$$x^2 - 6.45x + 8.9 = 0. \tag{13}$$

Solution: Write the equation as

$$x^2 = 6.45x - 8.9$$

and plot $y_1 = y^2$, $y_2 = 6.45x - 8.9$ (Fig. 7.10). Roots are seen to exist at approximately $x = 2$ and $x \approx 4.5$. We choose first to investigate the root near $x = 4.5$. When $x = 4.5$, $y_1 = 20.25$ and $y_2 = 20.125$. For this value of x, $y_2 < y_1$, so by inspecting Fig. 7.10 we see that this value of x is too large. Let us try $x = 4.4$.

When $x = 4.4$, $y_1 = 19.36$ and $y_2 = 19.480$. Here $y_2 > y_1$, so this value of x is too small. We might proceed by guessing some intermediate value; however, an organized attack on the problem is of greater usefulness. This general method is most conveniently applied to the original equation, (13). Let us plot this equation in the vicinity of the values $x = 4.4$ to $x = 4.5$. (See Fig. 7.11a.) Actually, only the values for $x = 4.4$ and $x = 4.5$ are calculated, and a straight line is used to approximate the function between the two calculated values. Using this straight line, we can interpolate to determine approximately where the curve intersects the x-axis. (The geometry is illustrated in Fig. 7.11b.)

$$\frac{OC}{DB} = \frac{OA}{DA},$$

or

$$OC = \frac{0\ 120}{0\ 120 + 0\ 125}(4\ 5 - 4\ 4) = 0\ 049$$

This is the amount which must be added to 4 4 to yield a new estimate of the root Thus our new estimate is

$$x = 4\ 449$$

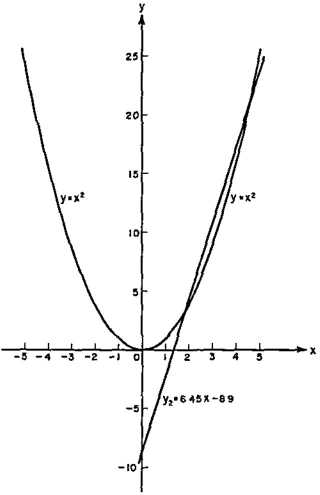
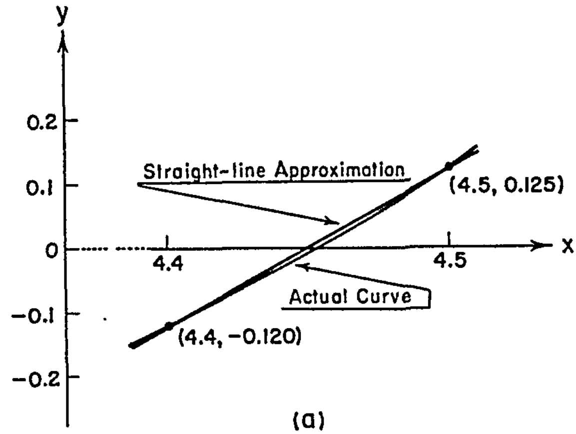


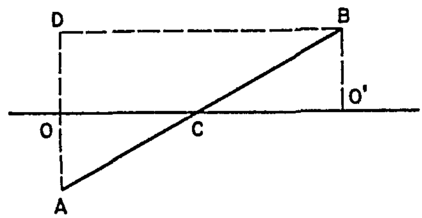
Fig 7 10

This procedure may be repeated as often as necessary to yield the desired degree of accuracy The actual value of the root is 4 45 The difference between this and the estimate is accounted for by the curvature of the function

The techniques demonstrated above are particularly useful in solving equations involving both trigonometric and algebraic functions of the variable



(a)



(b)

Fig. 7.11

Example 13. Find the roots of the equation

$$\sin x - \frac{x}{10} = 0. \tag{14}$$

Solution: The equation may be written

$$\sin x = \frac{x}{10}.$$

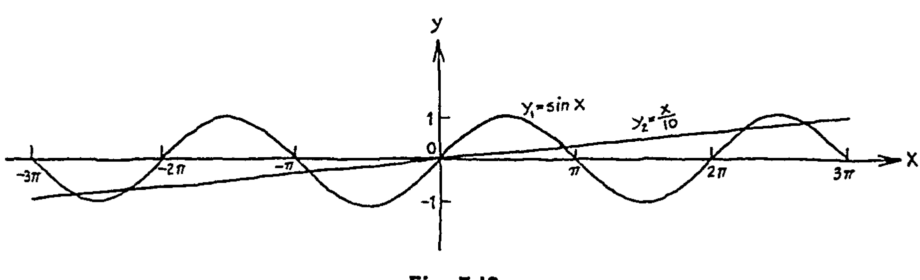


Fig. 7.12

Let $y_1 = \sin x$ and $y_2 = x/10$. These are sketched in Fig. 7.12. (These curves are certainly easier to draw than a graph of (14). Observe that a very accurate plot is not necessary.) Positive roots occur for values of x somewhat less than π, somewhat greater than 2π, somewhat less than 3π The symmetry shown in the drawing indicates that the negative roots will have the same values but be of opposite sign

Because the tables we have in the book are tabulated in degrees, it is convenient to estimate the roots in degrees The estimates are 164°, 405°, and 483° We consider in detail the estimate of the root near 164° To facilitate this, we write (14) with x expressed in degrees as

$$\sin x^\circ - \frac{x^\circ}{10}\frac{\pi}{180^\circ} = \sin x^\circ - 0\ 00174533\ x^\circ = 0$$

A judicious use of the tables, which is greatly facilitated by the use of a slide rule (see Section 9 5 of Chapter 9), enables us to find values of x which ' bracket ' the root

$$x = 163^\circ \quad \sin x^\circ - 0\ 0017453\ x^\circ = 0\ 2924 - 0\ 2845 = +0\ 0079$$

$$x = 164^\circ \quad \sin x^\circ - 0\ 0017453\ x^\circ = 0\ 2756 - 0.2862 = -0\ 0106$$

A first interpolation yields the approximation

$$x = 163^\circ + 0\ 43^\circ$$
$$= 163^\circ\ 26$$

EXERCISE 7-5

Solve for all values of x between $-\pi/2$ and $\pi/2$

1 $\sin x = \frac{\pi}{2}x$

2 $\sin x = \frac{2}{\pi}x$

3 $\sin x = x^2$

4 $\tan x = 2x$

5 $\sin x = x^3 - x$

6 $\cos x = x^3 - 1$

7 $\tan x = \sqrt{x}$

8 $\csc x = 15\ x^2$

9 $\sin^2 x = x$

10 $\cos^2 x = x^2 + \frac{1}{2}$

PROBLEMS

1 Find the smallest positive value of θ for which

(a) $6\ 90 \sin\theta + 2\ 56 \cos\theta = 4\ 86$

(b) $2\ 400 \sin\theta + 1\ 024 \cos\theta = 1\ 286$

(c) $-5\ 0 \sin\theta + 6\ 4 \cos\theta = 3\ 8$

2 Solve for x

$$\sin x + \cos x = \tfrac{1}{2}$$

3. Determine the smallest value of x which satisfies

$$\tan x + \tfrac{1}{10} = \cos x.$$

4. Find the smallest value of θ for which

$$\arctan \theta = \cos \theta.$$

5. Solve for θ:

$$\operatorname{arccot} \theta = \theta^2.$$

8

THE SOLUTION OF TRIANGLES

So far, we have considered the modern point of view of trigonometry as the study of trigonometric functions It must not be forgotten, however, that historically trigonometry was concerned with the measurement of triangles, as evidenced by its very name One of the most important uses of elementary trigonometry is the measurement or, more precisely, the calculation of triangles, their sides and angles The concern of the present chapter is to develop rules and procedures for determining all the sides and angles of a triangle when only some of them are specified

8.1. Right Triangles

The simplest triangles to treat are right triangles Indeed, all that is necessary to analyze right triangles has already been presented in Section 2 1 of Chapter 2 However, a discussion of the solution of right triangles does serve as an introduction to the calculation of more general triangles

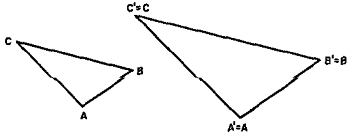


Fig 8 1

It will be recalled from geometry that two triangles are similar when corresponding angles are equal (Fig 8 1) For right triangles, this implies that two (right) triangles are known to be similar if an acute angle in one triangle is specified to be equal to an acute angle in the other triangle (Fig 8 2) For, the two right angles are also equal, and hence the remaining acute angles are

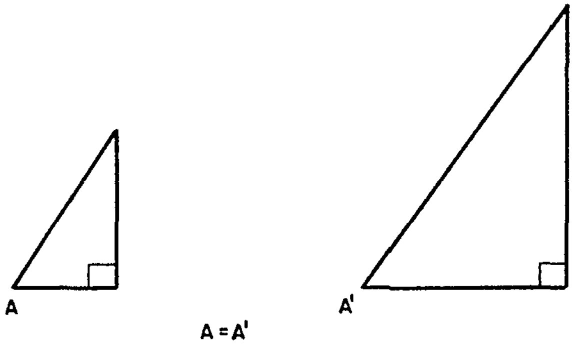


Fig. 8.2

equal since the sum of the angles of a triangle is always 180°. Accordingly, the "shape" of a right triangle is specified by specifying one of the acute angles. It remains, then, to specify its size.

With the angles known, the size of a right triangle is determined when any side is known, for then the other sides may rather simply be calculated in terms of the trigonometric functions.

Example 1. Determine the sides of the triangle of Fig. 8.3 when the hypotenuse is 3.7 feet.

Solution: By definition of the sine,

$$\sin 40° = \frac{a}{3.7},$$

whence $a = 3.7 \sin 40° = 3.7 \times 0.6428 = 2.378$. Also

$$\cos 40° = \frac{b}{3.7},$$

and

$$b = 3.7 \cos 40° = 3.7 \times 0.7660 = 2.834.$$

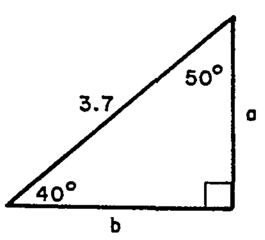


Fig. 8.3

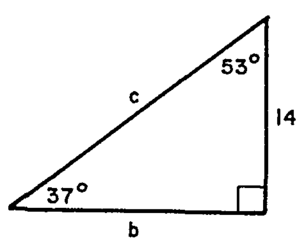


Fig. 8.4

Example 2. Determine the size of the hypotenuse and remaining side of the triangle of Fig. 8.4.

Solution: Here

$$\tan 37° = \frac{14}{b},$$

and

$$b = \frac{14}{\tan 37^\circ} = 18\ 58$$

Also

$$\sin 37^\circ = \frac{14}{c},$$

whence

$$c = \frac{14}{\sin 37^\circ} = 23\ 26$$

If two sides of a right triangle are specified, the third may be found from the Pythagorean theorem, and the angles may be readily determined (In order to avoid ambiguity, it is necessary to specify which, if either, of the sides is the hypotenuse)

Example 3 Determine the hypotenuse and the acute angles of the triangle shown in Fig 8 5

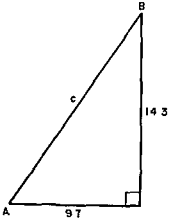


Fig 8 5

Solution From the figure

$$c = \sqrt{9\ 7^2 + 14\ 3^2} = \sqrt{298\ 58} = 17\ 279$$

and

$$A = \text{Arctan}\frac{14\ 3}{9\ 7} = \text{Arctan}\ 1\ 4742 = 55^\circ\ 51,$$

while

$$B = 90^\circ - 55^\circ\ 51 = 44^\circ\ 9$$

An alternative means of solving problems of this type particularly suitable for slide rule use (*see Section 9 5 of Chapter 9*) *is to find first the* angles and then determine the hypotenuse in terms of trigonometric functions This eliminates the use of the Pythagorean theorem and the necessity to extract the square root

Example 4 Repeat the previous problem without using the Pythagorean theorem

Solution We have

$$A = \text{Arctan}\frac{14\ 3}{9\ 7} = \text{Arctan}\ 1\ 4742 = 55^\circ\ 51,$$

and hence

$$c = \frac{14\ 3}{\sin A} = \frac{14\ 3}{\sin 55^\circ\ 51} = \frac{14\ 3}{0\ 8276} = 17\ 28$$

It is seen from the foregoing examples that there are alternative methods of performing the various determinations It is also seen, even with these simple examples, that the numerical work may become quite tedious. The use of logarithms, discussed in the next chapter (Sections 9.2 and 9.3) does much to alleviate this particular tedium.

EXERCISE 8-1

Determine the remaining sides and angle in a right triangle wherein:

1. $a = 5$	$B = 47° 10'$.	**6.** $a = 14$,	$B = 69° 55'$.
2. $c = 4$,	$A = 1° 0'$.	**7.** $b = 112$,	$B = 25° 24'$.
3. $b = 0.293$,	$A = 14° 20'$.	**8.** $c = 58.7$,	$A = 53° 18'$.
4. $c = 476$,	$B = 72° 42'$.	**9.** $b = 12.7$,	$B = \pi/8$.
5. $a = 5820$,	$A = 37° 31'$.	**10.** $a = 10$,	$A = \pi/9$.

Find the remaining side and angles of these right triangles. Use the Pythagorean theorem.

11. $a = 1$	$b = 2$.	**16.** $a = 97$	$b = 58$.
12. $b = 2$	$a = 5$.	**17.** $a = 0.21$	$b = 0.92$.
13. $a = 9$	$c = 11$.	**18.** $c = 4.32$	$a = 2.01$.
14. $c = 23$	$b = 17$.	**19.** $a = 57.2$	$b = 33.9$.
15. $c = 4.9$	$a = 3.2$.	**20.** $b = 927$	$c = 1249$.

Determine the remaining side and angles of the following right triangles without using the Pythagorean theorem:

21. $a = 10$	$b = 12$.	**26.** $b = 8.42$	$a = 5.76$.
22. $b = 27$	$a = 39$.	**27.** $a = 14.9$	$c = 23.7$.
23. $a = 2.3$	$b = 5.4$.	**28.** $a = 0.762$	$b = 0.538$.
24. $c = 76$	$a = 52$.	**29.** $c = 1972$	$a = 1266$.
25. $b = 0.49$	$c = 0.76$.	**30.** $b = 3972$	$c = 5899$.

8.2. Oblique Triangles

A triangle which does not contain a right angle is an oblique triangle. Since none of the sides is then unique (as was the hypotenuse in a right triangle), all the sides are merely called sides. The sides and angles of a triangle are called *parts* of the triangle. As is the case with right triangles, the standard problem with oblique triangles is to determine the remaining parts of the triangle when only some are specified. How many must be specified? There are six parts—three sides and three angles.* Two of them are insufficient to determine a triangle, for two angles specify only its shape, and two sides, or a side and an angle, specify nothing. Three parts (not all angles),

* Of course the sum of the angles is 180°. Hence knowing two angles uniquely determines the third.

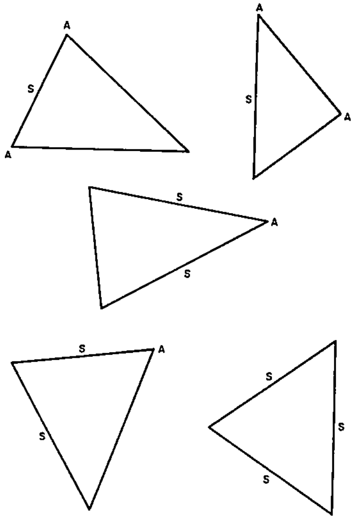


Fig 8 6

however, suffice to determine a triangle, for two angles and a side determine both the shape and size of the triangle, three sides also specify a unique triangle, and two sides and an angle determine a triangle (though with a possibility of ambiguity) In order to classify various cases for the method of solution differs depending on the information available, they are often referred to as ASA, SAA, SAS, SSA, and SSS These abbreviations mean that the known parts are, respectively two angles and the side common to both, a side and two angles, two sides and the included angle, two sides and one of the nonincluded angles, and three sides These cases are shown in Fig 8 6

The methods of treating these cases are developed in detail below

8.3. The Law of Cosines (SAS, SSS)

When two sides and the included angle of a triangle are specified, the situation is that shown in Fig. 8.7, with a, b, and θ known. Since we are already familiar with right triangles, we shall attempt the solution of this problem by augmenting the triangle as shown in Fig. 8.8 to form a right triangle. This figure shows two cases, corresponding to acute and obtuse values of θ. For either case, the Pythagorean theorem gives

$$c^2 = AB^2 = AD^2 + DB^2. \tag{1}$$

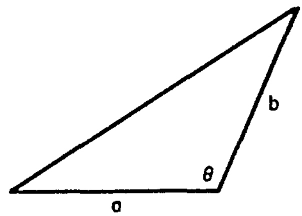


Fig. 8.7

Also, since $\sin(180° - \theta) = \sin\theta$,

$$DB = a \sin\theta. \tag{2}$$

When θ is acute,

$$AD = AC - DC = b - a\cos\theta, \tag{3}$$

and, when θ is obtuse,

$$\begin{aligned} AD = AC + CD &= b + a\cos(180° - \theta) \\ &= b - a\cos\theta. \end{aligned} \tag{4}$$

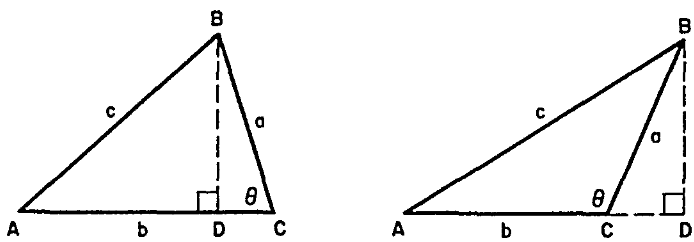


Fig. 8.8

Since (3) and (4) are identical, we may combine either with (1) and (2) without being concerned directly with whether θ is acute or obtuse. The resulting equation is

$$\begin{aligned} c^2 &= (b - a\cos\theta)^2 + (a\sin\theta)^2 \\ &= b^2 - 2ab\cos\theta + a^2\cos^2\theta + a^2\sin^2\theta \end{aligned} \tag{5}$$

and, since $\sin^2\theta + \cos^2\theta = 1$,

$$c^2 = a^2 + b^2 - 2ab\cos\theta. \tag{6}$$

This relation is known as "the law of cosines." It is a generalization of the Pythagorean theorem, since it reduces to that theorem when θ is a right angle.

Example 5. Determine the remaining side of a triangle in which two sides have length 3 and 4 and the included angle is 50°

Solution By (6),

$$c = \sqrt{3^2 + 4^2 - 2 \cdot 3 \cdot 4 \cos 50°} = \sqrt{9 + 16 - 24 \times 0.6428}$$
$$= \sqrt{9.573} = 3.094$$

The law of cosines is also useful when the three sides of a triangle are known and the angles are to be found.

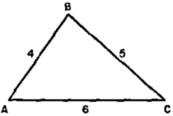


Fig. 8 9

Example 6. Determine the angles in the triangle shown in Fig. 8 9

Solution By (6),

$$\cos A = \frac{4^2 + 6^2 - 5^2}{2 \cdot 4 \cdot 6} = \frac{27}{48} = 0.5625$$

and

$$A = 55° 46'$$

Similarly,

$$\cos B = \frac{4^2 + 5^2 - 6^2}{2 \cdot 4 \cdot 5} = \frac{5}{40} = 0.1250$$

and

$$B = 82° 49',$$

while

$$\cos C = \frac{5^2 + 6^2 - 4^2}{2 \cdot 5 \cdot 6} = \frac{45}{60} = 0.7500$$

and

$$C = 41° 25'$$

Check $A + B + C = 180°$

EXERCISE 8-3

Determine the remaining parts of the following triangles:

1. $a = 5$	$c = 7$	$B = 64°$.
2. $a = 8$	$b = 4$	$C = 25°$
3. $b = 3$	$a = 5$	$C = 99°$
4. $c = 7$	$b = 6$	$A = 123°$
5. $a = 2.6$	$b = 5.9$	$C = 17° 10'$.
6. $c = 17$	$a = 11$	$B = 40° 30'$.
7. $b = 35$	$a = 59$	$C = 130° 40'$
8. $a = 0.13$	$c = 0.27$	$B = 58° 10'$
9. $a = 1.01$	$b = 0.76$	$C = 27° 20'$
10. $b = 43$	$c = 27$	$A = 43° 40'$
11. $b = 12$	$c = 19$	$A = 10° 27'$.

12. $a = 27$	$b = 41$	$C = 97° 11'$.
13. $c = 1.6$	$b = 2.4$	$A = 42° 1'$.
14. $b = 41$	$c = 11$	$A = 27° 51'$.
15. $c = 17$	$a = 10$	$B = 52° 9'$.
16. $a = 5.6$	$b = 4.2$	$C = 65° 44'$.
17. $b = 51$	$c = 80$	$A = 84° 22'$.
18. $b = 0.90$	$a = 0.57$	$C = 73° 29'$.
19. $a = 5.5$	$b = 6.5$	$C = 28° 32'$.
20. $a = 29$	$c = 49$	$B = 39° 19'$.

Find the angles in the triangles whose sides are as follows:

21. $a = 1$	$b = 3$	$c = 3$.
22. $a = 2$	$c = 3$	$b = 4$.
23. $b = 5$	$a = 7$	$c = 10$.
24. $c = 4$	$b = 5$	$a = 8$.
25. $b = 9$	$a = 12$	$c = 5$.
26. $c = 15$	$a = 9$	$b = 8$.
27. $a = 35$	$b = 49$	$c = 24$.
28. $a = 98$	$c = 103$	$b = 77$.
29. $b = 4.2$	$c = 3.9$	$a = 2.1$.
30. $c = 0.56$	$a = 0.32$	$b = 0.44$.

8.4. The Law of Sines (SAA, ASA, and SSA)

The law of cosines is inadequate, as it stands, to aid in solving triangles whose angles are specified, along with one side. (Only two angles need be specified; the third is then known since the sum of the angles of a triangle is 180°.) This case may be treated with the "law of sines," which states that

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C}. \tag{7}$$

That is, in any triangle the lengths of the sides are proportional to the sines of the opposite angles.

This may be proved with the aid of Fig. 8.10. This figure shows three cases, in which either or both of the angles of concern (that is, B or C) may be acute. The line h is dropped from the apex of the triangle perpendicular to the base. Then

$$h = c \sin B,$$

or alternatively

$$h = b \sin C.$$

Equating these expressions yields

$$c \sin B = b \sin C,$$

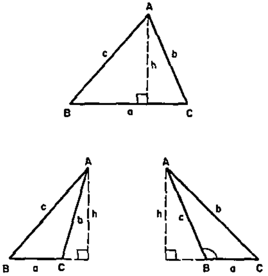


Fig 8 10

or

$$\frac{c}{\sin C} = \frac{b}{\sin B} \tag{8}$$

A similar construction using side c as the base shows that

$$\frac{a}{\sin A} = \frac{b}{\sin B} \tag{9}$$

Equations (8) and (9) together yield (7)

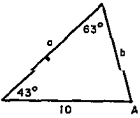


Fig 8 11

Example 7 Determine the remaining sides of the triangle shown in Fig 8 11 (This is the case SAA)

Solution The remaining angle is

$$A = 180° - (43° + 63°) = 74°$$

Then

$$\frac{a}{\sin 74°} = \frac{b}{\sin 43°} = \frac{10}{\sin 63°},$$

whence

$$a = \frac{10}{\sin 63°} \sin 74° = \frac{10}{0\,8910}\, 0\,9613 = 10\,79$$

and

$$b = \frac{10}{\sin 63°} \sin 43° = \frac{10}{0\,8910}\, 0\,6820 = 7\,654$$

Example 8. Find the other two sides of the triangle shown in Fig. 8.12 (case ASA).

Solution: Angle A is

$$A = 180° - (41° + 118°) = 21°.$$

Hence

$$b = \frac{7}{\sin 21°} \sin 118° = \frac{7}{\sin 21°} \sin 62° = \frac{7}{0.3584} 0.8829 = 17.24,$$

and

$$c = \frac{7}{\sin 21°} \sin 41° = \frac{7}{0.3584} 0.6561 = 12.81.$$

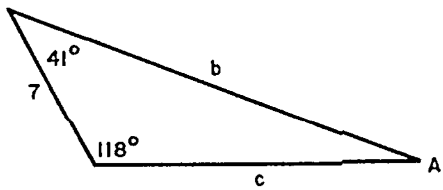


Fig. 8.12

The law of sines is also applicable to the case where two sides and the angle opposite one of them is specified. This is the situation shown in Fig. 8.13. A difficulty can arise in this case, however. If the given angle is

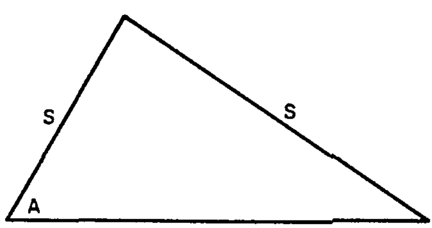


Fig. 8.13

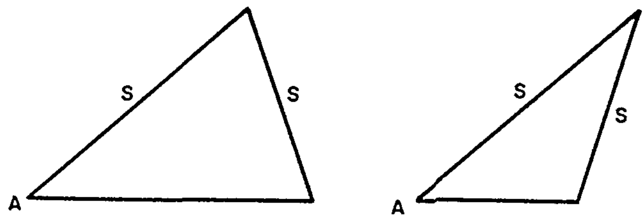


Fig. 8.14

opposite the shorter of the two given sides (which implies that the angle is acute) both of the triangles shown in Fig. 8.14 satisfy the given data; there exists an ambiguity. Observe that the two solutions correspond to the two points at which the arc of radius S_2 swung from the apex intercepts the base (Fig 8 15) (In a degenerate case, the arc is tangent to the base and the ambiguity ceases to exist If, in Fig 8 15, the radius S_2 does *not* intersect the base, there is no solution) In the ambiguous case, both solutions are correct—the one which may be appropriate in a particular practical problem depends on the other conditions of the problem

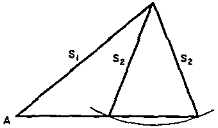


Fig 8 15

Example 9 Find the remaining parts of the triangle shown in Fig 8 16

Solution By the law of sines

$$\frac{10}{\sin C} = \frac{12}{\sin 51°},$$

whence

$$\sin C = \frac{10 \sin 51°}{12} = \frac{10 \quad 0\,7771}{12} = 0\,6476$$

and

$$C = 40° \, 22$$

Also

$$B = 180° - (51° + 40° \, 22') = 88° \, 38$$

so that

$$b = \frac{12}{\sin 51°} \sin 88° \, 38 = \frac{12}{0\,7771} 0\,9997 = 15\,44$$

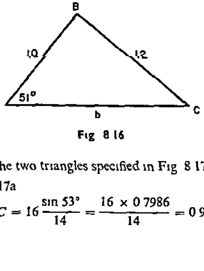


Fig 8 16

Example 10 Find the two triangles specified in Fig 8 17

Solution In Fig 8 17a

$$\sin C = 16 \frac{\sin 53°}{14} = \frac{16 \times 0\,7986}{14} = 0\,9127$$

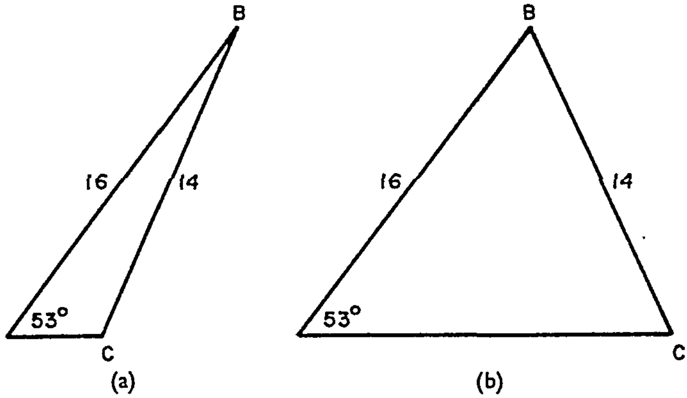


Fig. 8.17

and

$$C = 114° 7' \qquad \text{(not } 65° 53'\text{)}.$$

From

$$B = 180° - (114° 7' + 53°) = 12° 53',$$

we conclude that

$$b = \frac{14}{\sin 53°} \sin 12° 53' = \frac{14}{0.7986} 0.2230 = 3.909.$$

In Fig. 8.17b,

$$\sin C = 16 \frac{\sin 53°}{14} = \frac{16 \times 0.7986}{14} = 0.9127,$$

and

$$C = 65° 53' \qquad \text{(not } 114° 7'\text{)},$$

and

$$B = 180° - (65° 53' + 53°) = 61° 7'.$$

Thus

$$b = \frac{14}{\sin 53°} \sin 61° 7' = \frac{14}{0.7986} 0.8756 = 15.35.$$

EXERCISE 8-4

Evaluate the other sides and angle in the following triangles:

1. $a = 10$ $\quad A = 55° 10'$ $\quad B = 48° 40'$.
2. $b = 35$ $\quad B = 43° 10'$ $\quad C = 65° 20'$.
3. $c = 110$ $\quad A = 68° 17'$ $\quad C = 52° 19'$.
4. $a = 571$ $\quad B = 142° 27'$ $\quad A = 20° 11'$.
5. $b = 4632$ $\quad A = 105° 22'$ $\quad B = 50° 47'$.
6. $A = 98° 57'$ $\quad C = 65° 12'$ $\quad a = 7432$.
7. $B = 85° 17'$ $\quad b = 1593$ $\quad C = 40° 32'$.
8. $c = 1276$ $\quad A = 65° 12'$ $\quad C = 72° 11'$.
9. $B = 42° 19'$ $\quad C = 74° 12'$ $\quad b = 5189$.
10. $A = 58° 51'$ $\quad a = 2531$ $\quad B = 53° 14'$.

Find the remaining parts of the following triangles:

11.	$A = 41° 55'$	$b = 11\ 2$	$C = 48° 5'$
12.	$B = 62° 48'$	$c = 237$	$A = 71° 31'$
13.	$C = 42° 12'$	$a = 1\ 59$	$B = 63° 32'$.
14.	$B = 21° 39'$	$c = 365$	$A = 49° 25'$.
15.	$b = 29\ 8$	$C = 127° 1'$	$A = 30° 16'$.
16.	$A = 44° 24'$	$c = 4132$	$B = 92° 11'$
17.	$b = 31\ 47$	$A = 69° 44'$	$C = 81° 53'$.
18.	$B = 35° 29'$	$a = 159\ 6$	$C = 72° 6'$
19.	$C = 39° 38$	$A = 55° 17'$	$b = 4\ 283$
20.	$A = 19° 17'$	$c = 6591$	$B = 110° 22'$.

Find the remaining side of the following triangles

21.	$a = 5$	$c = 4$	$A = 43° 10'$.
22.	$b = 18$	$a = 13$	$B = 15° 17'$
23.	$c = 5\ 8$	$b = 4\ 2$	$C = 32° 12'$.
24.	$C = 40° 17'$	$c = 0\ 562$	$a = 0\ 317$
25.	$a = 195.4$	$A = 58° 12'$	$b = 137.6$
26.	$a = 2765$	$b = 4938$	$B = 72° 35'$
27.	$a = 923\ 4$	$A = 44° 1'$	$c = 767\ 2$
28.	$B = 100° 7'$	$b = 0\ 0438$	$c = 0\ 0392$
29.	$c = 11\ 97$	$C = 69° 54'$	$a = 8\ 65$
30.	$A = 125° 10'$	$b = 2\ 987$	$a = 11\ 23$

For each set of data given, determine whether there is any triangle which fits the data If so, determine the remaining parts of every triangle which fits the data; if not, explain why

31.	$A = 47° 30'$	$b = 74$	$a = 58$
32	$B = 58° 20$	$a = 15$	$b = 12$
33.	$C = 70° 10'$	$b = 29$	$c = 42$
34.	$A = 68° 14'$	$a = 9\ 38$	$b = 10\ 10$
35.	$C = 42° 12'$	$c = 0\ 554$	$a = 0\ 692$
36.	$B = 51° 27'$	$c = 4\ 97$	$b = 5\ 32$
37.	$a = 0\ 124$	$B = 109° 22'$	$b = 0\ 193$
38.	$a = 5\ 96$	$C = 75° 31'$	$c = 5\ 80$
39.	$c = 4\ 283$	$a = 3\ 386$	$A = 40° 40'$
40.	$a = 43\ 2$	$C = 57° 42'$	$c = 35\ 9$
41.	$a = 201$	$B = 123° 41'$	$b = 258$
42.	$B = 39° 9'$	$c = 15\ 00$	$b = 9\ 47$
43.	$c = 35\ 7$	$C = 22° 15'$	$a = 67\ 9$
44.	$b = 486$	$c = 692$	$C = 40° 12'$.
45.	$a = 4\ 32$	$A = 38° 29'$	$b = 5\ 13$
46	$b = 11\ 32$	$B = 96° 10'$	$a = 5\ 76$
47.	$C = 52° 24'$	$a = 2094$	$c = 2568$
48.	$B = 70° 59'$	$c = 15\ 11$	$b = 14\ 82$
49.	$b = 17\ 61$	$A = 30° 10'$	$a = 8\ 25$.
50.	$a = 2\ 477$	$b = 4\ 130$	$A = 36° 51'$.

8.5. The Law of Tangents (SAS)

The practical computational difficulty with the law of cosines is the necessity for extracting a square root. This difficulty can be eliminated by use of the "law of tangents," which applies to the same class of cases: two sides and the included angle specified. The law of tangents is given by (10) (see Fig. 8.18):

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)}, \tag{10}$$

that is, the difference of two sides is to their sum as the tangent of half the difference of the opposite angles is to the tangent of half their sum.

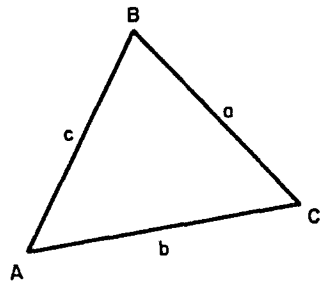


Fig. 8.18

The law of tangents may be derived rather simply from the law of sines. The law of sines states that

$$\frac{a}{b} = \frac{\sin A}{\sin B}.$$

Thus we may write

$$a = k \sin A,$$

$$b = k \sin B,$$

where k is a factor of proportionality. Then

$$\frac{a-b}{a+b} = \frac{k \sin A - k \sin B}{k \sin A + k \sin B} = \frac{\sin A - \sin B}{\sin A + \sin B}. \tag{11}$$

From (25) of Chapter 4, we may write

$$\frac{a-b}{a+b} = \frac{2 \cos \frac{1}{2}(A+B) \sin \frac{1}{2}(A-B)}{2 \sin \frac{1}{2}(A+B) \cos \frac{1}{2}(A-B)}.$$

But this is merely

$$\frac{a-b}{a+b} = \cot \tfrac{1}{2}(A+B) \tan \tfrac{1}{2}(A-B),$$

or

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(A+B)},$$

the desired relation.

In applying the law of tangents to the case SAS, what is specified is the angle C. Accordingly, for $A + B$ we write $180° - C$ or

$$\frac{a-b}{a+b} = \frac{\tan \frac{1}{2}(A-B)}{\tan \frac{1}{2}(180° - C)} = \frac{\tan \frac{1}{2}(A-B)}{\cot \frac{1}{2}C}, \tag{12}$$

since $\tan (90^\circ - \frac{1}{2}C) = \cot \frac{1}{2}C$ We use this law in the form

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \tfrac{1}{2}C \qquad (13)$$

Example 11 Using the law of tangents determine the remaining side of a triangle in which two sides have length 3 and 4 respectively and the included angle is 50° (This is Example 5, which was previously solved by the law of cosines)

Solution We have

$$\tan \tfrac{1}{2}(A - B) = \frac{a - b}{a + b} \cot \tfrac{1}{2}C = \frac{4 - 3}{4 + 3} \cot 25^\circ = \frac{1}{7} 2\ 1445 = 0\ 3064$$

and

$$\tfrac{1}{2}(A - B) = 17^\circ\ 2 \qquad (14)$$

Also

$$\tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2}C = 90^\circ - 25^\circ = 65^\circ \qquad (15)$$

The angle A can be found by adding (14) and (15)

$$A = 82^\circ\ 2$$

The angle B is found by subtracting (14) from (15)

$$B = 47^\circ\ 58$$

The law of sines is now used to determine side c

$$c = \frac{b}{\sin B} \sin C = \frac{3 \sin 50^\circ}{\sin 47^\circ\ 58} = \frac{3 \times 0\ 7660}{0\ 7428} = 3\ 094$$

Contrast the solution of this problem with the use of the law of cosines The advantage is particularly noticeable when the values are not simple numbers such as were selected for the examples Also the law of tangents is much more convenient to use with logarithms or a slide rule (Sections 9 4 and 9 5 of Chapter 9) than the law of cosines

EXERCISE 8 5

Evaluate the remaining parts of the following triangles

1	$a = 1\ 2$	$C = 43^\circ\ 12$	$b = 0\ 8$
2	$b = 5\ 9$	$A = 65^\circ\ 10$	$c = 8\ 7$
3	$c = 0\ 324$	$B = 14^\circ\ 20$	$a = 0\ 901$
4	$A = 33^\circ\ 27$	$b = 16\ 52$	$c = 31\ 97$
5	$a = 17\ 22$	$B = 58^\circ\ 48$	$c = 13\ 71$
6	$C = 49^\circ\ 45$	$a = 932\ 1$	$b = 476\ 8$
7	$a = 53\ 9$	$B = 60^\circ\ 17$	$c = 27\ 6$
8	$c = 41$	$b = 31$	$A = 29^\circ\ 52$
9	$b = 11\ 1$	$A = 16^\circ\ 11$	$c = 32\ 5$
10	$B = 43^\circ\ 72$	$c = 19\ 54$	$a = 35\ 76$

8.6. The Area of Triangles

Hitherto in this chapter, we have considered the problem of determining the sides and angles of triangles. In practical cases, it may also be important to find the areas of triangles. This can, of course, be done by determining the base and altitude of the triangle and then using the fundamental rule

$$S = \tfrac{1}{2}bh, \tag{16}$$

where S is the area, b the base, and h the altitude. This, in general, requires that we go through one of the procedures discussed above to find the appropriate parts of the triangle. Sometimes we have no other reason to determine

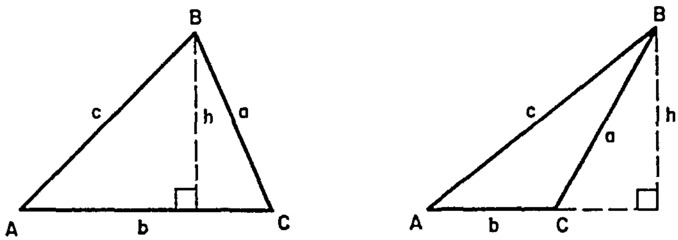


Fig. 8.19

these values explicitly, so the question arises: Can the area of a triangle be expressed directly in terms of the given parts? It can, and we shall now develop the appropriate formulas.

Consider the triangles shown in Fig. 8.19. Let b, A, and c be given. Then $h = c \sin A$ and

$$S = \tfrac{1}{2}bc \sin A. \tag{17}$$

Alternatively, if b is not given but the angles B and C are, we may use the law of sines

$$b = \frac{c \sin B}{\sin C}$$

to write

$$S = \frac{c^2 \sin A \sin B}{2 \sin C}, \tag{18}$$

or, since $C = 180° - (A + B)$,

$$S = \frac{c^2 \sin A \sin B}{2 \sin (A + B)}. \tag{19}$$

The formula to be used when only the sides of the triangle are known is determined as follows. From (17)

$$S = \tfrac{1}{2}bc \sin A = \tfrac{1}{2}bc\sqrt{1 - \cos^2 A}.$$

(The positive square root is taken because A must be less than 180°) On factoring under the radical we have

$$S = \tfrac{1}{2}bc\sqrt{(1 + \cos A)(1 - \cos A)}$$

An expression for $\cos A$ may be found from the law of cosines

$$\cos A = \frac{b^2 + c^2 - a^2}{2bc}$$

Thus

$$S = \tfrac{1}{2}bc\sqrt{\frac{(2bc + b^2 + c^2 - a^2)(2bc - b^2 - c^2 + a^2)}{4b^2c^2}}$$

$$= \tfrac{1}{4}\sqrt{[(b + c)^2 - a^2][a^2 - (b - c)^2]}$$

$$= \tfrac{1}{4}\sqrt{(b + c + a)(b + c - a)(a + b - c)(a - b + c)} \qquad (20)$$

Equation (20) is satisfactory, but a form which is simpler both to use and to remember is obtained by introducing the *semiperimeter s*, which is half the perimeter of the triangle That is,

$$s = \tfrac{1}{2}(a + b + c) \qquad (21)$$

Then (20) becomes

$$S = \tfrac{1}{4}\sqrt{(2s)(2s - 2a)(2s - 2c)(2s - 2b)}$$

$$= \sqrt{s(s - a)(s - b)(s - c)} \qquad (22)$$

This is known as *Heron's formula*

Because of the possible ambiguity in the case SSA, we shall not derive any area formula for it

EXERCISE 8-6

Find the area of each of the following triangles

1	$b = 11$	$A = 55° 17$	$c = 17$
2	$b = 23$	$C = 43° 49$	$a = 35$
3	$c = 51$	$B = 39\ 42$	$a = 31$
4	$A = 80° 6$	$b = 0\ 236$	$c = 0\ 952$
5	$a = 14\ 7$	$B = 121° 38$	$c = 16\ 3$
6	$C = 38° 55$	$b = 4\ 592$	$a = 6\ 875$
7	$a = 15\ 97$	$c = 23\ 21$	$B = 87° 29$
8	$c = 446\ 3$	$A = 22° 21$	$b = 325\ 7$
9	$C = 31°$	$b = 79\ 62$	$a = 51\ 86$
10	$c = 9\ 837$	$A = 19° 9$	$b = 6\ 972$

Determine the area of each of the following triangles:

11. $A = 23° 10'$ $b = 10$ $C = 40° 13'$.
12. $B = 31° 20'$ $c = 47.4$ $A = 39° 17'$.
13. $C = 54° 30'$ $a = 5.32$ $B = 76° 33'$.
14. $a = 4.96$ $B = 122° 0'$ $C = 41° 40'$.
15. $A = 15° 17'$ $b = 78.7$ $C = 92° 13'$.
16. $c = 0.9654$ $A = 32° 41'$ $B = 48° 28'$.
17. $C = 51° 52'$ $b = 11.62$ $A = 42° 22'$.
18. $C = 87° 59'$ $B = 59° 29'$ $a = 49.32$.
19. $A = 104° 7'$ $b = 5691$ $C = 42° 19'$.
20. $c = 44.39$ $A = 24° 41'$ $B = 77° 11'$.

What is the area of each of the triangles specified below?

21. $a = 4$ $b = 5$ $c = 6$.
22. $a = 10$ $b = 7$ $c = 12$.
23. $c = 16$ $b = 10$ $a = 17$.
24. $b = 4.32$ $a = 5.76$ $c = 9.11$.
25. $c = 33.72$ $a = 19.39$ $b = 40.71$.
26. $a = 963.4$ $b = 247.6$ $c = 901.7$.
27. $c = 7632$ $b = 8512$ $a = 6041$.
28. $b = 0.432$ $c = 0.298$ $a = 0.518$.
29. $b = 4.176$ $a = 6.932$ $c = 5.197$.
30. $c = 52.74$ $a = 33.97$ $b = 41.99$.

8.7. Half-Angle Formulas (SSS)

Just as the law of tangents provides an alternative to the law of cosines that offers some computational advantages in the case SAS, there is another method of treating the case wherein only the sides of the triangle are known (SSS). This method utilizes certain half-angle formulas.

As a preliminary to developing the half-angle formulas, we derive an expression for the radius of the inscribed circle (Fig. 8.20). Since this circle is tangent to the sides at points D, E, and F, the lines OD, OE, and OF are perpendicular to the sides.

We already have an expression for the area of the triangle, (22); let us find the area in terms of r and attempt to combine the two expressions to determine r. The area of the triangle ABC is equal to the sum of the areas AOE, BOE, BOF, COF, COD, and AOD. This is

$$\begin{aligned} S &= \tfrac{1}{2}(AE)r + \tfrac{1}{2}(EB)r + \tfrac{1}{2}(BF)r + \tfrac{1}{2}(FC)r + \tfrac{1}{2}(CD)r + \tfrac{1}{2}(DA)r \\ &= \tfrac{1}{2}r(AE + EB + BF + FC + CD + DA) = \tfrac{1}{2}r(AB + BC + CA). \end{aligned}$$

But $AB + BC + CA$ is the perimeter of the triangle or twice the semiperimeter s [see (21)]. Thus

$$S = \tfrac{1}{2}r(2s) = rs, \tag{23}$$

Equation (22) is an expression for S which combined with (23) yields

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \tag{24}$$

The half angle formula may be developed with the aid of Fig 8 20 It is first noted that there is a result from geometry which states that the center of the inscribed circle is the intersection of the angle bisectors of the triangle Thus the lines AO, BO, and CO bisect the angles A, B, and C respectively

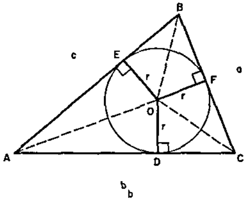


Fig 8 20

It is seen then that AE and AD must be equal since each is $AO \sin \frac{1}{2}A$ Similarly, BE and BF are equal as are CD and CF Thus the perimeter of the triangle is

$$2AE + 2BF + 2CD$$

Since this is twice the semiperimeter s, we may write

$$AE + BF + CD = s,$$

and

$$AE = s - (BF + CD) = s - (BF + CF) = s - a,$$

$$BF = s - (AE + CD) = s - (AD + CD) = s - b,$$

$$CD = s - (AE + BF) = s - (AE + BE) = s - c$$

Therefore from triangle AOE,

$$\tan \tfrac{1}{2}A = \frac{r}{AE} = \frac{r}{s-a}, \tag{25}$$

from triangle BOF,

$$\tan \tfrac{1}{2}B = \frac{r}{BF} = \frac{r}{s-b}, \tag{26}$$

and from triangle COD,

$$\tan \tfrac{1}{2}C = \frac{r}{CD} = \frac{r}{s-c}, \tag{27}$$

where r is determined from (24).

Example 12. Determine the angles in the triangle shown in Fig. 8.21. (This example is the same as Example 6, which previously was solved by the law of cosines.)

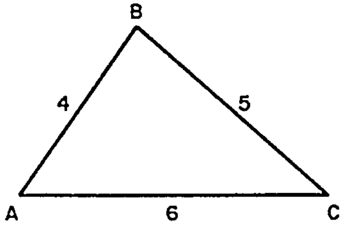


Fig. 8.21

Solution: With s as the semiperimeter,

$$2s = 4 + 5 + 6 = 15.$$
$$s = 7.5$$

and

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}}$$
$$= \sqrt{\frac{(2.5)(1.5)(3.5)}{7.5}} = \sqrt{1.75} = 1.323.$$

Thus

$$\tan \tfrac{1}{2}A = \frac{r}{s-a} = \frac{1.323}{2.5} = 0.5292,$$

or

$$\tfrac{1}{2}A = 27° 53',$$

and

$$A = 55° 46'.$$

Similarly,

$$\tan \tfrac{1}{2}B = \frac{r}{s-b} = \frac{1.323}{1.5} = 0.8820,$$

or

$$\tfrac{1}{2}B = 41° 25',$$

and

$$B = 82° 50',$$

while

$$\tan \tfrac{1}{2}C = \frac{r}{s-c} = \frac{1.323}{3.5} = 0.3780$$

implies that

$$\tfrac{1}{2}C = 20° 42',$$

or

$$C = 41° 24'.$$

The advantages of the half-angle method show up most strongly when the numbers are complicated and logarithms are used.

EXERCISE 8 7

Using the half angle formulas determine the angles of the triangles whose sides are the following

1	$a = 4\ 5$	$b = 3\ 2$	$c = 2\ 7$
2	$a = 10$	$b = 14$	$c = 12$
3	$a = 140$	$b = 76$	$c = 70$
4	$b = 1\ 32$	$c = 2\ 57$	$a = 1\ 56$
5	$c = 4\ 93$	$a = 5\ 72$	$b = 3\ 01$
6	$b = 0\ 763$	$a = 0\ 897$	$c = 1\ 231$
7	$a = 28$	$c = 43$	$b = 52$
8	$b = 932$	$a = 647$	$c = 593$
9	$c = 2\ 659$	$a = 4\ 782$	$b = 3\ 588$
10	$c = 437\ 6$	$b = 840\ 3$	$a = 621\ 9$
11	$b = 932$	$c = 876$	$a = 1592$
12	$a = 4\ 32$	$c = 7\ 86$	$b = 7\ 02$
13	$c = 79\ 68$	$b = 35\ 38$	$a = 99\ 54$
14	$b = 1\ 234$	$a = 0\ 863$	$c = 0\ 792$
15	$a = 0\ 176$	$c = 0\ 174$	$b = 0\ 159$

PROBLEMS

1 What is wrong with this problem Use the law of cosines to determine the angles in a triangle wherein $a = 14\ 7$, $b = 11\ 3$ $c = 28\ 6$?

2 Use the law of sines to find the second angle in Exercise 8 3, nos 1–20 after finding the third side with the law of cosines

3 Use the law of sines to find the second angle in Exercise 8 3 nos 21–30, after finding the first angle with the law of cosines

4 Use the law of tangents to find the remaining angles in Exercise 8 3 nos 1 20

5 Use the half angle formulas to solve Exercise 8 3 nos 21–30

6 Develop an expression for the area of a triangle wherein SAA are specified

7 Find the remaining parts of the following triangles

(a)	$A = 47°\ 35$	$b = 0\ 632$	$c = 1\ 40$
(b)	$A = 65°\ 28$	$B = 43°\ 12$	$c = 12\ 7$
(c)	$a = 193$	$b = 256$	$c = 89$
(d)	$A = 30\ \ 14$	$b = 43\ 1$	$C = 98°\ 22$
(e)	$A = 57°\ 23$	$b = 11\ 9$	$a = 11\ 0$
(f)	$a = 4\ 31$	$c = 7\ 62$	$C = 31°\ 19$
(g)	$a = 0\ 063$	$B = 73°\ 17$	$c = 0\ 041$
(h)	$a = 5\ 09$	$B = 114°\ 12$	$c = 40°\ 11$
(i)	$a = 16\ 3$	$b = 19\ 7$	$c = 12\ 2$
(j)	$a = 98\ 3$	$b = 15\ 2$	$C = 15°\ 10$

8. Show that the radius R of a circle circumscribed about a triangle is given by

$$R = \frac{abc}{4S}$$

where S is the area of the triangle (Fig. 8.22). [HINT: The center of the circumscribed circle is located on the perpendicular bisectors of the sides of the triangle.]

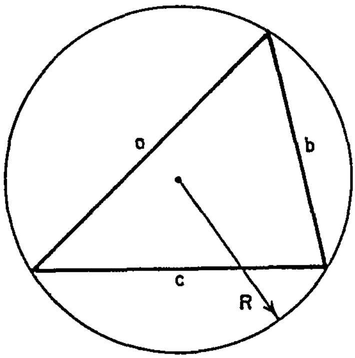


Fig. 8.22

9. Show that

$$\sin A = \frac{2}{bc}\sqrt{s(s-a)(s-b)(s-c)}.$$

10. Prove that

$$\sin\frac{A}{2} = \sqrt{\frac{(s-b)(s-c)}{bc}}.$$

11. Prove that, in any triangle,

$$\frac{a-b}{c} = \frac{\sin\frac{1}{2}(A-B)}{\cos\frac{1}{2}C}.$$

12. Demonstrate that

$$\frac{a+b}{c} = \frac{\cos\frac{1}{2}(A-B)}{\sin\frac{1}{2}C}.$$

(The above two expressions are known as *Mollweide's formulas*. They are very useful in checking solutions of triangles, since all six parts of the triangle appear in each formula.)

13. Two lighthouses protrude through the surface of a body of water. I locate myself in a boat so the tops of the lighthouses appear to be in a straight line. Let the angle of elevation of this straight line be α (Fig. 8.23). Let the angles of depression of the reflections be β and γ respectively. Then if the height of my eyes is h, show that the distance d between the two lighthouses is

$$d = 2h\frac{\cos^2\alpha\sin(\beta-\gamma)}{\sin(\beta-\alpha)\sin(\gamma-\alpha)}.$$

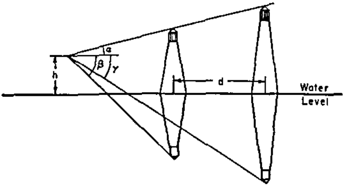


Fig. 8.23

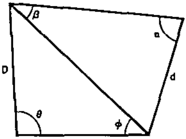


Fig 8.24

14. Show that, in Fig 8 24,

$$d = D\frac{\sin\theta\sin\beta}{\sin\phi\sin\alpha}$$

15. Derive the formula for cos $(A + B)$ by using the law of cosines. That is, let $\theta = (A + B)$ and find $\cos\theta$ [HINT Construct the triangle so the side common to the angles A and B is on one of the axes]

9

PRACTICAL PROBLEMS AND TECHNIQUES

In Chapter 8, methods were developed for solving triangles when various combinations of data were initially provided. Two features characterized the solutions considered: They were merely abstract problems in triangle solving, and they generally required extensive and tedious computation.

In this chapter, we will first consider techniques for reducing the labor of arithmetic manipulations, namely, logarithms and the slide rule. We will then discuss some practical situations in which the need to solve triangles arises.

9.1. The Law of Exponents

As a prelude to the discussion of logarithms, we first review some of the relationships governing the meaning and use of exponents.

When we say we raise a number $a \neq 0$ to the nth power, we mean we multiply the number by itself n times. Thus

$$a^n = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ times}}.$$

(In the expression a^n, the positive integer n is termed the *exponent*.) Accordingly,

$$a^n \times a^m = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n \text{ times}} \cdot \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{m \text{ times}} = \underbrace{a \cdot a \cdot a \cdot \cdots \cdot a}_{n+m \text{ times}},$$

or

$$a^n a^m = a^{n+m}. \tag{1}$$

Equation (1) assumes that n and m are positive integers. How shall we define a^p if p is zero or a negative integer? We define it so that (1) still holds true. For example, if $m = 0$ in (1),

$$a^n a^0 = a^{n+0} = a^n. \tag{2}$$

Thus a^0 must be one,

$$a^0 = 1 \tag{3}$$

Similarly, if $m = -n$,

$$a^n a^m = a^n a^{-n} = a^{n-n} = a^0 = 1$$

Thus for (1) to be consistent with negative integral exponents, we must have

$$a^{-n} = \frac{1}{a^n} \tag{4}$$

We also note that

$$(a^m)^n = \underbrace{a^m \; a^m \; \cdots \; a^m}_{n \text{ times}}$$

$$= \underbrace{\underbrace{a \; a \; \cdots \; a}_{m \text{ times}} \; \underbrace{a \; a \; \cdots \; a}_{m \text{ times}} \; \cdots \; \underbrace{a \; a \; \cdots \; a}_{m \text{ times}}}_{n \text{ times}}$$

$$= \underbrace{a \; a \; a \; \cdots \; a}_{m \times n \text{ times}} = a^{mn}$$

Thus

$$(a^m)^n = a^{mn} \tag{5}$$

We now turn to the problem of defining $a^{p/q}$ where p and q are integers and $q \neq 0$ First suppose $p = 1$ We shall define $a^{1/q}$ so that (5) holds

$$(a^q)^{1/q} = (a^{1/q})^q = a^{q/q} = a^1 = a$$

Thus $a^{1/q}$ is the qth root of a We may also write this as

$$a^{1/q} = \sqrt[q]{a}$$

Then $a^{p/q}$ is simply

$$(\sqrt[q]{a})^p$$

In general,

$$c = a^{p/q}$$

means that c is the pth power of the qth root of a Thus expressions like

$$10^{1.3762}$$

have meaning For, if worst comes to worst, we may consider it as the 13,762 power of the 10 000th root of 10

Finally suppose r is not a rational number, that is, not a number of the form p/q where p and q are integers ($q \neq 0$) For example, $\sqrt{2}$ is such a number We can still define a^r as a limit of the sequence $a^{r_1}, a^{r_2}, \ldots, a^{r_n}, \ldots$ where $r_1, r_2, \ldots, r_n, \ldots$ is a sequence of rational numbers approaching r (See Chapter 12)

EXERCISE 9-1

Simplify the following expressions:

1. $x^2x^3x^5$.
2. $\dfrac{x^4x^7}{x^5}$.
3. $(ax^2)(bx^3)(ax)^7$.
4. $(ax^2)^3(a^3x)^2(a^2x^2)^2$.
5. $\sqrt[3]{4}\,\sqrt[3]{16}$.
6. $\sqrt{3}\,\sqrt{27}$.
7. $\sqrt{\dfrac{a^2 - b^2}{a - b}}$.
8. $\dfrac{a^{-1} - b^{-1}}{(a - b)^{-1}}$.
9. $\dfrac{a^{-2} - b^{-2}}{a^{-1} - b^{-1}}$.
10. $64^{5/6}$.

9.2. The Basic Idea

Equation (1) gives us a clue how we may substitute the operation of addition for multiplication. Suppose we had two numbers, c and d, which we wished to multiply. Suppose further that we could express them in the form

$$c = a^r$$
$$d = a^s$$

where the a's are the same in both cases and r and s may be any real numbers. Then

$$cd = a^r a^s = a^{r+s}.$$

The problem is now merely to reduce a^{r+s} to an ordinary number. This may seem to be going in circles. However, suppose we had a table of powers of a. For concreteness, let $a = 2$ (Table 1). With this table, we may multiply simple numbers such as

$$4 \times 64 = 2^2 \times 2^6 = 2^8 = 256.$$

TABLE 1

n	0	1	2	3	4	5	6	7	8	9	10	11	12
2^n	1	2	4	8	16	32	64	128	256	512	1024	2048	4096

However, it is hardly a useful scheme. One difficulty lies in the fact that the values of n are too widely spaced and hence numbers of the form 2^n are also widely separated. If we could determine a number such as $2^{0.1}$ or $2^{0.01}$, we could remedy this difficulty (Table 2):

TABLE 2

n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
2^n	1	1.07	1.15	1.2	1.3	1.4	1.5	1.6	1.7	1.9	2.0

This table has fairly small spacing between successive values of 2^n, but it also appears that it will grow indefinitely long as n increases

There are two steps which can be taken to resolve this dilemma The first is to use a different base to our system than 2, we select 10 because it is the basis for our decimal numbering system The second step is related We introduce the so-called "scientific notation" for writing numbers

This system of writing numbers is used frequently in science to indicate the accuracy of measured values and also to reduce extremely small or extremely large numbers to a reasonable method of presentation Suppose we say Tom is twice as heavy as Jerry This probably means that Tom's weight is somewhere between 1 5 and 2 5 times that of Jerry's If we had more precise estimates of their weights, we might say that the ratio is 1 9 or 1 87 or 1 874, depending on how well the ratio is known Generally speaking, the first number (1 9) indicates that the ratio lies between 1 85 and 1 95, the next indicates narrower limits, 1 865 and 1 875, and the final number specifies that the ratio is between 1 8735 and 1 8745 Even if the ratio were more nearly two, the various numbers 2 0, 2 00, and 2 000 place successively narrower limits on the precision of the specification 1 95 and 2 05, 1 995 and 2 005, 1 9995 and 2 0005 respectively To indicate the precision of a number, we refer to the number of meaningful digits as the number of *significant figures* Thus 1 9 and 2 0 each have two significant figures, 1 87 and 2 00 each have three significant figures (Observe that here zero is a significant figure)

Suppose now we learn that the speed of light is 300,000,000 meters per second Does this imply that there are nine significant figures in the measurement? Or are the zeros present merely to make the number three hundred million rather than three or thirty? To avoid this source of ambiguity the scientific system of writing numbers is used

Any number may be written as a product of a number between one and ten and a power of ten Thus

$$30 = 3 \times 10,$$
$$31 = 3\ 1 \times 10,$$
$$3000 = 3 \times 10^3,$$
$$300{,}000{,}000 = 3 \times 10^8,$$
$$0\ 0000003 = 3 \times 10^{-7},$$
$$0\ 997 = 9\ 97 \times 10^{-1},$$
$$1\ 01 = 1\ 01\ (\times 10^0)$$

This notation may be used, then, to specify separately those zeros that are significant and those that are used merely to locate the decimal point For example,

$$3 \times 10^8$$

has one significant figure, while

$$3\ 00 \times 10^8$$

has three significant figures. (In fact, the speed of light is 3.00×10^8 meters per second.) Also, this notation is generally more compact than conventional notation and less susceptible to generating errors in reading. (Is 430000000000 greater than or less than 50000000000?)

TABLE 3.

n	0	0.1	0.2	0.3	0.4	0.5	0.6	0.7	0.8	0.9	1.0
10^n	1.00	1.26	1.58	2.00	2.51	3.16	3.98	5.01	6.31	7.94	10.00

Let us now construct a table of powers of 10 (Table 3).* With the aid of this table, we may express any positive number as a power of 10 (although this brief table limits us to about two significant figures) by writing the number in the scientific notation as a product of a number between one and ten and a power of 10. For example,

$$200 = 2.00 \times 10^2 = 10^{0.3} \times 10^2 = 10^{2.3},$$

$$63.1 = 6.31 \times 10 = 10^{0.8} \times 10 = 10^{1.8},$$

$$0.00316 = 3.16 \times 10^{-3} = 10^{0.5} \times 10^{-3} = 10^{-2.5},$$

$$2 = 2 \times 10^0 = 10^{0.3} \times 10^0 = 10^{0.3},$$

$$2250 = 2.25 \times 10^3 = 10^{0.35} \times 10^3 = 10^{3.35}.$$

(In the last example, simple interpolation is necessary.)

To return to our original purpose in considering powers of 10, let us use this procedure to multiply 251 by 31.6:

$$251 = 2.51 \times 10^2 = 10^{0.4}10^2 = 10^{2.4},$$

$$31.6 = 3.16 \times 10 = 10^{0.5}10 = 10^{1.5}.$$

Therefore

$$251 \times 31.6 = 10^{2.4} \times 10^{1.5} = 10^{3.9} = 10^{0.9} \times 10^3$$
$$= 7.94 \times 10^3 = 7940 \qquad \text{(to three significant figures).}$$

It is not clear from this trivial example that it is necessarily advantageous to use this method. The systematic exploitation of this technique, using more extensive tables, does indeed result in a great saving in computational labor. This systematic use is explained in the next section.

* For the present, we may assume that we determined the tenth root of 10 (that is, $10^{0.1}$) by trial and error.

EXERCISE 9 2

Evaluate the following expressions

1. 2^{15}
2 2^{-5}
3 $10^{1\ 7}$
4 $10^{4\ 8}$
5 $10^{-6\ 3}$
6 Determine, by any means, the value of $10^{0\ 1}$ to four significant figures

9.3. The Use of Logarithms

In actually utilizing the methods outlined in the previous section, it proves convenient to deal not with powers of a number but with a related function, termed the *logarithm* The logarithm of a number is the exponent to which another number, called the *base*, must be raised to obtain that number Thus

$$\log_{10} 100 = 2$$

(read "logarithm to the base 10 of 100") means that 10 must be raised to the exponent 2 (that is, 10^2) to obtain 100 From Tables 1, 2, and 3, we see that

$$\log_2 32 = 5$$
$$\log_{10} 3\ 98 = 0\ 6$$
$$\log_2 1\ 3 = 0\ 4$$

Because of the convenience of 10 as a base when scientific notation is used, this is the one most commonly used for computation, logarithms to the base 10 are called *common logarithms* Logarithms to the base 2 have some application in the field of information theory * Another base which is of great theoretical importance is 2 71828 , represented by the letter e The strange-looking number occurs quite naturally in advanced work (see Chapters 12 and 13), so logarithms to the base e are referred to as *natural logarithms*

Because of the frequent use of common and natural logarithms, the convention exists that the symbol $\log x$ (without explicitly specifying the base) means a common logarithm, $\log_{10} x$, and $\ln x$ means $\log_e x$

In order to develop the properties of logarithms, we shall relate them explicitly to exponential functions By the definition of a logarithm,

$$a^{\log_a x} = x, \tag{6}$$

so

$$10^{\log x} = x \tag{7}$$

* If one of a set of N objects is to be selected by a game resembling "twenty questions $\log_2 N$ is the number of questions which on the average must be asked

Since $a^n a^m = a^{n+m}$ [see (1)],

$$xy = 10^{\log x}\, 10^{\log y} = 10^{(\log x + \log y)}. \tag{8}$$

But by (7),

$$xy = 10^{\log xy}. \tag{9}$$

On comparing (8) and (9), we have

$$10^{\log xy} = 10^{(\log x + \log y)},$$

or

$$\log xy = \log x + \log y \tag{10}$$

(since the logarithm is a single-valued function). Similarly,

$$\log \frac{x}{y} = \log x - \log y. \tag{11}$$

It is also easy to show that

$$\log x^y = y \log x. \tag{12}$$

One further trivial property is

$$\log 1 = 0. \tag{13}$$

Let us now investigate the use of logarithms in computation. In the back of the book will be found tables of logarithms to four decimal places, for values of n between 1.00 and 9.99. (These tables are related to but are more extensive than Table 3. Observe the different significance of n in the two tables.) It is customary in printing such tables to neglect the decimal point. The values of n are arranged with the first two significant figures in the left-hand column, with the third figure heading the appropriate vertical column. Thus

$$\log 1.47 = 0.1673,$$
$$\log 3.96 = 0.5977.$$

Logarithms of numbers having four significant figures may be determined by interpolation:

$$\log 4.556 = 0.6586,$$
$$\log 9.064 = 0.9573,$$
$$\log 3.257 = 0.5128.$$

When we wish to find the logarithm of numbers less than one or greater than ten, we first write the number in scientific notation. Let N be such a number. Write this as

$$N = n \times 10^c \tag{14}$$

where n is a number between 1 and 10, and c is an integer, positive or negative. Then

$$\log N = \log n + \log 10^c = \log n + c. \tag{15}$$

The value of $\log n$ may be found from the table For example,

$$\log 347 = \log 3\ 47 + 2 = 2\ 5403,$$
$$\log 12\ 34 = \log 1\ 234 + 1 = 1\ 0913,$$
$$\log 13900 = \log 1\ 39 + 4 = 4\ 1430$$

The number c is often termed the *characteristic* of the logarithm, $\log n$ is termed the *mantissa*

If c is negative, it often proves convenient to leave the subtraction indicated without actually carrying it out

$$\log 0\ 219 = 0\ 3404 - 1,$$
$$\log 1\ 427 \times 10^{-4} = 0\ 1544 - 4,$$
$$\log 0\ 0763 = 0\ 8825 - 2$$

Logarithms may be used for multiplication with the aid of (10)

Example 1 Multiply 23 76 by 302 4 with the aid of logarithms

Solution

$$\begin{array}{rll} 23\ 76 & \rightarrow \log 23\ 76 & = 1\ 3758 \\ \times 302\ 4 & \rightarrow \log 302\ 4 & = 2\ 4806 \\ \hline 7185 & \leftarrow \log\text{(product)} & = 3\ 8564 \end{array}$$

Interpolation has been used to determine the fourth digit of the product which has four significant figures

Example 2 Find the product of the factors 492 3, 5 629, 0 006230, 98 15 with the aid of logarithms

Solution

$$\begin{array}{lll} 492\ 3 & \rightarrow \log 492\ 3 & = 2\ 6922 \\ \times 5\ 629 & \rightarrow \log 5\ 629 & = 0\ 7504 \\ \times 0\ 006230 & \rightarrow \log 0\ 006230 & = 0\ 7945 - 3 \\ \times 98\ 15 & \rightarrow \log 98\ 15 & = 1\ 9919 \\ \hline & & \quad 6\ 2290 - 3 \\ 1694 & \leftarrow \log\text{(product)} & = 3\ 2290 \end{array}$$

This example also illustrates the value of writing log 0 006230 as 0 7945 − 3 rather than as −2 2055 The latter requires two extra subtractions

Division may also be performed, using (11)

Example 3 Find the quotient 493 7 − 33 29

Solution

$$\begin{array}{rll} 493\ 7 & \rightarrow \log 493\ 7 & = 2\ 6934 \\ -33\ 29 & \rightarrow -\log 33\ 29 & = -1\ 5223 \\ \hline 14\ 83 & \leftarrow \log\text{(quotient)} & = 1\ 1711 \end{array}$$

Example 4. Divide 53.29 by 147.6, using logarithms.

Solution:

$$\begin{aligned} \log 53.29 &= 1.7266 \\ \log 147.6 &= 2.1691 \\ \hline \log(\text{quotient}) &= -0.4425. \end{aligned}$$

Here the logarithm of the quotient is a negative number (implying that the quotient is less than one). However, since tables list only logarithms between one and ten, we must contrive to reduce this logarithm to a form suitable for use with the tables. This may be done by both adding one to and subtracting one from the logarithm:

$$\log(\text{quotient}) = -0.4425 = 1 - 0.4425 - 1 = 0.5575 - 1,$$

whence

$$\text{quotient} = 0.3610.$$

Operations involving combinations of multiplication and division may also be performed conveniently with the aid of logarithms.

Example 5. Evaluate

$$\frac{19.72 \times 0.4931 \times 1.590 \times 10^{-7}}{4732 \times 3.988 \times 10^{-11}}.$$

Solution:

$$\begin{aligned} \log 19.72 &= 1.2949 & \log 4732 &= 3.6751 \\ \log 0.4931 &= 0.6929 - 1 & \log 3.988 &= 0.6008 \\ \log 1.590 &= 0.2014 & \log 10^{-11} &= 0.0000 - 11 \\ \log 10^{-7} &= 0.0000 - 7 & & \\ \hline \log(\text{numerator}) &= 2.1892 - 8 & \log(\text{denominator}) &= 4.2759 - 11 \end{aligned}$$

$$\begin{aligned} \log(\text{numerator}) &= 2.1892 - 8 \\ -\log(\text{denominator}) &= -4.2759 + 11 \\ \hline \log(\text{quotient}) &= -2.0867 + 3 = 0.9133 \\ \text{quotient} &= 8.190. \end{aligned}$$

Equation (12) is also very useful in determining powers and roots of numbers with the aid of logarithms.

Example 6. Determine $\sqrt{893.7}$.

Solution: $\sqrt{893.7} = 893.7^{1/2}$.

$$\begin{aligned} \log 893.7 &= 2.9512 \\ & \times \tfrac{1}{2} \\ \hline \log \sqrt{893.7} &= 1.4756 \\ \sqrt{893.7} &= 29.89. \end{aligned}$$

This technique is quite powerful since the exponent does not have to be a simple fraction.

Example 7 Evaluate $14\ 53^{2\ 761}$

Solution

$$\log 14\ 53 = 1\ 1623$$
$$\times 2\ 761$$
$$\log 14\ 53^{2\ 761} = 3\ 210$$
$$14\ 53^{2\ 761} = 1620\ ^*$$

(Logarithms were also used in performing the multiplication indicated

$$\log 1\ 1623 = 0\ 0654$$
$$\log 2\ 761 = 0\ 4411$$
$$\log(\text{product}) = 0\ 5065$$
$$\text{product} = 3\ 210$$

Observe the loss of one significant figure in performing the multiplication)

In connection with determining powers of numbers note that

$$\log_a n = (\log_a 10)(\log_{10} n) \tag{16}$$

for

$$a^{\log_a n} = 10^{\log_{10} n}$$

and

$$10 = a^{\log_a 10}$$

so that

$$n = a^{\log_a n} = (a^{\log_a 10})^{\log_{10} n} = a^{(\log_a 10)(\log_{10} n)}$$

which implies (16)

Equation (16) may be generalized to

$$\log_a n = \log_a b \log_b n \tag{17}$$

However (16) is of more practical value since it enables one to determine logarithms to any base from a table of common logarithms

Example 8 In determining logarithms with base 2 it is necessary to know $\log_2 10$ Determine $\log_2 10$

Solution From (17)

$$\log_{10} 10 = \log_{10} 2 \log_2 10$$
$$1 = 0\ 3010 \log_2 10$$

and

$$\log_2 10 = \frac{1\ 0000}{0\ 3010} = 3\ 322$$

(This division was performed using logarithms $\log\ 3\ 322 = \log\ 1\ 0000 - \log 0\ 3010 = 0 - 0\ 4786 - 0\ 5214 - 1$)

---

* More precisely we should write $1\ 62 \times 10^3$

In the foregoing example we implicitly developed an alternative form of (16):

$$\log_a n = \frac{\log_{10} n}{\log_{10} a}. \tag{18}$$

EXERCISE 9-3

Find the value of each logarithm:

1. $\log_3 27$.
2. $\log_2 64$.
3. $\log_2 1$.
4. $\log_{1/2} 32$.
5. $\log_{0.1} 100$.
6. $\log_{100} 0.1$.
7. $\log_{3/4} \frac{9}{16}$.
8. $\log_2 \frac{1}{8}$.
9. $\log_9 81$.
10. $\log_5 625$.

Find the value of each of the following numbers:

11. log 1.03.
12. log 8.43.
13. log 2.29.
14. log 1.63.
15. log 7.08.
16. log 3.53.
17. log 9.71.
18. log 5.49.
19. log 4.68.
20. log 6.15.

Evaluate each logarithm:

21. log 9.375.
22. log 6.838.
23. log 3.162.
24. log 1.494.
25. log 7.727.
26. log 5.047.
27. log 4.569.
28. log 1.833.
29. log 8.124.
30. log 5.491.

Determine the common logarithm of each of the following numbers:

31. 1.76×10.
32. 3.31×10^2.
33. 4.41×10^{-2}.
34. 2.59×10^5.
35. 5.26×10^8.
36. 7.53×10^{-2}.
37. 6.58×10^6.
38. 3.27×10^{-7}.
39. 1.18×1.
40. 4.28×10^0.
41. 57.7.
42. 2.49.
43. 0.643.
44. 0.011.
45. 863.
46. 2120.
47. 54100.
48. 817000.
49. 0.00241.
50. 0.000692.

Find the following logarithms:

51. log 3142.
52. log 765.9.
53. log 0.004311.
54. log 63,780,000,000.
55. log 0.0000005932.
56. log 5.713.
57. log 3157.
58. log 5,492,000.
59. log 27.96.
60. log 0.2165.

Multiply the following numbers by using logarithms

61. 7 49 × 3 15
62. 2 71 × 828
63. 592 × 0 653
64. 4 83 × 71 7
65. 16 6 × 17 7
66. 3976 × 4392
67. 1895 × 0 3422
68. 13590000 × 3 721
69. 53 88 × 721 2
70. 0 1433 × 0 005981

Determine the following quotients by using logarithms

71. 6 92 − 5 87
72 4 72 − 3 92
73. 581 − 0 632
74. 0 0777 − 41 3
75. 0 698 − 0 721
76. 4 397 − 5 388
77. 76 32 − 0 7559
78. 437 8 − 6 392
79. 81 03 − 779 4
80. 1976 − 432100

Perform the following operations

81. 432 × 965 × 8 11

82. 3 14 × 197 × 4 32

83. $\frac{211 \times 0\ 752}{0\ 00681}$

84. $\frac{4\ 32 \times 75\ 5}{33\ 7}$

85. $\frac{7\ 68 \times 0\ 00971}{145\ 2 \times 0\ 189}$

86 $\frac{3\ 000 \times 10^{9} \times 0\ 7652}{15\ 97 \times 0\ 01561}$.

87. $\frac{43\ 71 \times 199\ 9}{0\ 001771 \times 286500}$

88. $\frac{0\ 02134 \times 15\ 99}{4967 \times 14\ 98}$

89. $\frac{1\ 234 \times 9872 \times 8\ 113}{0\ 2761 \times 4\ 199 \times 432500}$.

90 $\frac{5928 \times 7\ 132 \times 0\ 006777}{4362 \times 0\ 8195 \times 0\ 9988}$.

Evaluate the following quantities using logarithms

91. $\sqrt{2}$
92. $\sqrt[3]{3}$
93. $10^{0\ 1}$
94 $\sqrt[4]{7}$
95. $\sqrt[11]{2}$
96. $1\ 732^{4\ 387}$
97. $4986^{7\ 314}$
98 $0\ 2185^{0\ 97813}$
99. $5\ 621^{43\ 76}$
100. $61\ 77^{0\ 7166}$
101. $\log_2 7$
102. $\log_3 15$
103. $\log_{2\ 5} 8$
104. $\log_{2\ 18} 9\ 32$
105 $\log_{0\ 176} 1\ 54$

9.4. The Solution of Triangles with Logarithms

Much of the labor of solving triangles is the vast amount of multiplication and division which must be performed Clearly this can be simplified greatly by the use of logarithms, merely by using the rules already explained. However, there is one additional simplification that renders almost trivial the solution of triangles. This is the availability of tables of logarithms of trigonometric functions. Thus to find, for example, log (sin 44° 10′), it is not necessary first to find sin 44° 10′ (= 0.6967) and then determine log 0.6967 = 0.8430 − 1. One need merely find log sin 44° 10′ directly in the tables (at the end of the book) as 9.8431 − 10. (Observe how a round-off error occurs when two steps are used.) Interpolation, when necessary, may be performed directly in the tables of logarithms of trigonometric functions.

There are two facts to note especially when using these tables. Since the trigonometric functions often have values less than one, the corresponding logarithms will be negative. In order to avoid the inconvenience of dealing with negative logarithms, each of the entries has had ten added to it. Thus when copying a logarithm from the tables, one must subtract ten from it. This subtraction should be indicated, rather than actually performed, in order to reduce manipulative labor. (Any remaining subtraction can be performed at the end of the problem, as will be illustrated in the examples.) Thus log sin 44° 10′ is written 9.8431 − 10 rather than −0.1569.

The second point to consider is that the trigonometric functions often take on negative values themselves. Negative numbers do not have real logarithms (see, however, Chapter 13). Thus we may multiply the magnitudes of the numbers and append the appropriate signs later.

Example 9. Evaluate 47.3 cos 100° 20′.

Solution: cos 100° 20′ = −cos 79° 40′.

Therefore, log cos 100° 20′ should be found by determining log cos 79° 40′:

$$\begin{aligned} \log 47.3 &= 1.6749 \\ \log \cos 79^\circ 40' &= \underline{9.2538 - 10} \\ \log(\text{product}) &= 0.9287 \\ 47.3 \cos 79^\circ 40' &= 8.486. \end{aligned}$$

So

$$47.3 \cos 100^\circ 20' = -8.486.$$

We now illustrate the solution of triangles with the aid of logarithms.

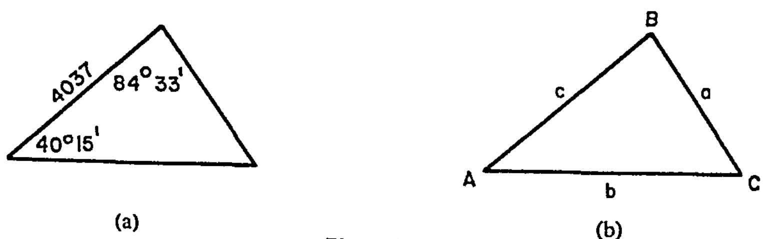


Fig. 9.1

Example 10. Find the remaining parts of the triangle shown in Fig. 9.1a.

Solution· The parts of the triangle are first labeled for identification as in Fig 9 1b The law of sines is the appropriate rule for solving this triangle·

$$C = 180° - (40° 15' + 84° 33') = 55° 12'.$$

$$a = 4037 \frac{\sin 40° 15'}{\sin 55° 12'},$$

$$\begin{aligned} \log 4037 &= \quad 3\ 6061 \\ \log \sin 40° 15' &= \quad \underline{9\ 8104 - 10} \\ & \quad\ 13\ 4165 - 10 \\ -\log \sin 55° 12' &= \quad \underline{-9\ 9144 + 10} \\ \log a &= \quad 3\ 5021 \\ a &= 3178 \end{aligned}$$

$$b = 4037 \frac{\sin 84° 33'}{\sin 55° 12'},$$

$$\begin{aligned} \log 4037 &= \quad 3\ 6061 \\ \log \sin 84° 33' &= \quad \underline{9.9980 - 10} \\ & \quad\ 13\ 6041 - 10 \\ -\log \sin 55° 12' &= \quad \underline{-9\ 9144 + 10} \\ \log b &= \quad 3\ 6897 \\ b &= 4894 \end{aligned}$$

Example 11. Determine the remaining parts of the triangle shown in Fig 9 2

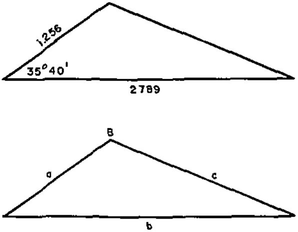


Fig. 9.2

Solution· Two solutions are demonstrated so they may be contrasted The first utilizes the law of cosines, the second the law of tangents

I. *By the law of cosines*

$$c^2 = a^2 + b^2 - 2ab\cos C$$

$$\begin{aligned} \log 1.256 &= 0.0990 \\ &\quad \times\ 2 \\ \log 1.256^2 &= 0.1980 \\ 1.256^2 &= 1.578 \end{aligned}$$

$$\begin{aligned} \log 2.789 &= 0.4454 \\ &\quad \times\ 2 \\ \log 2.789^2 &= 0.8908 \\ 2.789^2 &= 7.777 \end{aligned}$$

$$\begin{aligned} \log 2 &= 0.3010 \\ \log 1.256 &= 0.0990 \\ \log 2.789 &= 0.4454 \\ \log\cos 35^\circ 40' &= 9.9098 - 10 \\ \log(2ab\cos C) &= 0.7552 \\ 2ab\cos C &= 5.691 \end{aligned}$$

$$\begin{aligned} c^2 = &\ \ \ 1.578 \\ &+ 7.777 \\ &\ \ \ 9.355 \\ &- 5.691 \\ &\ \ \ 3.664 \end{aligned}$$

$$\begin{aligned} \log c^2 &= 0.5640 \\ &\quad \times\ \tfrac{1}{2} \\ \log c &= 0.2820 \\ c &= 1.914. \end{aligned}$$

The angles may now be found using the law of sines:

$$\sin B = \frac{2.789}{1.914}\sin 35^\circ 40'$$

$$\begin{aligned} \log 2.789 &= \ \ 0.4454 \\ \log\sin 35^\circ 40' &= \ \ 9.7657 - 10 \\ &\ \ \ \ 0.2111 \\ -\log c &= -0.2820 \\ \log\sin B &= \ \ 9.9291 - 10 \\ B &= 58^\circ 9' \text{ or } 121^\circ 51'. \end{aligned}$$

(This is an ambiguous case. The ambiguity could have been avoided by determining A instead of B. However, a reasonably well-made drawing shows that B is 121° 50′ and

$$A = 180^\circ - (35^\circ 40' + 121^\circ 51') = 22^\circ 29'.)$$

II. *By the law of tangents*

$$\tan\tfrac{1}{2}(A - B) = \frac{a - b}{a + b}\cot\tfrac{1}{2}C = -\frac{1.533}{4.045}\cot 17^\circ 50'.$$

(The addition and subtraction were done mentally)

$$
\begin{aligned}
\log 1\ 533 &= 0\ 1855 \\
\log \cot 17^\circ\ 50' &= \underline{10\ 4925 - 10} \\
 & 0\ 6780 \\
-\log 4\ 045 &= \underline{-0\ 6070} \\
\log |\tan \tfrac{1}{2}(A - B)| &= 10\ 0710 - 10 \\
\tfrac{1}{2}(A - B) &= -49^\circ\ 40'
\end{aligned}
$$

(Observe the reintroduction of the minus sign)

But

$$
\tfrac{1}{2}(A + B) = 90^\circ - \tfrac{1}{2}C = 90^\circ - 17^\circ\ 50 = 72^\circ\ 10',
$$

$$
A = 72^\circ\ 10' - 49^\circ\ 40' = 22^\circ\ 30'
$$

$$
B = 72^\circ\ 10' + 49^\circ\ 40 = 121^\circ\ 50
$$

(There is no ambiguity here)

$$
c = 2\ 789 \frac{\sin 35^\circ\ 40'}{\sin 121^\circ\ 50'}
$$

$$
\begin{aligned}
\log 2\ 789 &= 0\ 4454 \\
\log \sin 35^\circ\ 40 &= \underline{9\ 7657 - 10} \\
 & 0\ 2111 \\
-\log \sin 121^\circ\ 50 = -\log \sin 58^\circ\ 10' &= \underline{-9\ 9292 - 10} \\
\log c &= 0\ 2819 \\
c &= 1\ 914
\end{aligned}
$$

EXERCISE 9-4

Evaluate the following quantities Indicate by (—) when the trigonometric function is negative and the log of its magnitude is being determined

1. $\log \sin 104^\circ\ 11'$
2. $\log \cos 95^\circ\ 5'$
3. $\log \tan 186^\circ\ 53$
4. $\log \csc 58^\circ\ 21'$
5. $\log \sec 247^\circ\ 34'$
6. $\log \cot 82^\circ\ 42'$.
7. $\log \sin 356^\circ\ 4'$.
8. $\log \cos 35^\circ\ 31'$
9. $\log \cos 323^\circ\ 23'$.
10. $\log \sin 18^\circ\ 6'$
11. $\log \tan 128^\circ\ 5'$
12. $\log \sec 207^\circ\ 19'$
13. $\log \sin 116^\circ\ 12'$
14. $\log \csc 86^\circ\ 56'$
15. $\log \cos 10^\circ\ 30'$
16. $\log \csc 195^\circ\ 5'$
17. $\log \tan 243^\circ\ 28'$.
18. $\log \sin 51^\circ\ 3'$.
19. $\log \sin 287^\circ\ 59'$
20. $\log \cos 305^\circ\ 47'$

s are de
the

Find the values of all angles less than 360° for which the following relationships hold:

21. $\log \sin x = 9.1850 - 10.$
22. $\log \cos \theta = 8.9500 - 10.$
23. $\log \sin \phi = 9.6973 - 10.$
24. $\log \cos \alpha = 9.3765 - 10.$
25. $\log \tan x = 9.1000 - 10.$
26. $\log \sin \beta = 9.7474 - 10.$
27. $\log \cot x = 11.6500 - 10.$
28. $\log \sin y = 9.6601 - 10.$
29. $\log \cot \theta = 9.5342 - 10.$
30. $\log \cos \phi = 9.8348 - 10.$
31. $\log \cos \theta = 9.6230 - 10.$
32. $\log \tan \theta = 9.6794 - 10.$
33. $\log \cos \phi = 9.5321 - 10.$
34. $\log \sec \phi = 11.2341 - 10.$
35. $\log \sin x = 9.4522 - 10.$
36. $\log \csc \alpha = 12.7652 - 10.$
37. $\log \sin y = 9.6544 - 10.$
38. $\log \sin z = 9.4120 - 10.$
39. $\log \sec \beta = 10.0976 - 10.$
40. $\log \cos \theta = 8.1053 - 10.$

Evaluate the remaining parts of the following triangles with the aid of logarithms:

41.	$a = 4.793$	$B = 34° 17'$	$c = 2.385.$
42.	$a = 59.62$	$b = 43.87$	$c = 30.99.$
43.	$a = 2.357$	$b = 1.143$	$A = 137° 11'.$
44.	$A = 120° 37'$	$b = 0.003145$	$C = 21° 17'.$
45.	$A = 107° 12'$	$B = 57° 46'$	$c = 78.32.$
46.	$A = 98° 43'$	$b = 0.1439$	$c = 0.1767.$
47.	$a = 1059$	$b = 1593$	$c = 2156.$
48.	$a = 576.4$	$B = 89° 1'$	$C = 49° 16'.$
49.	$A = 67° 52'$	$b = 23.91$	$a = 47.31.$
50.	$a = 0.01326$	$B = 35° 47'$	$b = 0.01193.$

9.5. The Slide Rule

An extremely useful and versatile instrument for performing computations, not only trigonometric but also algebraic, is the slide rule. This is essentially a simple mechanical device for adding and subtracting logarithms, but its simplicity and convenience make it an indispensable tool for the practicing scientist and engineer.

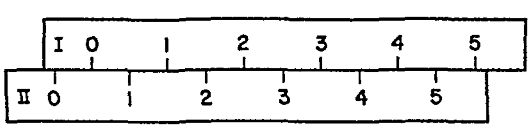


Fig. 9.3

In this section, we explain the basic principles of slide rule operation. A detailed discussion of its use is not given, nor are the special tricks and computing techniques, which vary from one model slide rule to another and are well explained in the instruction manuals provided with each slide rule.

In its simplest form, the slide rule consists of two identical scales which slide against each other (Fig. 9.3). These scales may be used to add numbers. For example, in Fig. 9.4, the scales are arranged to add 1.50 to 2.75, yielding

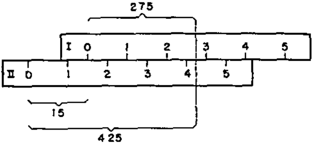


Fig 9 4

4 25 as a result The zero on Scale I is placed against 1 50 on Scale II, opposite 2 75 on Scale I is read 4 25 on Scale II The actual manipulation of such a slide rule is facilitated by the sliding hairline (*cursor*) shown in Fig 9 5

The simple slide rule just described can be used to add logarithms of numbers, but such a use is cumbersome To increase the convenience of use, the

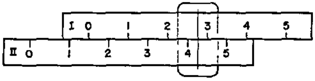


Fig 9 5

numbers themselves, rather than the corresponding logarithms, are marked on the scales (Fig 9 6) The only problem now is how long to make the scale It is pointless to make it so long that any conceivable problem may be performed on it, because it will either be unwieldy or impossible to read

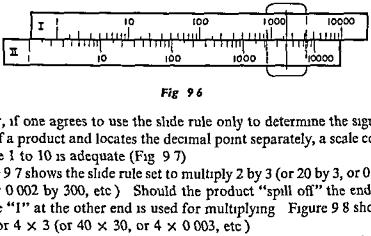


Fig 9 6

However, if one agrees to use the slide rule only to determine the significant figures of a product and locates the decimal point separately, a scale covering the range 1 to 10 is adequate (Fig 9 7)

Figure 9 7 shows the slide rule set to multiply 2 by 3 (or 20 by 3, or 0 002 by 0 003, or 0 002 by 300, etc) Should the product "spill off" the end of the scale, the "1" at the other end is used for multiplying Figure 9 8 shows the setting for 4 × 3 (or 40 × 30, or 4 × 0 003, etc)

An actual slide rule (Fig 9 9) has many more scales than the two discussed above (which are normally designated "C" and "D") Some of these are used

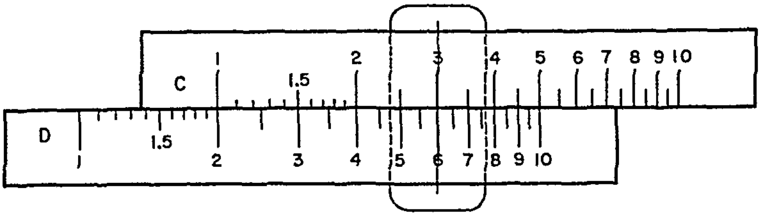


Fig. 9.7

to facilitate computations by taking reciprocals (CI,DI), multiplying by π (CF,DF), taking square roots (A,B) and cube roots (K), and the like. Of particular interest here are the scales of the trigonometric functions. These generally include sines of angles between 5.7° and 90° (S) and tangents between 5.7° and 45° or 84.3° (one or two T-scales). These scales are actually

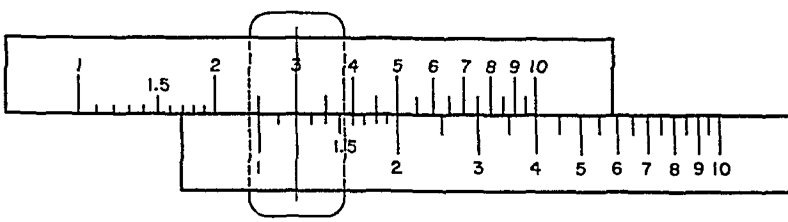


Fig. 9.8

logarithms of trigonometric functions, but the angles are marked directly. Thus, by using auxiliary markings the sine scale may be used for evaluating cosines. For angles less than 5.7°, sin θ and tan θ are both approximated by θ (in radians) and the ST scale is used; for angles near 90°, cos θ is approximated by $(\pi/2) - \theta$.

Trigonometric scales are marked either in degrees and minutes or degrees and decimal fractions of a degree. The choice of markings depends on the use to be made of the rule. For surveying and navigation problems, the calibration in degrees and minutes is probably more useful; in most other branches of engineering and science, the decimal calibration is customarily used. The slide rule shown in Fig. 9.9 has all the scales likely to be useful to one pursuing a career in science or engineering. It is recommended that such a person acquire first this type of slide rule, rather than a simpler one, so he will become familiar with it early, and not have to "unlearn" the short cuts and tricks of its use. More elaborate slide rules exist, but the authors do not believe their limited advantages warrant their extra cost. On the other hand, much simpler slide rules, of the so-called "Trig" type, are adequate for the solution of any problem posed in this book, and for most ordinary purposes also.

The usual 10-inch slide rule can be read to an accuracy of 1/10 percent; errors accumulate, however, and the accuracy of an ordinary calculation is

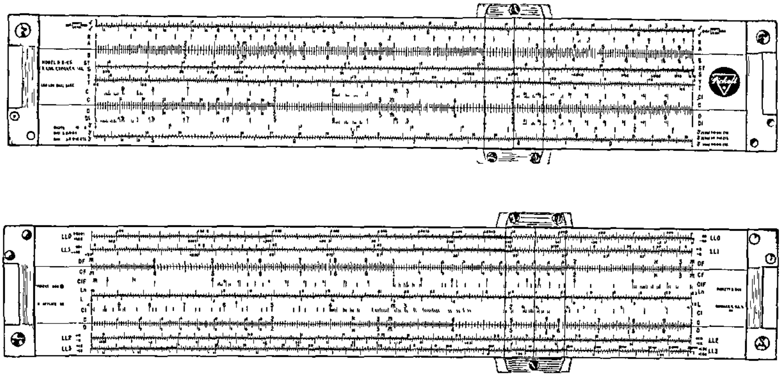


Fig. 99. (Courtesy of Pickett and Eckel, Inc.)

somewhat better than 1 percent. Results are usually given to three significant figures. It is remarkable how many real computations need be carried out only to this precision*—very few measurements are made more precisely. Even when tables of logarithms are used for more precise computations, the slide rule can be used in performing interpolation and is often used to perform rough calculations to uncover any gross mistakes in the precise computation.

Because the slide rule permits rapid multiplication, there are many short-cut methods of solving triangles which can be used. One example is shown.

Example 12. Determine the remaining parts of the triangle shown in Fig. 9.10a. (This is the same as Example 11, two solutions of which have already been demonstrated.)

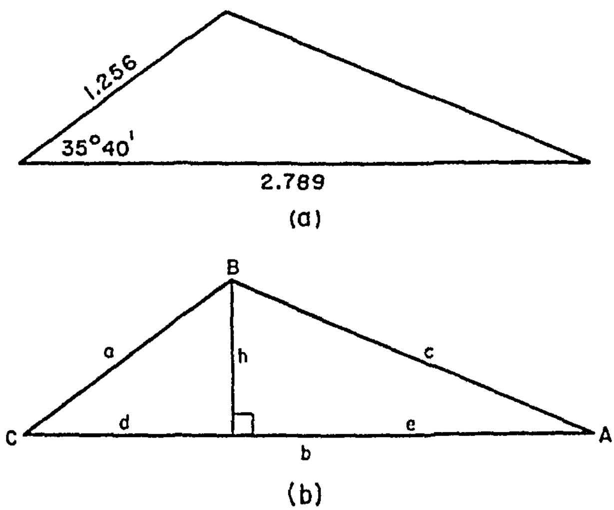


Fig. 9.10

Solution: First erect the line h, perpendicular to the base, and dividing the base into two segments d and e:

$$h = 1.256 \sin 35°\,40' = 0.732$$

$$d = 1.256 \cos 35°\,40' = 1.020.$$

Hence

$$e = 2.789 - 1.020 = 1.769,$$

$$A = \text{Arctan}\,\frac{h}{e} = \text{Arctan}\,\frac{0.732}{1.769} = 22°\,30',$$

$$B = 180° - (35°\,40' + 22°\,30') = 121°\,50',$$

$$c = \frac{h}{\sin A} = \frac{0.732}{\sin 22°\,30'} = 1.914.$$

* One of the authors uses a slide rule almost daily (and regularly carries a pocket slide rule) and refers to a table of logarithms perhaps once a year.

EXERCISE 9 5

As exercises in the use of the slide rule, all the exercises in Chapter 8 may be solved, with lower accuracy than by using logarithms Exercise 9 3 may also be used

9 6. Vectors

In many applications of trigonometry the quantities of concern are *vectors* A vector is a quantity having both magnitude and direction For example, displacement is a vector ("2 inches to the right," "6 miles northeast") as is force ("7 pounds downward," "18 dynes at an angle of 50° with the horizontal ') Velocity is a vector quantity ("65 miles per hour due west"), but speed is not Speed indicates magnitude but not direction

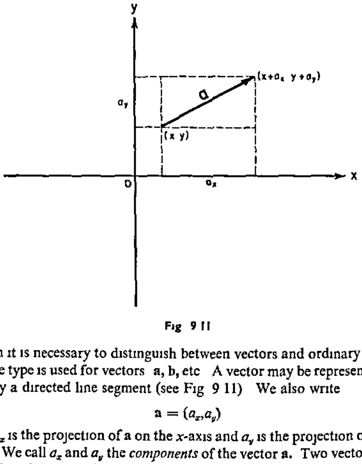


Fig 9 11

When it is necessary to distinguish between vectors and ordinary numbers, boldface type is used for vectors **a**, **b**, etc A vector may be represented in the plane by a directed line segment (see Fig 9 11) We also write

$$\mathbf{a} = (a_x,a_y) \tag{19}$$

where a_x is the projection of **a** on the x-axis and a_y is the projection of **a** on the y-axis We call a_x and a_y the *components* of the vector **a**. Two vectors **a** and **b** are said to be equal if they have the same length and the same direction (Fig 9 12) We write this as

$$\mathbf{a} = \mathbf{b}. \tag{20}$$

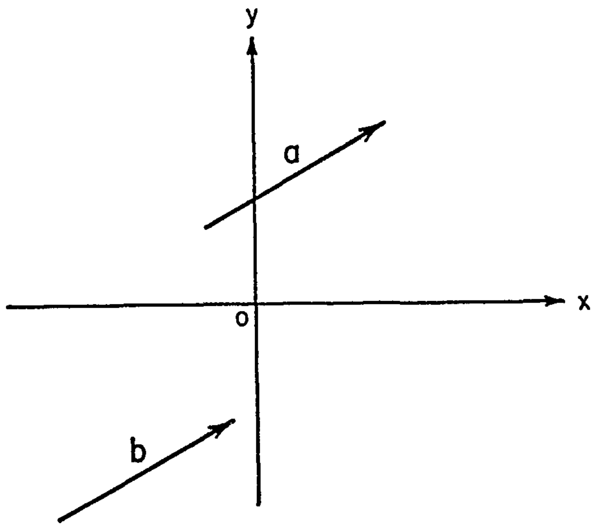


Fig. 9.12

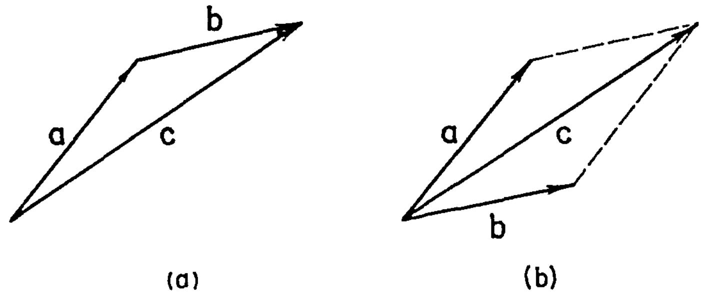


Fig. 9.13

If

$$\mathbf{b} = (b_x, b_y) \tag{21}$$

and **a** is given by (19), then (20) is equivalent to the two statements

$$a_x = b_x \quad \text{and} \quad a_y = b_y. \tag{22}$$

Equation (20) or (22) implies two results, namely, the equivalence of length and the equivalence of direction.

By the sum of two vectors **a** and **b** we mean a third vector **c**, defined by Fig. 9.13a. We write this as

$$\mathbf{c} = \mathbf{a} + \mathbf{b}. \tag{23}$$

If **a** and **b** are given in terms of components as (19) and (21), then

$$\mathbf{a} + \mathbf{b} = (a_x + b_x, a_y + b_y) = \mathbf{c} \tag{24}$$

and if $\mathbf{c} = (c_x, c_y)$, (24) is equivalent to

$$c_x = a_x + b_x \quad \text{and} \quad c_y = a_y + b_y.$$

(The physical rationale behind this definition may be seen by considering a and b as displacements Then c is a displacement equivalent to the two separate displacements a and b.) It is sometimes convenient to indicate the sum of two vectors by using the alternative but equivalent form of Fig 9 13b

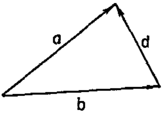


Fig 9 14

If $\mathbf{a} = (a_x, a_y)$ is a vector, then by $-\mathbf{a}$ we mean a vector whose length is the same as that of $\mathbf{a}$ but whose direction is opposite Thus $-\mathbf{a} = (-a_x, -a_y)$ Clearly

$$\mathbf{a} + (-\mathbf{a}) = -\mathbf{a} + \mathbf{a} = (0,0)$$

We call $\mathbf{0} = (0,0)$ the *zero vector* Its length is zero and it may be considered to have any direction By $\mathbf{a} + (-\mathbf{b})$ or simply $\mathbf{a} - \mathbf{b}$ we mean a vector $\mathbf{d}$ such that $\mathbf{a} = \mathbf{b} + \mathbf{d}$ (Fig 9 14)

Vectors in the plane have the following properties

I The sum of two vectors is again a vector That is, if $\mathbf{a}$ and $\mathbf{b}$ are vectors, then so is $\mathbf{a} + \mathbf{b}$.

II Vector addition is associative That is, if $\mathbf{a}$, $\mathbf{b}$, and $\mathbf{c}$ are vectors, then

$$\mathbf{a} + (\mathbf{b} + \mathbf{c}) = (\mathbf{a} + \mathbf{b}) + \mathbf{c}.$$

III There is a zero vector, $\mathbf{0}$. That is,

$$\mathbf{a} + \mathbf{0} = \mathbf{0} + \mathbf{a} = \mathbf{a}$$

for all vectors $\mathbf{a}$

IV There is an inverse vector That is, for every vector $\mathbf{a}$, there exists a vector $-\mathbf{a}$ such that

$$\mathbf{a} + (-\mathbf{a}) = -\mathbf{a} + \mathbf{a} = \mathbf{0}.$$

We have discussed I, III, and IV above We may easily prove II by writing $\mathbf{a} = (a_x\ a_y)$, $\mathbf{b} = (b_x, b_y)$, $\mathbf{c} = (c_x\ c_y)$ and noting that

$$a_x + (b_x + c_x) = (a_x + b_x) + c_x,$$

and

$$a_y + (b_y + c_y) = (a_y + b_y) + c_y$$

(That is, addition of real numbers is associative) Any algebraic system satis fying I–IV is known as a *group* Furthermore, since

$$a_x + b_x = b_x + a_x \quad \text{and} \quad a_y + b_y = b_y + a_y$$

V $$\mathbf{a} + \mathbf{b} = \mathbf{b} + \mathbf{a},$$

that is, vector addition is *commutative* Thus we say that vectors form a *commutative* (or *abelian*) *group*

When discussing vectors, we often refer to real numbers as *scalars* Thus the components a_x, a_y of the vector a are scalars If α is a number (scalar) and

$\mathbf{a} = (a_x, a_y)$ is a vector, we define the multiplication of α by $\mathbf{a}$ as the vector with components αa_x and αa_y:

$$\alpha\mathbf{a} = (\alpha a_x, \alpha a_y).$$

Thus $\alpha\mathbf{a}$ is a vector whose length is $|\alpha|$ times the length of $\mathbf{a}$ and whose direction is the same as that of $\mathbf{a}$ if $\alpha > 0$ and the opposite to that of $\mathbf{a}$ if $\alpha < 0$.

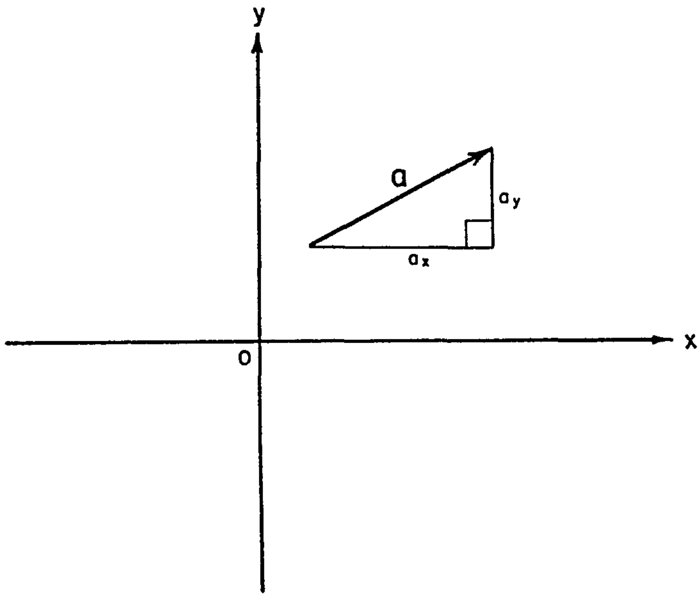


Fig. 9.15

We see therefore that

VI. $$1\mathbf{a} = \mathbf{a},$$

$$(-1)\mathbf{a} = -\mathbf{a},$$

and leave to the reader the proof of

VII. $$(\alpha + \beta)\mathbf{a} = \alpha\mathbf{a} + \beta\mathbf{a},$$

VIII. $$\alpha(\beta\mathbf{a}) = (\alpha\beta)\mathbf{a},$$

IX. $$\alpha(\mathbf{a} + \mathbf{b}) = \alpha\mathbf{a} + \alpha\mathbf{b},$$

for all vectors $\mathbf{a}$, $\mathbf{b}$ and scalars α, β. An algebraic system with properties I–IX is called a *linear vector space over the field of real numbers.*

The length of a vector $\mathbf{a}$ is written $|\mathbf{a}|$ and by the Pythagorean theorem (Fig. 9.15) we see that

$$|\mathbf{a}| = \sqrt{a_x^2 + a_y^2}.$$

Sometimes $|\mathbf{a}|$ is called the *norm* of the vector.

If **a**, **b** are any vectors and α is a scalar then the student may readily verify that

X $\quad |\mathbf{a}| \geqq 0 \qquad |\mathbf{a}| = 0$ if and only if $\mathbf{a} = 0$

XI $\quad |\alpha\mathbf{a}| = |\alpha|\,|\mathbf{a}|$

XII $\quad |\mathbf{a} + \mathbf{b}| < |\mathbf{a}| + |\mathbf{b}|$ (triangle inequality)

Algebraic systems with properties I–XII are called *normed vector spaces*

After this rather abstract introduction to vectors let us consider some simple concrete examples

Example 13 A vector **b** of length 7 3 units makes an angle of 37 4° with the horizontal (Fig 9 16) Determine the components of **b**

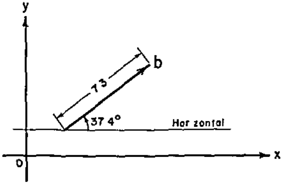


Fig 9 16

Solution

$$b_x = 7\,3 \cos 37°\,24$$
$$= 7\,3 \times 0\,7944$$
$$= 5\,799$$

and

$$b_y = 7\,3 \sin 37°\,24$$
$$= 7\,3 \times 0\,6074$$
$$= 4\,434$$

Example 14 A vector **d** has components

$$d_x = 4\,72$$
$$d_y = -3\,97$$

Determine the length of the vector and the angle it makes with the positive direction of the x axis

Solution Refer to Fig 9 17 Then

$$\tan\theta = \frac{-3\,97}{4\,72}$$

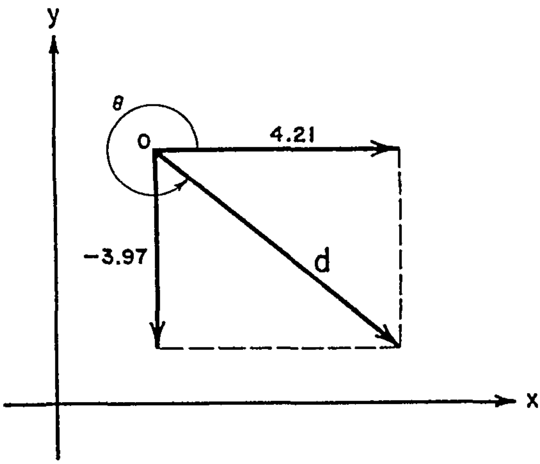


Fig. 9.17

and

$$\log|\tan\theta| = \log 3.97 - \log 4.72 = 0.5988 - 0.6739$$
$$= -0.0751 = 9.9249 - 10,$$
$$\theta = 319^\circ\,56' = -40^\circ\,4'.$$

The length may be found by the Pythagorean theorem as

$$|\mathbf{d}| = \sqrt{(4.72)^2 + (3.97)^2}.$$

It is simpler, though, to note that

$$|\mathbf{d}| = \frac{-3.97}{\sin\theta} = \frac{-3.97}{\sin(-40^\circ\,4')} = \frac{3.97}{\sin 40^\circ\,4'}$$

and

$$\log|\mathbf{d}| = \log 3.97 - \log\sin 40^\circ\,4'$$
$$= 0.5988 - 9.8087 + 10 = 0.7901$$

or

$$|\mathbf{d}| = 6.167.$$

EXERCISE 9-6

Determine the direction and magnitude of vectors having components as follows:

1. $A_x = 73.2$ $\quad A_y = 16.5.$
2. $F_x = 4.76$ $\quad F_y = -9.32.$
3. $E_x = -0.762$ $\quad E_y = 1.596.$
4. $C_x = 432$ $\quad C_y = -511.$
5. $Z_x = 65.6$ $\quad Z_y = -99.9.$
6. $W_x = -5.92$ $\quad W_y = 12.7.$
7. $V_x = 33.8$ $\quad V_y = -17.7.$
8. $P_x = -556$ $\quad P_y = 401.$
9. $Q_x = 5630$ $\quad Q_y = -7710.$
10. $L_x = -0.0132$ $\quad L_y = -0.1490.$

Determine the components of the vectors listed below (The angle is measured counterclockwise from the positive x axis)

11	$A = 976$	$\theta = 38° 10$	16	$\lambda = 768$	$\theta = 135° 40$
12	$E = 143$	$\theta = 86° 30$	17	$S = 4 32$	$\theta = 275° 0'$
13	$J = 20 7$	$\theta = 233° 50$	18	$M = 19 7$	$\theta = 152° 30$
14	$B = 1 55$	$\theta = 72° 20$	19	$Z = 8 87$	$\theta = 199° 50$
15	$W = 1430$	$\theta = 316° 10$	20	$Q = 1 99$	$\theta = 28° 40$

9 7. Applications to Surveying and Navigation

Among the most ancient applications of trigonometry are those to surveying Indeed geometry, the parent science of trigonometry, means "earth measurement Many of the problems that arise in surveying and in the related science of navigation are most conveniently approached by vector techniques Some simple examples suffice to demonstrate this approach

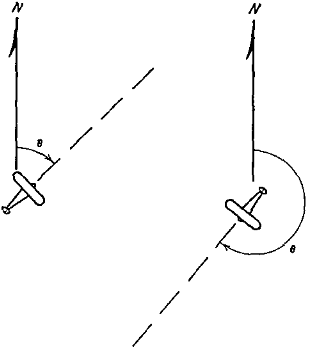


Fig 9 18

More elaborate examples might be more realistic, but their introduction would require a discussion of methods and nomenclature that would take us far from the discussion of simple trigonometry Accordingly, we here define only a minimum number of new terms, most of which are useful in everyday speech

The direction in which a ship or aircraft is pointed is known as its *heading*. This is generally measured as the clockwise angle between north and that direction (Fig. 9.18). Ordinarily, however, the path of the ship or aircraft does not have the same direction as the heading, since currents or winds may cause a sidewise motion. To distinguish the direction of the path from the heading, the former is referred to as the *course* (Fig. 9.19).

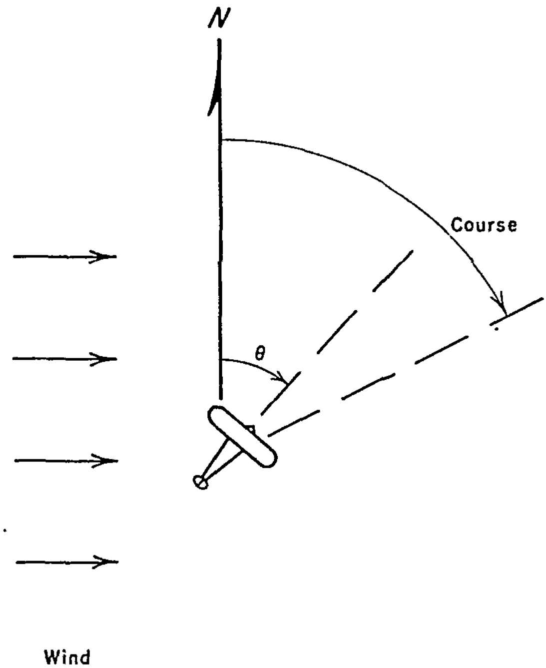


Fig. 9.19

In the case of an aircraft, the actual velocity relative to the ground is the sum of its velocity relative to the air mass and the velocity of the air mass relative to the ground. The latter is merely the wind velocity. Expressed symbolically,

$$\mathbf{V}_g = \mathbf{V}_a + \mathbf{V}_w \tag{25}$$

where $\mathbf{V}_g$ is the velocity with respect to ground ($|\mathbf{V}_g|$, the magnitude of $\mathbf{V}_g$ is the ground speed), $\mathbf{V}_a$ is the velocity of the aircraft with respect to the air mass ($|\mathbf{V}_a|$ is the air speed), and $\mathbf{V}_w$ is the wind velocity. The heading is the direction of $\mathbf{V}_a$; the course is the direction of $\mathbf{V}_g$. In the case of a ship, instead of $\mathbf{V}_w$ we consider $\mathbf{V}_c$, the velocity of the current (though in a real situation the wind may also influence the ship's behavior—but this goes beyond simple trigonometric analysis).

Example 15. An airplane has an air speed of 218 knots* and a heading of 36°.

* One knot = one nautical mile per hour.

The wind has a speed of 30 knots at 98° (Fig 9 20) Find the course and ground speed of the aircraft

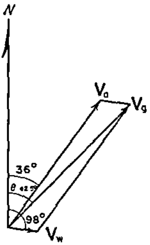


Fig 9 20

Solution We approach this problem by resolving the air and wind velocities into their E–W and N–S (or x- and y-) components

$$V_{a-x} = 218 \sin 36° = 128 \text{ knots},$$

$$V_{w-x} = 30 \sin 98° = 30 \sin 82° = 30 \text{ knots}$$

Accordingly, the total x-component of ground speed is $128 + 30 = 158$ knots Also

$$V_{a-y} = 218 \cos 36° = 176 \text{ knots},$$

$$V_{w-y} = 30 \cos 98° = -30 \cos 82° = -4 \text{ knots},$$

and the y-component of ground speed is $176 - 4 = 172$ knots Hence velocity with respect to ground is

$$\mathbf{V}_g = (158, 172)$$

Thus

$$\theta = \text{Arctan} \frac{158}{172} = 42\ 5°,$$

and

$$|\mathbf{V}_g| = \frac{158}{\sin \theta} = 234 \text{ knots}$$

The computations in this example were carried out to slide-rule accuracy, since neither speeds nor angles are actually known to a high degree of accuracy In fact, navigators often use a special form of slide-rule or graphical calculator to solve these particular problems

The vector method is not the only approach to problems such as the above The law of cosines or the law of tangents could also be used, but the vector approach is more straightforward

Another type of problem in which the vector approach is applicable is *traversing* In surveying, it is often necessary to find the relative location of several points (the corners of a lot, the tops of mountains, and so forth) This is generally done by measuring the distances between the points and the angles formed by the lines joining the points (Fig 9 21) Such a set of measurements is termed a *traverse*, and the process of making these measurements is called *traversing*. In Fig. 9.21, in order to locate points B, C, D, and E with respect to A, it is necessary to measure the distances a, b, c, d and the angles α, β, γ, δ. (The angle α orients the set of points with respect to north.) However, as a check against gross errors and to permit an estimate of over-all accuracy, it is customary to *close* the traverse by measuring the distance e and the angle ε (and perhaps the angle ζ). With the complete set of measurements, one can add the displacements AB, BC, CD, DE, and EA, and ideally would would have zero total displacement. In fact, this is rarely the case, and the resultant (small) displacement, termed the *error of closure*, is a measure of the quality of the survey.

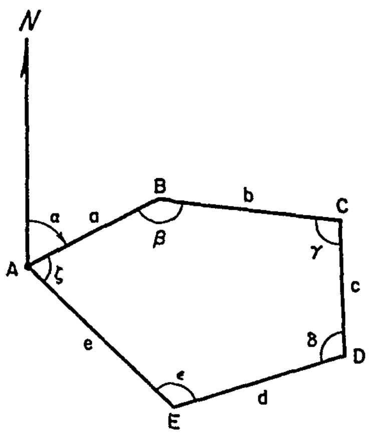


Fig. 9.21

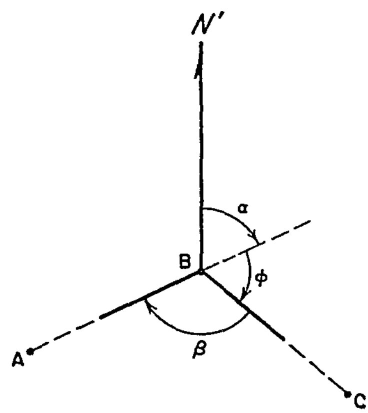


Fig. 9.22

The procedure to be followed is quite straightforward. Only one additional comment need be made: In surveying, the direction of a line is specified by its *bearing*, which is the angle the line makes with respect to north, measured clockwise from north. Thus in Fig. 9.21, the bearing of line AB is the angle α. The bearing of the other lines may be calculated as in Fig. 9.22, which for concreteness illustrates point B. Line BN' is directed North from point B (and is parallel to AN), so angle $N'BC$ is the bearing of line BC. Clearly

$$\measuredangle N'BC = \alpha + \phi \tag{26}$$

and ϕ is $180° - \beta$, so the bearing of line BC is

$$\alpha + 180° - \beta.$$

Similarly, the bearing of line CD is

$$\measuredangle N'BC + 180° - \gamma.$$

The method is best illustrated by means of a simple example.

Example 16. A surveyor records the following measurements while making a traverse (Fig 9 21)

Line	*Distance (feet)*	*Angle*	
AB	587 2	α	63° 47
BC	704 6	β	147° 22
CD	518 6	γ	98° 41′
DE	699 4	δ	105° 17
EA	809 0	ε	119° 12
		ζ	69° 28

Determine the error of closure

Solution The first step is to check the interior angles by using the rule that the sum of the interior angles in an n-sided polygon is equal to $(n - 2)\ 180°$, in this case, 540°

$$\beta + \gamma + \delta + \varepsilon + \zeta = 147°\ 22' + 98°\ 41' + 105°\ 17' + 119°\ 12' + 69°\ 28' = 540°$$

Next determine the bearings of the various lines

AB 63° 47

BC 63° 47′ + 180° − 147° 22 = 96° 25′

CD 96° 25′ + 180° − 98° 41′ = 177° 44′

DE 177° 44′ + 180° − 105° 17 = 252° 27′

EA 252° 27 + 180° − 119° 12′ = 313° 15

The x- and y-components of each displacement may be computed from the relationships

$$x \text{ component} = (\text{distance}) \times \text{sine (bearing)}$$

$$y\text{-component} = (\text{distance}) \times \text{cosine (bearing)}$$

These computations are presented below in tabular form

Line	*log Distance*	*log sin Bearing**	*log cos Bearing**	*log* x	x	*log* y	y
AB	2 7688	9 9528	9 6452	2 7216	526 8	2 4140	259 4
BC	2 8480	9 9972	9 0482	2 8452	700 2	1 8962	−78 7
CD	2 7148	8 5969	9 9996	1 3117	20 5	2 7144	−518 1
DE	2 8447	9 9793	9 4793	2 8240	−666 8	2 3240	−210 9
EA	2 9079	9 8624	9 8358	2 7703	−589 2	2 7437	554 2

The total x-displacement is

526 8	
700 2	
20 5	
	−666 8
	−589 2
1247 5	−1256 0
−1256 0	
−8 5 feet	

* Ten must be subtracted from each logarithm shown

The y-displacement is

259.4	
	−78.7
	−518.1
	−210.9
554.2	
813.6	−807.7
−807.7	
+5.9 feet.	

The error of closure is $\sqrt{8.5^2 + 5.9^2} = 10.3$ feet. This is an intolerable error, when one considers that the distances were measured to 0.1 foot, and hence probably represents a gross mistake. After rechecking the computations (which, to locate an error of this magnitude, can be done on a slide rule), the surveyor returns to the field to recheck his measurements. This time he obtains the following results.

Line	*Distance (feet)*	*Angle*	
AB	587.2	α	63° 47′
BC	704.6	β	147° 22′
CD	518.6	γ	98° 41′
DE	699.4	δ	105° 17′
EA	799.0	ε	119° 12′
		ζ	69° 28′

The error appears to have been an incorrect measurement of the distance *EA*. The angles check as before; we now recompute the error of closure.

Line	*log Distance*	*log sin Bearing*	*log cos Bearing*	*log* x	x	*log* y	y
AB	2.7688	9.9528	9.6452	2.7216	526.8	2.4140	259.4
BC	2.8480	9.9972	9.0482	2.8452	700.2	1.8962	−78.7
CD	2.7148	8.5969	8.9996	1.3117	20.5	2.7144	−518.1
DE	2.8447	9.9793	9.4793	2.8240	−666.8	2.3240	−210.9
EA	2.9025	9.8624	9.8358	2.7649	−582.0	2.7383	547.4

The components of error are found as before:

x-component		*y-component*	
526.8		259.4	
700.2			−78.7
20.5			−518.1
	−666.8		−210.9
	−582.0	547.4	
1247.5	−1248.8	806.8	−807.7
−1248.8		−807.7	
−1.3		−0.9	

The error of closure is $\sqrt{1.3^2 + 0.9^2} = 1.6$ foot. (Even this is not particularly good.)

EXERCISE 9-7

Find the ground speed and course of the aircraft for which (slide-rule accuracy is sufficient):

1. $V_a = 180$ knots $\theta = 22°$ $V_w = 10$ knots $\theta_w = 0°$.
2. $V_a = 200$ knots $\theta = 195°$ $V_w = 20$ knots $\theta_w = 90°$.
3. $V_a = 100$ knots $\theta = 275°$ $V_w = 15$ knots $\theta_w = 120°$.
4. $V_a = 230$ knots $\theta = 98°$ $V_w = 25$ knots $\theta_w = 40°$.
5. $V_a = 300$ knots $\theta = 215°$ $V_w = 10$ knots $\theta_w = 130°$
6. $V_a = 195$ knots $\theta = 236°$ $V_w = 35$ knots $\theta_w = 100°$
7. $V_a = 400$ knots $\theta = 55°$ $V_w = 60$ knots $\theta_w = 250°$
8. $V_a = 220$ knots $\theta = 315°$ $V_w = 55$ knots $\theta_w = 210°$.
9. $V_a = 150$ knots $\theta = 138°$ $V_w = 30$ knots $\theta_w = 160°$.
10. $V_a = 240$ knots $\theta = 347°$ $V_w = 40$ knots $\theta_w = 330°$.

The following sets of measurements were taken on traverses. Assuming the given data are without errors, determine the expected length of the remaining leg, and the values of the interior angles

11. Bearing of $AB = 43° 17'$,	$AB = 453.7$	$\measuredangle ABC = 70° 38'$.
	$BC = 389.2$	$\measuredangle BCD = 120° 11'$.
	$CD = 576.5$	$\measuredangle CDA = ?$
	$DA = ?$	$\measuredangle DAB = ?$
12. Bearing of $AB = 87° 16'$,	$AB = 219.3$	$\measuredangle ABC = 106° 16'$.
	$BC = 304.2$	$\measuredangle BCD = 91° 12'$
	$CD = 476.3$	$\measuredangle CDA = ?$
	$DA = ?$	$\measuredangle DAB = ?$
13. Bearing of $AB = 163° 11'$,	$AB = 563.4$	$\measuredangle ABC = 111° 16'$
	$BC = 497.6$	$\measuredangle BCD = ?$
	$CD = ?$	$\measuredangle CDA = ?$
	$DA = 311.7$	$\measuredangle DAB = 122° 41'$.
14. Bearing of $AB = 257° 43'$;	$AB = 611.3$	$\measuredangle ABC = ?$
	$BC = ?$	$\measuredangle BCD = ?$
	$CD = 754.3$	$\measuredangle CDA = 85° 16'$.
	$DA = 933.5$	$\measuredangle DAB = 65° 43'$.
15. Bearing of $AB = 93° 11'$,	$AB = 518.7$	$\measuredangle ABC = ?$
	$BC = 498.3$	$\measuredangle BCD = ?$
	$CD = ?$	$\measuredangle CDA = 75° 12'$.
	$DA = 550.2$	$\measuredangle DAB = 60° 13'$.
16. Bearing of $AB = 14° 3'$,	$AB = 352.1$	$\measuredangle ABC = 105° 42'$
	$BC = 298.4$	$\measuredangle BCD = 88° 12'$.
	$CD = ?$	$\measuredangle CDA = ?$
	$DA = 383.1$	$\measuredangle DAB = ?$
17. Bearing of $AB = 0° 0'$;	$AB = 198.3$	$\measuredangle ABC = 106° 14'$.
	$BC = 227.6$	$\measuredangle BCD = 91° 12'$.
	$CD = 413.9$	$\measuredangle CDE = 127° 43'$.
	$DE = 511.3$	$\measuredangle DEA = ?$
	$EA = ?$	$\measuredangle EAB = ?$

18. Bearing of $AB = 350° 1'$;	$AB = 406.2$	$\measuredangle ABC = 118° 16'$.
	$BC = 519.7$	$\measuredangle BCD = 80° 42'$.
	$CD = 312.4$	$\measuredangle CDE = 123° 37'$.
	$DE = 298.7$	$\measuredangle DEA = ?$
	$EA = ?$	$\measuredangle EAB = ?$
19. Bearing of $AB = 260° 51'$;	$AB = 563.2$	$\measuredangle ABC = 90° 42'$.
	$BC = 419.8$	$\measuredangle BCD = 116° 55'$.
	$CD = 377.6$	$\measuredangle CDE = 140° 22'$.
	$DE = 717.3$	$\measuredangle DEA = ?$
	$EA = ?$	$\measuredangle EAB = ?$
20. Bearing of $AB = 191° 47'$;	$AB = 419.3$	$\measuredangle ABC = 60° 37'$.
	$BC = 672.4$	$\measuredangle BCD = 196° 22'$.
	$CD = 761.7$	$\measuredangle CDE = 111° 7'$.
	$DE = 598.3$	$\measuredangle DEA = ?$
	$EA = ?$	$\measuredangle EAB = ?$

9.8. Applications to Mechanics

Mechanics is the science that deals with forces and motion. In this introductory discussion, we only consider some applications of that branch of mechanics which deals with bodies at rest, namely, *statics*. Hence we consider only forces.

Force is a vector in that it has both magnitude and direction. Thus when several forces act on a body, their total effect is found by adding the forces by vector addition.

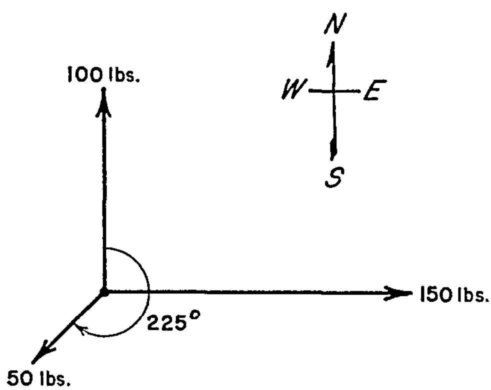


Fig. 9.23

Example 17. Three horses are tethered to a post. One horse pulls due north with a force of 100 lb, one pulls due east with a force of 150 lb, and one pulls southwest with a force of 50 lb (Fig. 9.23). What is the total force exerted on the post by the horses?

Solution Designate these forces F_A, F_B, and F_C respectively The components of each are

$$F_{A-x} = 0$$
$$F_{A-y} = 100$$
$$F_{B-x} = 150$$
$$F_{B-y} = 0,$$
$$F_{C-x} = 50 \sin 225° = -35\ 4,$$
$$F_{C-y} = 50 \cos 225° = -35\ 4$$

The components of the total force are

$$F_x = 150 - 35\ 4 = 114\ 6,$$
$$F_y = 100 - 35\ 4 = 64\ 6$$

The magnitude of this force is

$$\sqrt{114\ 6^2 + 64\ 6^2} = 132 \text{ lb}$$

Its bearing is

$$\text{Arctan}\, \frac{114\ 6}{64\ 6} = 60°\ 35$$

In problems such as the above it is generally adequate to compute with only three significant figures (slide rule accuracy), since it is improbable that the basic data are known any more accurately

A fundamental law of statics is the rule that the *total* force on a body must be zero if the body is at rest In the preceding example, the force applied by the horses is 132 lb at a bearing of 60° 35 , however, an additional force is exerted on the post by the ground—a force equal in magnitude but opposite in direction to the force of the horses Thus the ground exerts a force of 132 lb at a bearing of $60° 35' + 180° = 240° 35'$ on the post Should the horses pull harder on the post, the ground would exert a greater force (until, of course, the post broke or pulled free—in which case it would no longer be at rest)

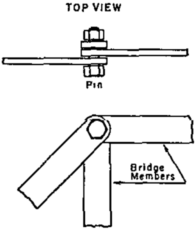


Fig. 9.24

A particularly interesting application of this rule is to the case of *pin-connected* structures In analyzing simple bridges, roof trusses, and the like, it is usual to consider them to be made from straight members connected at their ends by *pins* * Pins (Fig 9 24) can withstand sidewise push and pull

* In fact, pins are rarely used, but the simplification of analysis permitted by this assumption generally justifies its use

but do not resist twisting. Thus a pin-connected bridge of the form shown in Fig. 9.25a is rigid but that in Fig. 9.25b would collapse (Fig. 9.25c). The analysis of pin-connected structures is quite simple in terms of the rule that the total force on any pin is zero and that the directions of the forces are the same as those of the bridge members. While the orientation of the force is the

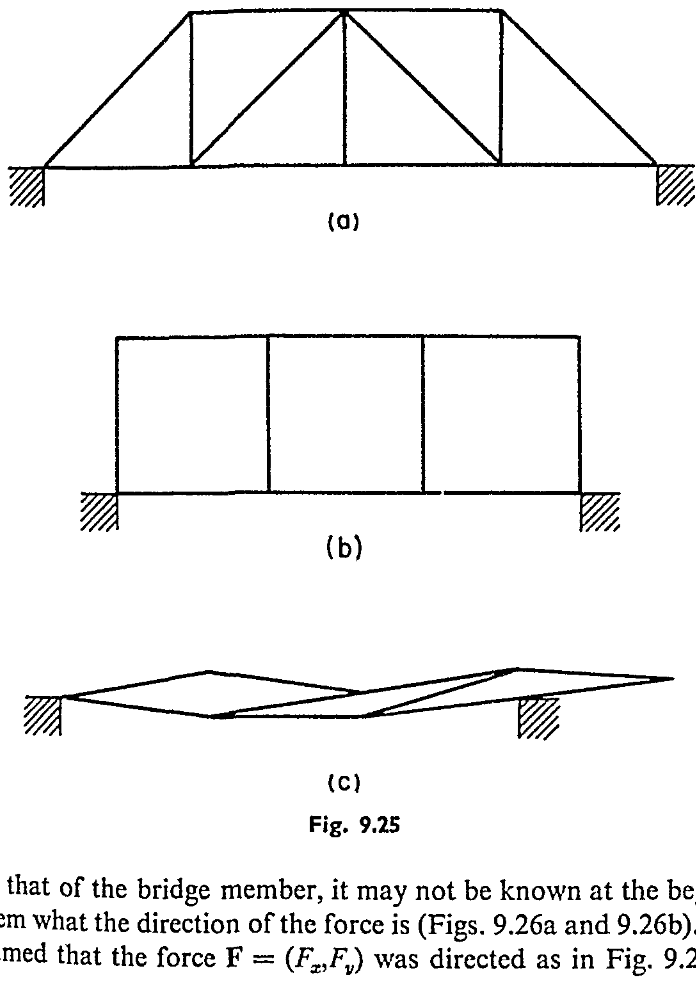


Fig. 9.25

same as that of the bridge member, it may not be known at the beginning of a problem what the direction of the force is (Figs. 9.26a and 9.26b). Suppose we assumed that the force $\mathbf{F} = (F_x, F_y)$ was directed as in Fig. 9.26a. Then

$$F_x = |\mathbf{F}| \cos \theta.$$

$$F_y = |\mathbf{F}| \sin \theta.$$

Suppose further that after the analysis has been completed it turns out that $|\mathbf{F}|$ is negative. This simply means that we should have chosen the direction as $180° + \theta$ (Fig. 9.26b). With this interpretation of $|\mathbf{F}|$, we can now present a nontrivial example, with comments.

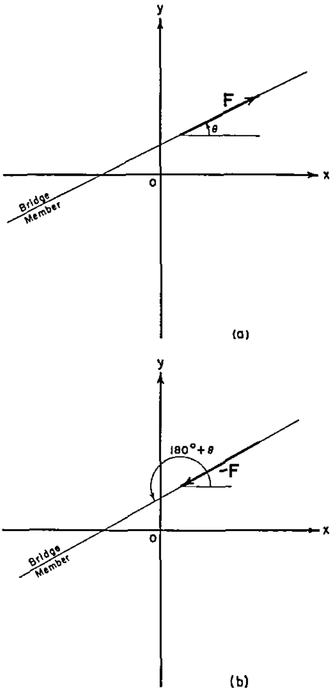


Fig. 9.26

Example 18. Determine the force in each member of the bridge shown in Fig. 9.27. (Slide-rule accuracy is sufficient.) The dimensions of the bridge are given in Fig. 9.27a; the design loading is shown in Fig. 9.27b (in pounds). It is assumed that the load is applied to the lower pins. In addition to the downward forces due to loads, there is an upward force of 20,000 lb on each of pins A and H. These are the forces exercised by the supports on the ends of the bridge.

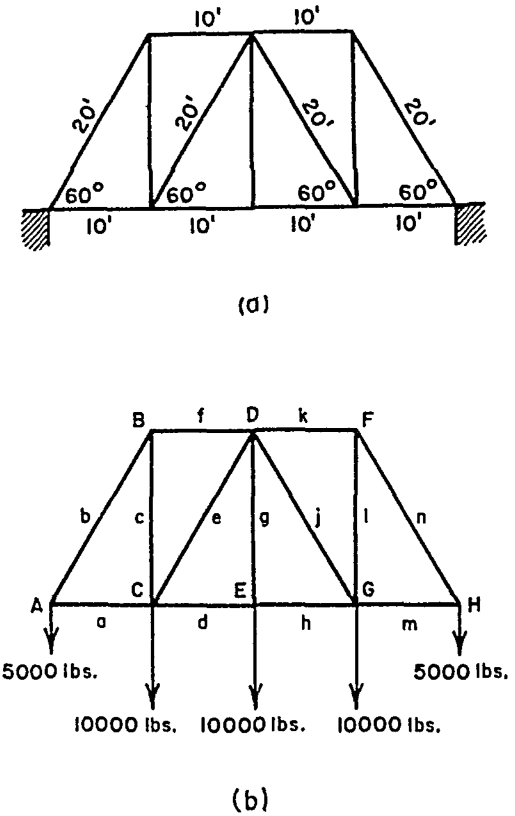


Fig. 9.27

Solution: Consider first pin A (Fig. 9.28). The forces acting on this pin are the (downward) weight of 5000 lb, the (upward) supporting force of 20,000 lb, the force $\mathbf{F}_a$ in member a, and the force $\mathbf{F}_b$ in member b.

Since the total force on the pin is zero, the total horizontal force is zero and the total vertical force is also zero:

$$|\mathbf{F}_H| = |\mathbf{F}_a| + |\mathbf{F}_b| \cos 60° = |\mathbf{F}_a| + 0.500\,|\mathbf{F}_b| = 0 \tag{27a}$$

$$|\mathbf{F}_V| = 20{,}000 - 5000 + |\mathbf{F}_b| \sin 60° = 15{,}000 + 0.866\,|\mathbf{F}_b| = 0. \tag{27b}$$

From (27b), it follows that

$$|\mathbf{F}_b| = -\frac{15{,}000}{0.866} = -17{,}320 \text{ lb},$$

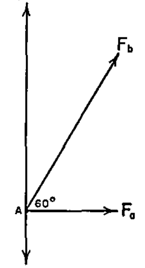


Fig. 9.28

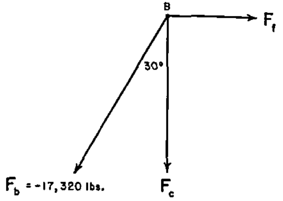


Fig. 9.29

so that (27a) becomes

$$|\mathbf{F}_a| + 0.500\,(-7320) = 0,$$

and

$$|\mathbf{F}_a| = 8660 \text{ lb}$$

Thus $\mathbf{F}_a$ is a force to the right, hence a pull on pin *A*. The force $\mathbf{F}_b$ is inclined *downward* and is a push against pin *A*. We describe this situation by saying member *a* is under *tension*, whereas member *b* undergoes *compression*.

Now consider pin *B* (Fig. 9.29). The force in member *b* is known. Observe carefully that since member *b* pushes on pin *A* it must also push on pin *B*. Thus the sense (direction) of $\mathbf{F}_b$ is reversed when it acts on pin *B*. Now

$$|\mathbf{F}_H| = |\mathbf{F}_f| + |\mathbf{F}_b| \cos 60° = |\mathbf{F}_f| + 17{,}320 \times 0.500 = 0,$$

whence

$$|\mathbf{F}_f| = -17{,}320 \times 0.500 = -8660 \text{ lb}$$

Also

$$|\mathbf{F}_V| = -|\mathbf{F}_c| + |\mathbf{F}_b| \sin 60° = -|\mathbf{F}_c| + 17{,}320 \times 0.866 = 0$$

and

$$|\mathbf{F}_c| = 17{,}320 \times 0.866 = 15{,}000 \text{ lb}$$

Next we analyze the forces at pin C (Fig. 9.30). Observe carefully the directions of $\mathbf{F}_a$ and $\mathbf{F}_c$.

$$|\mathbf{F}_V| = 15{,}000 - 10{,}000 + |\mathbf{F}_e| \sin 60^\circ = 0$$

so

$$|\mathbf{F}_e| = -\frac{5000}{0.866} = -5774 \text{ lb}$$

(thus the member e is in compression). Also

$$|\mathbf{F}_H| = |\mathbf{F}_d| - 8660 + |\mathbf{F}_e| \cos 60^\circ = |\mathbf{F}_d| - 8660 - 5774 \times 0.500 = 0,$$

and

$$|\mathbf{F}_d| = 8660 + 5774 \times 0.500 = 8660 + 2887 = 11{,}550 \text{ lb}.$$

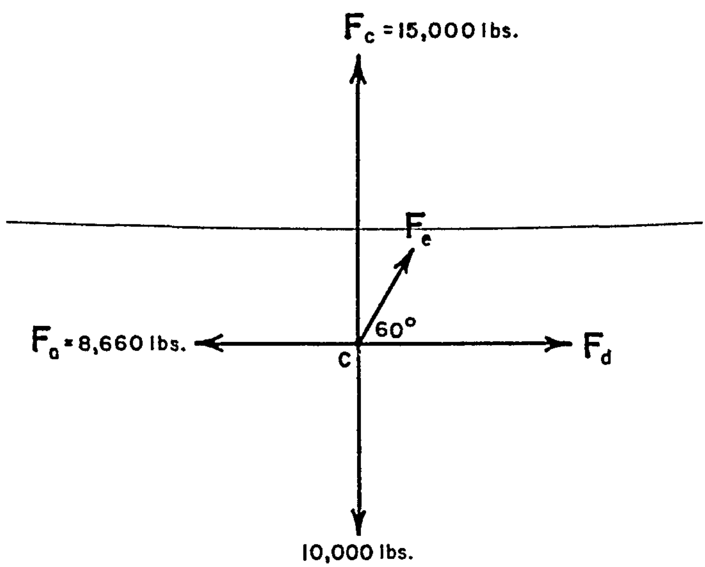


Fig. 9.30

It might appear that the next step is to analyze the forces on pin D. However, there are three unknown forces acting on that pin ($\mathbf{F}_g$, $\mathbf{F}_j$, $\mathbf{F}_k$) and there are only two equations* on $\mathbf{F}_H$ and $\mathbf{F}_V$. Hence we proceed to pin E (Fig. 9.31). It is clear from an inspection of the figure that

$$|\mathbf{F}_h| = 11{,}550 \text{ lb},$$
$$|\mathbf{F}_g| = 10{,}000 \text{ lb}.$$

* Additional equations may be written, however. If we invoke symmetry, we recognize that the force in member k equals that in member f, and that in member e equals that in member j.

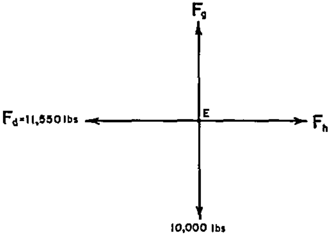


Fig 9 31

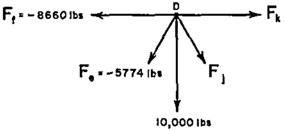


Fig 9 32

Now we may consider pin D (Fig 9 32)

$$|\mathbf{F}_V| = -10{,}000 - |\mathbf{F}_e| \sin 60^\circ - |\mathbf{F}_j| \sin 60^\circ = 0$$

$$-10\,000 + 5774 \times 0\,866 - |\mathbf{F}_j| \times 0\,866 = 0$$

$$-10{,}000 + 5000 - |\mathbf{F}_j| \times 0\,866 = 0$$

so

$$|\mathbf{F}_j| = -\frac{5000}{0\,866} = -5774 \text{ lb}$$

and member j is in compression Also

$$|\mathbf{F}_H| = -|\mathbf{F}_f| + |\mathbf{F}_k| - |\mathbf{F}_e| \cos 60^\circ + |\mathbf{F}_j| \cos 60^\circ = 0$$

$$8660 + |\mathbf{F}_k| + 5000 \times 0\,500 - 5000 \times 0\,500 = 0$$

and

$$|\mathbf{F}_k| = -8660 \text{ lb}$$

so that member k is also in compression

The situations at pins F, G, and H are the same as at pins B, C, and A respectively; so the remaining forces are found to be:

member l: 15,000 lb tension

member m: 8660 lb tension

member n: 17,320 lb compression.

EXERCISE 9-8

Determine the sums of the following sets of forces. Forces are specified in pounds. All angles are measured counterclockwise from the positive x-axis. Slide-rule accuracy is sufficient.

1. $F_1 = 93$ $\theta_1 = 27°$ $F_2 = 53$ $\theta_2 = 107°$.
2. $F_1 = 155$ $\theta_1 = 11°$ $F_2 = 585$ $\theta_2 = 180°$ $F_3 = 123$ $\theta_3 = 138°$.
3. $F_1 = 167$ $\theta_1 = 233°$ $F_2 = 467$ $\theta_2 = 250°$ $F_3 = 442$ $\theta_3 = 321°$.
4. $F_1 = 245$ $\theta_1 = 109°$ $F_2 = 179$ $\theta_2 = 122°$ $F_3 = 141$ $\theta_3 = 148°$.
5. $F_1 = 412$ $\theta_1 = 222°$ $F_2 = 351$ $\theta_2 = 171°$ $F_3 = 62$ $\theta_3 = 305°$.
6. $F_1 = 590$ $\theta_1 = 104°$ $F_2 = 221$ $\theta_2 = 32°$ $F_3 = 558$ $\theta_3 = 285°$ $F_4 = 322$ $\theta_4 = 274°$.
7. $F_1 = 67$ $\theta_1 = 199°$ $F_2 = 743$ $\theta_2 = 241°$ $F_3 = 947$ $\theta_3 = 129°$ $F_4 = 527$ $\theta_4 = 58°$.
8. $F_1 = 634$ $\theta_1 = 19°$ $F_2 = 615$ $\theta_2 = 39°$ $F_3 = 210$ $\theta_3 = 93°$ $F_4 = 857$ $\theta_4 = 193°$.
9. $F_1 = 911$ $\theta_1 = 162°$ $F_2 = 536$ $\theta_2 = 170°$ $F_3 = 630$ $\theta_3 = 81°$. $F_4 = 282$ $\theta_4 = 183°$.
10. $F_1 = 788$ $\theta_1 = 211°$ $F_2 = 368$ $\theta_2 = 265°$ $F_3 = 577$ $\theta_3 = 65°$ $F_4 = 718$ $\theta_4 = 51°$.

In the situations shown in Fig. 9.33, determine the tensions in the various cables and the compressive force in the boom:

11. Weight = 1000 lb $\theta = 33°$ $\phi = 90°$ (Fig. 9.33a).
12. Weight = 1500 lb $\theta = 41°$ $\phi = 60°$ (Fig. 9.33b).
13. Weight = 2000 lb $\theta = 27°$ $\phi = 100°$ (Fig. 9.33c).
14. Weight = 5000 lb $\theta = 80°$ $\phi = 50°$ (Fig. 9.33d).
15. Weight = 4000 lb $\theta = 47°$ $\phi = 55°$ $\alpha = 90°$ $\beta = 60°$ (Fig. 9.33e).

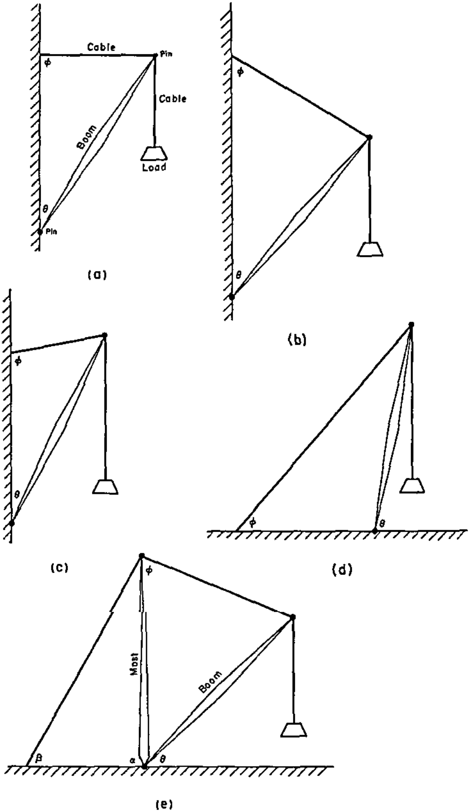


Fig. 9 33

PROBLEMS

1. Solve the exercises of Chapter 8 with the aid of logarithms.

2. In surveying, it is often desired to find the distance between two points, but it is impractical to measure it directly. This occurs, for example, when the points are inaccessible or when the terrain surrounding them is rough. In such a case, the method of *triangulation* is used, wherein the unknown distance is referred to a known distance by measuring suitable angles. An example of this is Fig. 9.34 where the distance d is desired and the distance D and the angles α, β, ϕ, θ are measured. Develop an expression for d in terms of the measured quantities.

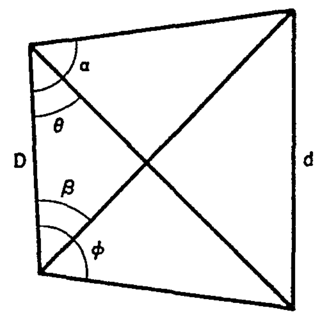


Fig. 9.34

3. Use a slide rule to check whether there is no solution or two solutions to the triangles of Exercise 8-3, nos 41–50.

4. In Exercise 9-7, nos. 11–20, what is the role played by the bearing of the line AB?

5. A traverse is run with the following measurements:

Bearing of $AB = 19° 30'$;	$AB = 1150.8$ ft	$ABC = 78° 42'$.
	$BC = 990.6$ ft	$BCD = 232° 6'$.
	$CD = 1117.4$ ft	$CDE = 65° 23'$.
	$DE = 1477.4$ ft	$DEA = 80° 42'$.
	$EA = 2244.2$ ft	$EAB = 83° 7'$.

Determine the error of closure.

6. Find the force in each member of the bridge shown in Fig. 9.35. The arrows represent the load on the bridge. The force exerted by the abutments on each end of the bridge is not shown. It is 10,000 lb upward at each abutment.

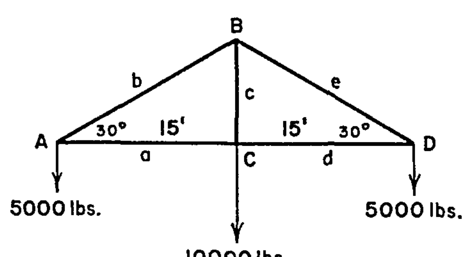


Fig. 9.35

7. Determine the force on each member of the bridge shown in Fig. 9.36. The assumed load is shown by the arrows.

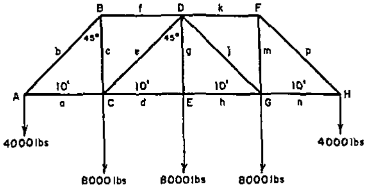


Fig. 9.36

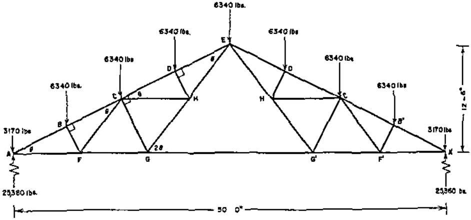


Fig. 9 37

8. Figure 9 37 shows a roof truss The loads and support forces are shown by the arrows Determine the force in each member.

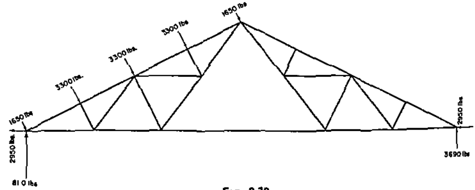


Fig. 9 38

9. Figure 9 38 shows the same truss as Fig 9 37, but now only the wind load is shown and the support forces resulting from wind Determine the force in each member resulting from wind loading alone

II
ADVANCED THEORY

10
SUMS, AREAS, AND TANGENTS

In Part I, we considered certain elementary applications of trigonometric theory. The most conspicuous of these was the solution of triangles. The thoughtful student has probably wondered whether there are any other properties of the trigonometric functions to be investigated. In this chapter, we shall examine three such problems. Later chapters will deal with more sophisticated applications.

10.1. Statement of the Problems

Summation problem. In elementary algebra, the student learned that the sum of the *arithmetic series*

$$a + (a + d) + (a + 2d) + \cdots + (a + Nd) \equiv \sum_{n=0}^{N} (a + nd)$$

was

$$\frac{N + 1}{2}(2a + Nd), \tag{1}$$

and that the sum of the *geometric series*

$$a + ar + ar^2 + \cdots + ar^N \equiv \sum_{n=0}^{N} ar^n$$

was

$$\frac{a(1 - r^{N+1})}{1 - r}, \qquad r \neq 1. \tag{2}$$

We have also looked at these equations from the point of view of induction (see Section 4.3 of Chapter 4).

It is, therefore, not unreasonable to ask whether the series

$$\cos\theta + \cos 2\theta + \cdots + \cos N\theta \equiv \sum_{n=1}^{N} \cos n\theta$$

(as well as similar sums involving trigonometric functions) may be expressed in "closed form" By this we mean a formula analogous to (1) or (2) which does not involve the use of " " or "Σ" This is the first problem and it will be solved in Section 10 2

Area problem In courses in arithmetic, the student learned how to find the area of triangles, simple polygons, and circles, In fact, we can even compute the shaded area of the circle illustrated in Fig 10 1 by elementary techniques Can we do the same for a sine or cosine curve? That is, can we

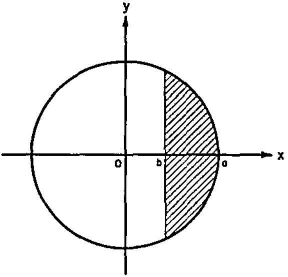


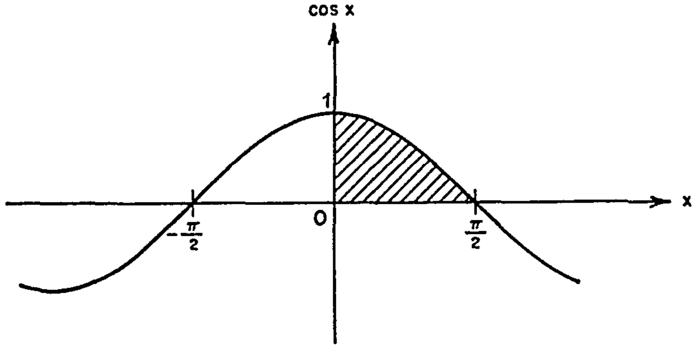
Fig 10 1

compute the shaded areas indicated in Fig 10 2? Clearly the shaded area under the sine curve is twice that under the cosine curve since

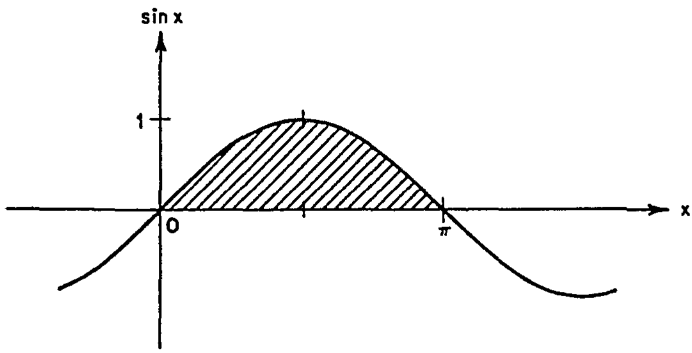
$$\sin\left(x + \frac{\pi}{2}\right) = \cos x$$

This problem and certain generalizations will be solved in Section 10 3

Tangent problem In plane geometry, the student learned how to construct a tangent to a circle (see Fig 10 3) We recall that a tangent to a circle is a straight line that cuts the circle at only one point and is perpendicular to the radius drawn to this point The tangent at the point P can also be specified by giving the angle α which l makes with the positive direction of the x axis Consider now the sine curve of Fig 10 4 We know intuitively that the line l is tangent to the sine curve at the point P_0 However, there is no "radius" that we can draw perpendicular to l in order to specify α_0 Thus we have a twofold problem (1) to give a precise definition to the meaning of "tangent line" and (2) to find the tangent at any point of a sine (cosine) curve We shall answer these questions in Section 10 4



(a)



(b)

Fig. 10.2

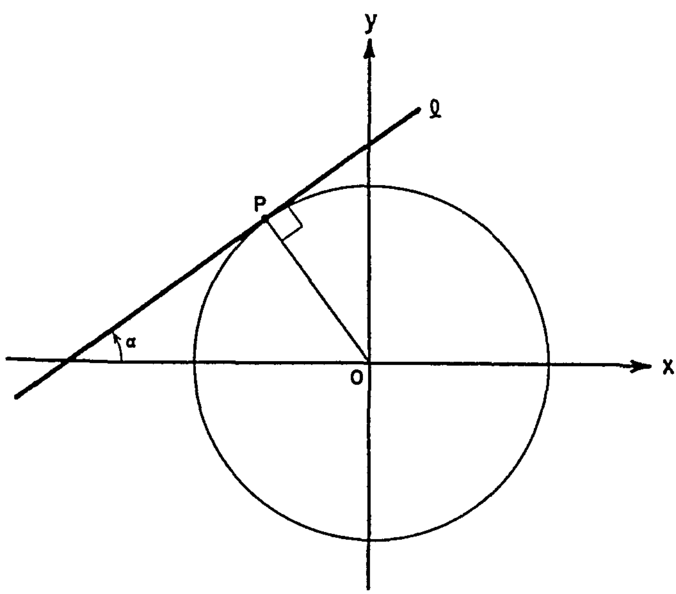


Fig. 10.3

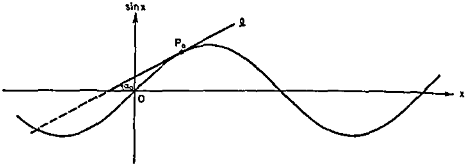


Fig 10 4

10 2 Solution of the Summation Problem

The summation problem may be solved in a variety of ways In Chapter 11 we shall solve it by the use of complex numbers Here we shall present an interesting solution using an alternative method The student will immediately be aware that the simple techniques we are going to introduce have applications beyond trigonometry

Suppose we have a function $f(x)$ defined for all x Then we define an *operator* Δ by the equation

$$\Delta f(x) = f(x + 1) - f(x)$$

That is if we apply the operator to $f(x)$ we obtain $f(x + 1) - f(x)$ For example

$$\Delta x = (x + 1) - (x) = 1$$

while

$$\begin{aligned} \Delta(2x^3 - 3x^2 + x) &= [2(x + 1)^3 - 3(x + 1)^2 + (x + 1)] - [2x^3 - 3x^2 + x] \\ &= [2x^3 + 3x^2 + x] - [2x^3 - 3x^2 + x] \qquad (3) \\ &= 6x^2 \end{aligned}$$

Also

$$\Delta \sin x = \sin (x + 1) - \sin x = 2 \sin \tfrac{1}{2} \cos (x + \tfrac{1}{2})$$

and

$$\Delta \cos x = \cos (x + 1) - \cos x = -2 \sin \tfrac{1}{2} \sin (x + \tfrac{1}{2})$$

More generally

$$\begin{aligned} \Delta \sin (ax + b) &= \sin (a(x + 1) + b) - \sin (ax + b) \\ &= 2 \sin \frac{a}{2} \cos \left(ax + b + \frac{a}{2}\right) \end{aligned}$$

If in particular we let $b = -a/2$ then the above formula becomes

$$\Delta \sin \left(ax - \frac{a}{2}\right) = 2 \sin \frac{a}{2} \cos ax \qquad (4)$$

Similarly,

$$\begin{aligned}\Delta \cos(ax+b) &= \cos(a(x+1)+b) - \cos(ax+b)\\ &= -2\sin\frac{a}{2}\sin\left(ax+b+\frac{a}{2}\right)\end{aligned}$$

and, if we let $b = -a/2$,

$$\Delta \cos\left(ax-\frac{a}{2}\right) = -2\sin\frac{a}{2}\sin ax. \tag{5}$$

Of course, we could find $\Delta f(x)$ for quite a variety of functions and in certain theories this is a useful thing to do. However, our immediate problem is to find the sum of trigonometric functions, namely,

$$\cos\theta + \cos 2\theta + \cdots + \cos N\theta = \sum_{x=1}^{N} \cos x\theta. \tag{6}$$

It is no more difficult to formulate the more general problem of finding

$$F(\theta) + F(2\theta) + \cdots + F(N\theta) = \sum_{x=1}^{N} F(x\theta). \tag{7}$$

In fact, the notation is even simpler. The method is to find a function $f(\theta x)$ such that $\Delta f(\theta x) = F(\theta x)$, that is, such that

$$f(\theta(x+1)) - f(\theta x) = F(\theta x). \tag{8}$$

If this can be done, then

$$\begin{aligned}F(\theta) &= f(2\theta) - f(\theta),\\ F(2\theta) &= f(3\theta) - f(2\theta),\\ F(3\theta) &= f(4\theta) - f(3\theta),\\ &\cdots\cdots\cdots\cdots\cdots\\ F((N-1)\theta) &= f(N\theta) - f((N-1)\theta),\\ F(N\theta) &= f((N+1)\theta) - f(N\theta).\end{aligned}$$

If we add up the left-hand column, we get the left-hand member of (7). Adding up the right-hand column, we see that all terms cancel except $f(\theta)$ and $f((N+1)\theta)$. Thus

$$\sum_{x=1}^{N} F(x\theta) = f((N+1)\theta) - f(\theta).$$

Now let us return to (6). We see from (4) that

$$\frac{1}{2\sin\frac{a}{2}}\Delta \sin\left(ax-\frac{a}{2}\right) = \cos ax, \qquad a \neq 0, \pm 2\pi, \pm 4\pi, \cdots.$$

Thus in the notation of (8),

$$F(x\theta) = \cos x\theta,$$

and

$$f(x\theta) = \frac{\sin\left(\theta x - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

Hence

$$\sum_{x=1}^{N} \cos x\theta = \frac{\sin\left(\theta(N+1) - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}} - \frac{\sin\left(\theta - \frac{\theta}{2}\right)}{2\sin\frac{\theta}{2}}$$

$$= \frac{1}{2}\left[\frac{\sin\theta\left(N+\frac{1}{2}\right)}{\sin\frac{\theta}{2}} - 1\right],$$

or we may write, alternatively,

$$\sum_{x=1}^{N} \cos x\theta = \frac{\cos\frac{\theta(N+1)}{2}\sin\frac{\theta N}{2}}{\sin\frac{\theta}{2}}, \tag{9}$$

if θ is not a multiple of 2π [If θ is a multiple of 2π, then (6) is an arithmetic series] Similarly, we find from (5) that

$$\sum_{x=1}^{N} \sin x\theta = \frac{\cos\left(\theta(N+1) - \frac{\theta}{2}\right)}{-2\sin\frac{\theta}{2}} + \frac{\cos\frac{\theta}{2}}{2\sin\frac{\theta}{2}}$$

$$= \frac{\sin\frac{\theta(N+1)}{2}\sin\frac{\theta N}{2}}{\sin\frac{\theta}{2}} \tag{10}$$

The student has probably already noted other applications For example, we can deduce from (3) that

$$\sum_{x=1}^{N} x^2 = \tfrac{1}{6}\{[2(N+1)^3 - 3(N+1)^2 + (N+1)] - [2 - 3 + 1]\}$$

$$= \tfrac{1}{6}\{[2N^3 + 3N^2 + N] - [0]\}$$

$$= \frac{N(2N^2 + 3N + 1)}{6} = \frac{N(N+1)(2N+1)}{6} \tag{11}$$

Other applications to trigonometric and nontrigonometric functions will be found in the problems at the end of this chapter.

A systematic exploitation of the techniques used in this section is part of that branch of mathematics known as the *calculus of finite differences.*

10.3. Solution of the Area Problem

We now turn to the second problem mentioned at the beginning of the chapter, namely, that of finding the area under one arch of a sine curve.

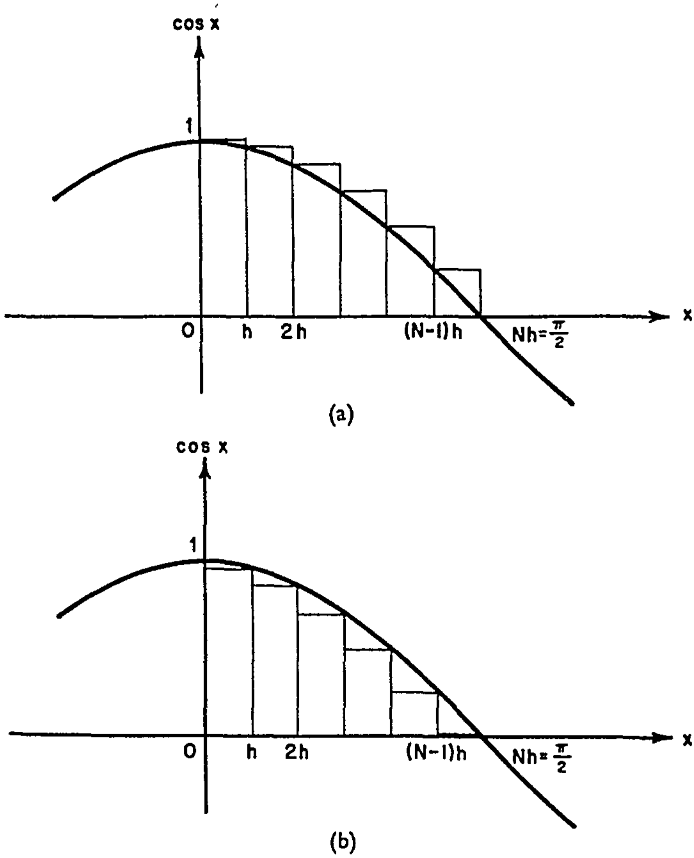


Fig. 10.5

Consider then Fig. 10.2a. Suppose we divide the interval $\pi/2$ into N equal pieces by the points of subdivision

$$0, h, 2h, \cdots, Nh,$$

where $h = \pi/2N$. If we then draw the rectangles illustrated in Fig. 10.5, we see that the area under the cosine curve between 0 and $\pi/2$ is less than the sum of the areas of the rectangles of Fig 10 5a and exceeds the sum of the areas of the rectangles of Fig 10 5b If we call S_N the sum of the areas of the rectangles in Fig 10 5a and call s_N the sum of the areas of the rectangles in Fig 10 5b, then for any N

$$S_N \geq A \geqq s_N, \tag{12}$$

where A is the shaded area of Fig 10 2a

Now

$$S_N = h\cos 0 + h\cos h + h\cos 2h + \cdots + h\cos (N-1)h$$

$$= h\sum_{x=0}^{N-1} \cos xh,$$

and

$$s_N = h\cos h + h\cos 2h + \cdots + h\cos Nh$$

$$= h\sum_{x=1}^{N} \cos xh$$

From the previous section, we find that

$$S_N = h\frac{\cos\dfrac{h(N-1)}{2}\sin\dfrac{hN}{2}}{\sin\dfrac{h}{2}}, \tag{13}$$

and

$$s_N = h\frac{\cos\dfrac{h(N+1)}{2}\sin\dfrac{hN}{2}}{\sin\dfrac{h}{2}} \tag{14}$$

Now, no matter how large N is, (12) is still valid Thus

$$\lim_{N\to\infty} S_N \geq A \geqq \lim_{N\to\infty} s_N$$

We shall show that

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} s_N = 1 \tag{15}$$

Thus the area A under half an arch of the cosine (or sine) curve is 1, and, of course, the shaded area of Fig 10 2b is therefore 2

In order to establish (15), we write

$$\cos\frac{h(N-1)}{2} = \cos\frac{hN}{2}\cos\frac{h}{2} + \sin\frac{hN}{2}\sin\frac{h}{2}$$

If we recall that $h = \pi/2N$, (13) becomes

$$S_N = \frac{h}{\sin\frac{h}{2}} \sin\frac{\pi}{4}\left(\cos\frac{\pi}{4}\cos\frac{h}{2} + \sin\frac{h}{2}\sin\frac{\pi}{4}\right).$$

But $\cos \pi/4 = \sin \pi/4 = \sqrt{2}/2$. Thus

$$S_N = \frac{\frac{h}{2}}{\sin\frac{h}{2}}\left(\cos\frac{h}{2} + \sin\frac{h}{2}\right).$$

Now we saw in Section 3.2 of Chapter 3 (Part I) that $\lim_{\xi\to 0} \frac{\xi}{\sin\xi} = 1$, and, more trivially,

$$\lim_{\xi\to 0} \cos\xi = 1, \qquad \lim_{\xi\to 0} \sin\xi = 0.$$

Thus

$$\lim_{N\to\infty} S_N = \lim_{h\to 0} S_N = 1\cdot(1+0) = 1.$$

Similarly, if we write

$$\cos\frac{h(N+1)}{2} = \cos\frac{hN}{2}\cos\frac{h}{2} - \sin\frac{hN}{2}\sin\frac{h}{2}$$

and recall that $h = \pi/2N$, (14) becomes

$$s_N = \frac{h}{\sin\frac{h}{2}}\sin\frac{\pi}{4}\left(\cos\frac{\pi}{4}\cos\frac{h}{2} - \sin\frac{\pi}{4}\sin\frac{h}{2}\right)$$

$$= \frac{\frac{h}{2}}{\sin\frac{h}{2}}\left(\cos\frac{h}{2} - \sin\frac{h}{2}\right),$$

and

$$\lim_{N\to\infty} s_N = \lim_{h\to 0} s_N = 1\cdot(1-0) = 1.$$

The student has probably thought of other applications. For example, we can find the shaded area under the curve $f(x) = x^2$, see Fig. 10.6. (This curve is known as a *parabola*.) Using the methods described above,

$$S_N = h\{h^2 + (2h)^2 + \cdots + (Nh)^2\}$$
$$s_N = h\{0^2 + h^2 + \cdots + [(N-1)h]^2\},$$

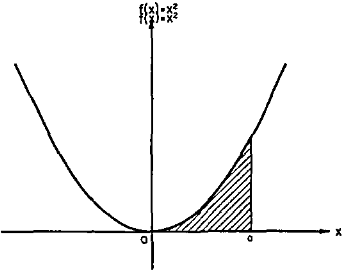


Fig. 10.6

where $h = a/N$ Thus if A is the indicated shaded area,

$$S_N \geqq A \geqq s_N$$

for any N. But from (11),

$$\sum_{x=1}^{N} x^2 = \frac{N(N+1)(2N+1)}{6}$$

Hence

$$S_N = h^3 \sum_{x=1}^{N} x^2 = \frac{h^3 N(N+1)(2N+1)}{6}$$

and

$$s_N = h^3 \sum_{x=1}^{N-1} x^2 = \frac{h^3(N-1)(N)(2N-1)}{6}.$$

Recalling that $h = a/N$, we obtain

$$\lim_{N\to\infty} S_N = \lim_{N\to\infty} \frac{a^3}{6}\left(1 + \frac{1}{N}\right)\left(2 + \frac{1}{N}\right) = \frac{a^3}{3}$$

and

$$\lim_{N\to\infty} s_N = \lim_{N\to\infty} \frac{a^3}{6}\left(1 - \frac{1}{N}\right)\left(2 - \frac{1}{N}\right) = \frac{a^3}{3}.$$

Thus the area under the parabola is

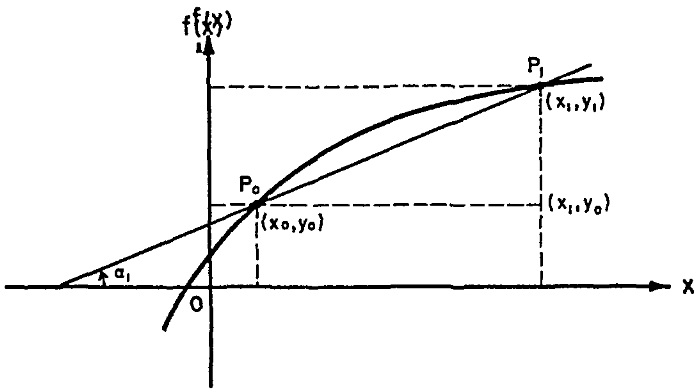
$$A = \frac{a^3}{3}$$

Other applications to trigonometric and nontrigonometric functions will be found in the problems at the end of this chapter.

A systematic exploitation of the techniques used in this section is part of that branch of mathematics known as the *integral calculus.*

10.4. Solution of the Tangent Problem

Finally we consider the problem of finding the tangent to a sine curve. We recall from the introduction that the first difficulty to overcome is that of



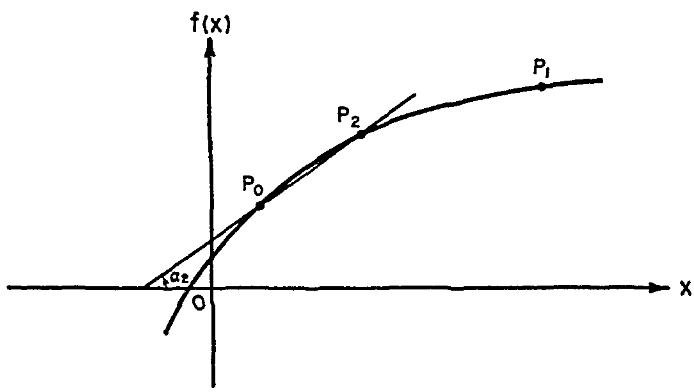


Fig. 10.7

giving a precise meaning to the word "tangent." Suppose then we wish to find the tangent to the curve $f(x)$ at the point P_0 which has coordinates (x_0,y_0), see Fig. 10.7a. At our present level of generality, we need not assume that $f(x)$ is a sine curve. Suppose we draw a line $\overline{P_0P_1}$ between P_0 and any other point P_1 on the curve. (We call such a line a *secant.*) The line $\overline{P_0P_1}$

forms an angle α_1 with the positive direction of the x axis The tangent of this angle is called the *slope* m_1 of the line,

$$m_1 = \tan \alpha_1$$

Clearly we may also write

$$m_1 = \frac{y_1 - y_0}{x_1 - x_0} = \frac{f(x_1) - f(x_0)}{x_1 - x_0}$$

If we choose any other point P_2 closer to P_0 than P_1 (see Fig 10 7b), then we may again draw a secant $\overline{P_0P_2}$ and the slope of this line is

$$m_2 = \tan \alpha_2 = \frac{y_2 - y_0}{x_2 - x_0} = \frac{f(x_2) - f(x_0)}{x_2 - x_0}$$

Similarly, we may choose a point P_3 closer to P_0 than P_2 and find *its* slope

Now if a point P is very close to P_0 (on either side), it appears, geometrically, that the line through P and P_0 is very close to the tangent at P_0 In fact, we *define* the slope of the tangent to the curve $f(x)$ at the point P_0 as

$$m_0 = \tan \alpha_0 = \lim_{x \to x_0} \frac{f(x) - f(x_0)}{x - x_0},$$

provided this limit exists *

Armed with a precise definition of tangent, let us compute the tangent to the sine curve (see Fig 10 4) at the point $P_0 = (x_0, y_0)$ We have by definition

$$\begin{aligned} m_0 = \tan \alpha_0 &= \lim_{x \to x_0} \frac{\sin x - \sin x_0}{x - x_0} \\ &= \lim_{x \to x_0} \frac{2 \cos \frac{1}{2}(x + x_0) \sin \frac{1}{2}(x - x_0)}{x - x_0} \\ &= \lim_{x \to x_0} \left[\frac{\sin \frac{1}{2}(x - x_0)}{\frac{1}{2}(x - x_0)}\right] \cos \frac{1}{2}(x + x_0) \end{aligned}$$

But we have seen before that $\lim_{\xi \to 0} \frac{\sin \xi}{\xi} = 1$, and clearly $\lim_{x \to x_0} \cos \frac{1}{2}(x + x_0) = \cos \frac{1}{2}(x_0 + x_0) = \cos x_0$ Thus

$$m_0 = \cos x_0 \tag{16}$$

In other words, the slope at any point (x_0, y_0) of a sine curve is equal to the cosine of x_0 For example, the slope of the tangent to $\sin x$ at $x = \pi/6$ is $\cos \pi/6 = \sqrt{3}/2$ Indeed, a remarkable property of the trigonometric

---

* For example, if $f(x) = |x|$ then $\lim_{x \to 0} \frac{|x| - |0|}{x - 0}$ does not exist (In fact it equals +1 as x approaches zero through positive values and −1 as x approaches zero through negative values)

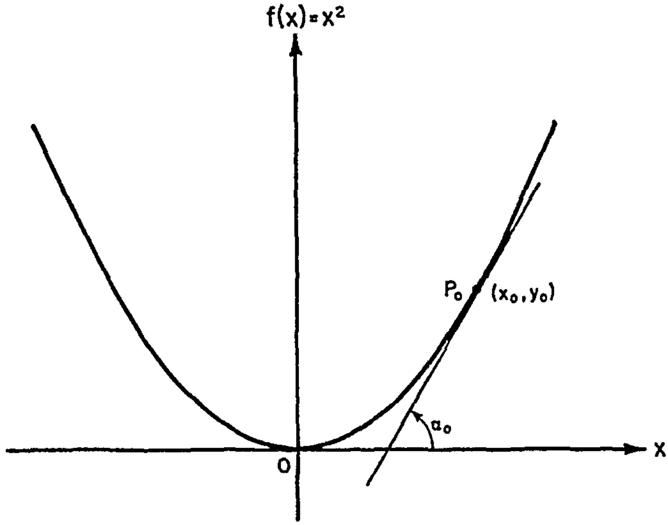


Fig. 10.8

functions! Similarly, the slope of the tangent to a cosine curve at the point $P_0 = (x_0,y_0)$ is, by definition,

$$\begin{aligned} m_0 = \tan \alpha_0 &= \lim_{x\to x_0} \frac{\cos x - \cos x_0}{x - x_0} \\ &= \lim_{x\to x_0} \frac{-2 \sin \frac{1}{2}(x + x_0) \sin \frac{1}{2}(x - x_0)}{x - x_0} \\ &= \lim_{x\to x_0} \left[\frac{\sin \frac{1}{2}(x - x_0)}{\frac{1}{2}(x - x_0)}\right][-\sin \tfrac{1}{2}(x + x_0)], \end{aligned}$$

and as above we find that this limit is $-\sin x_0$. Thus

$$m_0 = -\sin x_0. \tag{17}$$

The student has probably already noted other applications. For example, we can find the tangent to the parabola $f(x) = x^2$ at the point P_0 with coordinates (x_0,y_0) (see Fig. 10.8). For, by the definition of tangent, the slope at P_0 is

$$\begin{aligned} m_0 &= \lim_{x\to x_0} \frac{x^2 - x_0^2}{x - x_0} = \lim_{x\to x_0} \frac{(x - x_0)(x + x_0)}{x - x_0} \\ &= \lim_{x\to x_0} (x + x_0) = 2x_0. \end{aligned}$$

Other applications to trigonometric and nontrigonometric functions will be found in the problems at the end of this chapter.

A systematic exploitation of the techniques used in this section is part of that branch of mathematics known as the *differential calculus*.

PROBLEMS

1 Find $\Delta f(x)$ for the following functions $f(x)$

(a) $\tan(ax + b)$
(b) $\cot(ax + b)$
(c) $5x^4$
(d) $x(x - 1)(x - 2)(x - 3)$
(e) 2^x

2 If n is a positive integer and we define $x^{(n)}$ as

$$x^{(n)} = x(x - 1)(x - 2) \quad (x - n + 1),$$

prove that

$$\Delta x^{(n)} = nx^{(n-1)}$$

(We define $x^{(0)}$ as 1)

3 For the following functions $F(x)$ find a function $f(x)$ such that $F(x) = \Delta f(x)$

(a) $\sin 3x$
(b) $\cos^2 5x$
(c) $6x^3$
(d) $\dfrac{1}{x(x + 1)}$
(e) 3^x

4 Find

$$\sum_{x=1}^{N} x \sin x$$

(HINT See if you can find an A and a B such that

$$\Delta[A \cos(x - \tfrac{1}{2}) + B \sin x] = x \sin x\ )$$

5. Using the methods of finite differences, derive the formula for the sum of an arithmetic series

6. Find

$$\sum_{x=1}^{N} x^3$$

See if you can generalize this to $\sum_{x=1}^{N} x^p$ where p is a positive integer

7. Find the shaded area under the arch of the sine curve illustrated in Fig 10 9 where $0 < \alpha < \beta < \pi$

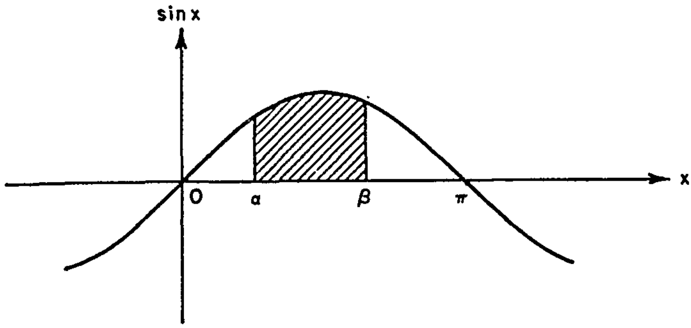


Fig. 10.9

8. Find the algebraic area between 0 and π under the curves

   (a) $\sin mx$
   (b) $\cos mx$

   where m is a positive integer. (By algebraic area we mean that the area below the x-axis is to be considered as negative. Thus for example, the area under $y = \sin x$ between 0 and 2π is zero.)

9. Find the algebraic area between 0 and 2π under

   (a) $\sin nx \sin mx$
   (b) $\cos nx \cos mx$
   (c) $\cos nx \cos mx$

   where n and m are integers. (HINT: Write the product of trigonometric functions as a sum. For example, $\sin nx \sin mx = \frac{1}{2}[\cos (n - m)x - \cos (n + m)x]$.)

10. Find the area under the curve $y = x^3$ between $x = 1$ and $x = 5$.

11. Find the slope of the curve

$$y = \tan x$$

   at any point (x_0,y_0) where $\tan x_0$ is finite.

12. Repeat Problem 11 for the function $y = \cot x$.

13. Find the slope of the curve $y = x^3$ at any point (x_0,y_0). See if you can generalize this to $y = x^p$ where p is a positive integer.

14. Let $y = \sqrt{x}$ for $x \geqq 0$. Find the slope of this curve at any point (x_0,y_0) where $x_0 > 0$.

15. Find the equation of the line tangent to the curve $y = 4x^2$ at the point (1,4).

11 THE ALGEBRA OF COMPLEX NUMBERS

In elementary algebra, we learned that the *quadratic equation*

$$ax^2 + bx + c = 0$$

had two solutions which were given by

$$x = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

For example, the equation

$$x^2 - 3x + \tfrac{7}{2} = 0 \tag{1}$$

has

$$x = \frac{3 \pm \sqrt{9 - 14}}{2}$$

or

$$x_1 = \tfrac{3}{2} + \tfrac{1}{2}\sqrt{-5}, \qquad x_2 = \tfrac{3}{2} - \tfrac{1}{2}\sqrt{-5}$$

as its two solutions The number $\sqrt{-5}$ is called an *imaginary number* since there is no real number whose square is -5 The term "imaginary number" is perhaps unfortunate, but the name is now well established Actually, imaginary numbers have very real and concrete uses (for example, in electric circuit theory) In the present chapter, we shall demonstrate some remarkable results that can be achieved by the use of such numbers

11.1. Complex Numbers

It is customary to denote $\sqrt{-1}$ by the symbol i, although in electrical engineering j is generally used since the letter i is reserved for current Thus the solutions to (1) may be written

$$x_1 = \frac{3}{2} + \frac{\sqrt{5}}{2}\, i, \qquad x_2 = \frac{3}{2} - \frac{\sqrt{5}}{2}\, i$$

A number of the form $a + ib$ where a and b are real is called a *complex number*. If $b = 0$ we have a *real number* and if $a = 0$ we have a *purely imaginary number*.

One manipulates complex numbers just as one would real numbers. For example, the *sum* of two complex numbers $a + ib$ and $c + id$ is defined as

$$(a + ib) + (c + id) = (a + c) + i(b + d).$$

Also, the product is defined as

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd.$$

But $i^2 = (\sqrt{-1})^2 = -1$. Thus we may write

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Hence we see that the sums and products of complex numbers are again complex numbers. For example,

$$(1 + 2i) + (3 - 4i) = 4 - 2i,$$

and

$$(1 + 2i)(3 - 4i) = 11 + 2i.$$

As a special case, the square, cube, or any power of a complex number is again a complex number. For example,

$$\begin{aligned}(1 + i\sqrt{3})^3 &= (1 + i\sqrt{3})(1 + i\sqrt{3})^2 = (1 + i\sqrt{3})(-2 + 2i\sqrt{3}) \\ &= -8. \end{aligned} \tag{2}$$

The reciprocal of a nonzero complex number is also a complex number. For if we write

$$\alpha = \frac{1}{a + ib}$$

with a and b not both zero and multiply numerator and denominator by $a - ib$, then

$$\alpha = \frac{1}{a + ib}\,\frac{a - ib}{a - ib} = \frac{a - ib}{a^2 - i^2b^2} = \frac{a - ib}{a^2 + b^2} = \frac{a}{a^2 + b^2} - i\,\frac{b}{a^2 + b^2}.$$

For example, if

$$\alpha = \frac{1}{-1 + 3i},$$

then we may write

$$\alpha = \frac{1}{-1 + 3i}\,\frac{-1 - 3i}{-1 - 3i} = \frac{-1 - 3i}{1 + 9} = -0.1 - 0.3i.$$

Thus sums, products, powers, and quotients of complex numbers are again complex numbers A typical problem would be to write

$$\beta = (\tfrac{1}{2} - 2i)^2 - \frac{(7 + 2i)(-4 - 3i)}{2 + 2i} \tag{3}$$

in the form $a + ib$ We have

$$(7 + 2i)(-4 - 3i) = -22 - 29i,$$

and

$$\frac{1}{2 + 2i} = \frac{1}{2 + 2i}\,\frac{2 - 2i}{2 - 2i} = \tfrac{1}{4} - \tfrac{1}{4}i$$

Thus

$$(-22 - 29i)(\tfrac{1}{4} - \tfrac{1}{4}i) = -\tfrac{51}{4} - \tfrac{7}{4}i$$

and

$$(\tfrac{1}{2} - 2i)^2 = \tfrac{1}{4} - 4 - 2i = -\tfrac{15}{4} - 2i$$

Hence (3) becomes

$$\beta = (-\tfrac{15}{4} - 2i) - (-\tfrac{51}{4} - \tfrac{7}{4}i) = 9 - \tfrac{1}{4}i \tag{4}$$

One could also give an axiomatic treatment of complex numbers as is done in courses in the *theory of functions* However, the formal rules boil down to those illustrated above

11.2. Relation to Trigonometric Functions

If we consider a coordinate system and label the x-axis the "real axis" and the y-axis the "imaginary axis" (see Fig 11 1), then any complex number

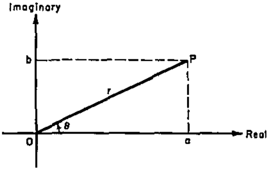


Fig. 11.1

$a + ib$ may be represented by a point P in the plane where the abscissa of P is a and the ordinate b However, from an inspection of Fig 11 1 it is clear that we can define P by giving r and θ where r is the length of the line from the origin to P and θ is the angle it makes with the positive direction of the real axis

The relations between r, θ and a, b are very simple. In fact, (r,θ) is the polar coordinate representation of (a,b) (see Section 5.6 of Chapter 5). Thus

$$\begin{aligned} a &= r \cos \theta, \\ b &= r \sin \theta, \end{aligned} \tag{5}$$

and

$$\begin{aligned} r &= +\sqrt{a^2 + b^2}, \\ \theta &= \arctan \frac{b}{a}. \end{aligned} \tag{6}$$

In (6), the *positive* square root of $a^2 + b^2$ is always to be taken. We use the notation $+\sqrt{a^2 + b^2}$ for emphasis. Thus we may represent the complex number

$$\alpha = a + ib$$

in the form

$$\alpha = r(\cos \theta + i \sin \theta). \tag{7}$$

Equation (7) is called the *polar form* of the complex number. The real number $r = +\sqrt{a^2 + b^2}$ is called the *modulus* of the complex number and is frequently denoted by $|\alpha|$. That is,

$$|\alpha| = r.$$

Note that if $\alpha = a + ib$ and $\bar{\alpha} = a - ib$, then

$$\alpha\bar{\alpha} = a^2 + b^2 = |\alpha|^2. \tag{8}$$

(The number $\bar{\alpha}$ is called the *conjugate* of α.) Thus in words, (8) says that the modulus of a complex number is the square root of the product of a complex number by its conjugate.

As an example, let us express the number

$$\alpha = -3 + 4i \tag{9}$$

in polar form (see Fig. 11.2). We see from (6) that

$$r = +\sqrt{(-3)^2 + (4)^2} = 5$$

and

$$\theta = \arctan \frac{4}{-3}.$$

Since we are in the second quadrant (see Fig. 11.2) and Arctan $\frac{4}{3} = 53° \, 8'$,

$$\theta = 180° - 53° \, 8' = 126° \, 52'.$$

Thus

$$\alpha = 5(\cos 126° \, 52' + i \sin 126° \, 52'). \tag{10}$$

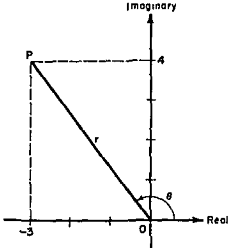


Fig 11.2

11.3. De Moivre's Theorem

The notation cis θ is sometimes used to represent $\cos\theta + i\sin\theta$ Thus

$$\text{cis}\,\theta \equiv \cos\theta + i\sin\theta \tag{11}$$

We shall prove the remarkable result that

$$\text{cis}\,(\theta + \phi) = (\text{cis}\,\theta)(\text{cis}\,\phi), \tag{12}$$

which is known as *De Moivre's theorem*

For, from (11), we have

$$\begin{aligned}(\text{cis}\,\theta)(\text{cis}\,\phi) &= (\cos\theta + i\sin\theta)(\cos\phi + i\sin\phi)\\ &= (\cos\theta\cos\phi - \sin\theta\sin\phi) + i(\sin\theta\cos\phi + \sin\phi\cos\theta)\\ &= \cos(\theta+\phi) + i\sin(\theta+\phi) = \text{cis}\,(\theta+\phi)\end{aligned}$$

—which proves (12) Now from (12),

$$\text{cis}\,\theta = \frac{\text{cis}\,(\theta+\phi)}{\text{cis}\,\phi}, \tag{13}$$

provided cis $\phi \neq 0$ If we let $\theta + \phi = \psi$, then (13) becomes

$$\text{cis}\,(\psi - \phi) = \frac{\text{cis}\,\psi}{\text{cis}\,\phi} \tag{14}$$

These formulas make it very easy to multiply and divide complex numbers when they are written in polar form For example, if

$$\alpha = r\,\text{cis}\,\theta$$

and

$$\beta = s \operatorname{cis} \phi,$$

then

$$\alpha\beta = rs(\operatorname{cis} \theta)(\operatorname{cis} \phi) = rs \operatorname{cis} (\theta + \phi),$$

and

$$\frac{\alpha}{\beta} = \frac{r \operatorname{cis} \theta}{s \operatorname{cis} \phi} = \frac{r}{s} \operatorname{cis} (\theta - \phi), \qquad (\beta \neq 0).$$

As a numerical example, if

$$\alpha = 2 \operatorname{cis} \frac{\pi}{6} = \sqrt{3} + i,$$

$$\beta = 3 \operatorname{cis} \frac{2\pi}{3} = -\frac{3}{2} + \frac{3\sqrt{3}}{2} i,$$

then

$$\alpha\beta = 6 \operatorname{cis} \left(\frac{2}{3}\pi + \frac{\pi}{6}\right) = 6 \operatorname{cis} \tfrac{5}{6}\pi$$

$$= 6(\cos \tfrac{5}{6}\pi + i \sin \tfrac{5}{6}\pi) = 6\left(-\frac{\sqrt{3}}{2} + i\,\frac{1}{2}\right) = -3\sqrt{3} + 3i,$$

and

$$\frac{\alpha}{\beta} = \frac{2}{3} \operatorname{cis} \left(\frac{\pi}{6} - \frac{2}{3}\pi\right) = \frac{2}{3} \operatorname{cis} \left(-\frac{\pi}{2}\right)$$

$$= \frac{2}{3}\left[\cos \left(-\frac{\pi}{2}\right) + i \sin \left(-\frac{\pi}{2}\right)\right] = \tfrac{2}{3}[0 + i(-1)] = -\tfrac{2}{3}i.$$

From (14), we can deduce that

$$\operatorname{cis} (-\phi) = \frac{1}{\operatorname{cis} \phi} \tag{15}$$

by letting $\psi = 0$ and noting that

$$\operatorname{cis} 0 = \cos 0 + i \sin 0 = 1.$$

One could, of course, prove this formula directly by writing

$$\frac{1}{\operatorname{cis} \phi} = \frac{1}{\cos \phi + i \sin \phi}$$

and multiplying numerator and denominator by $\cos \phi - i \sin \phi$:

$$\frac{1}{\operatorname{cis} \phi} = \frac{1}{\cos \phi + i \sin \phi} \, \frac{\cos \phi - i \sin \phi}{\cos \phi - i \sin \phi} = \frac{\cos \phi - i \sin \phi}{\cos^2 \phi + \sin^2 \phi}$$

$$= \cos \phi - i \sin \phi = \cos (-\phi) + i \sin (-\phi) = \operatorname{cis} (-\phi).$$

Now let us consider (12) when $\theta = \phi$ Then

$$\text{cis } 2\theta = (\text{cis } \theta)^2 \tag{16}$$

By induction, we can easily show that

$$\text{cis } n\theta = (\text{cis } \theta)^n \tag{17}$$

where n is a positive integer For suppose (17) is valid for $n = k$ Then

$$(\text{cis } \theta)^{k+1} = (\text{cis } \theta)^k(\text{cis } \theta) = (\text{cis } k\theta) \text{ cis } \theta$$

by our induction hypothesis But from (12) with $\phi = k\theta$,

$$(\text{cis } k\theta)(\text{cis } \theta) = \text{cis } (k + 1)\theta$$

Thus (17) is established

More easily, we show that

$$(\text{cis } \theta)^{-n} = \text{cis } (-n\theta) \tag{18}$$

where n is a positive integer For

$$(\text{cis } \theta)^{-n} = \frac{1}{(\text{cis } \theta)^n} = \frac{1}{\text{cis } n\theta}$$

and, by (15),

$$\frac{1}{\text{cis } n\theta} = \text{cis } (-n\theta)$$

We have shown now that

$$\text{cis } n\theta = (\text{cis } \theta)^n \tag{19}$$

where n is a nonzero integer Since

$$\text{cis } 0\theta = \cos 0 + i \sin 0 = 1$$

and

$$(\text{cis } \theta)^0 = 1, \tag{20}$$

we see that (19) also holds when $n = 0$ Thus we have the remarkable result that cis θ obeys the same laws as an exponential such as a^θ We shall show in Section 13 6 of Chapter 13 (Euler's formula) that this is more than an analogy That is, we shall actually find a (complex) number a such that

$$a^\theta \equiv \text{cis } \theta \tag{21}$$

11.4. Some Applications of De Moivre's Theorem

We recall that two complex numbers $a + ib$ and $c + id$ are equal if and only if $a = c$ and $b = d$, that is, if their real and imaginary parts respectively are equal From this simple remark we can deduce some fascinating formulas

For example, consider the identity

$$\text{cis}\, n\theta = (\text{cis}\, \theta)^n \tag{22}$$

where n is a positive integer. In expanded form, (22) becomes

$$(\cos n\theta + i \sin n\theta) = (\cos \theta + i \sin \theta)^n. \tag{23}$$

Now let us expand the right-hand side of (23) and collect real and imaginary terms. We have, by the binomial theorem, that

$$(\cos \theta + i \sin \theta)^n = \sum_{k=0}^{n} \binom{n}{k} \cos^{n-k} \theta (i \sin \theta)^k. \tag{24}$$

If k is even, $i^k = (-1)^{k/2}$ and, if k is odd, $i^k = i(-1)^{(k-1)/2}$. Thus we can separate real and imaginary parts of (24) by writing

$$(\cos \theta + i \sin \theta)^n = \sum_{\substack{k=0 \\ (k \text{ even})}}^{n} (-1)^{k/2} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta$$
$$+ i \sum_{\substack{k=0 \\ (k \text{ odd})}}^{n} (-1)^{(k-1)/2} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta.$$

Returning to (23) and equating real and imaginary parts, we have

$$\cos n\theta = \sum_{\substack{k=0 \\ (k \text{ even})}}^{n} (-1)^{k/2} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta, \tag{25}$$

and

$$\sin n\theta = \sum_{\substack{k=0 \\ (k \text{ odd})}}^{n} (-1)^{(k-1)/2} \binom{n}{k} \cos^{n-k} \theta \sin^k \theta. \tag{26}$$

Thus we have expressed the sine and cosine of a multiple angle in terms of powers of the sine and cosine of that angle! For example,

$$\cos 8\theta = \sum_{\substack{k=0 \\ (k \text{ even})}}^{8} (-1)^{k/2} \binom{8}{k} \cos^{8-k} \theta \sin^k \theta$$
$$= \binom{8}{0} \cos^8 \theta - \binom{8}{2} \cos^6 \theta \sin^2 \theta + \binom{8}{4} \cos^4 \theta \sin^4 \theta$$
$$- \binom{8}{6} \cos^2 \theta \sin^6 \theta + \binom{8}{8} \sin^8 \theta$$
$$= \cos^8 \theta - 28 \cos^6 \theta \sin^2 \theta + 70 \cos^4 \theta \sin^4 \theta$$
$$- 28 \cos^2 \theta \sin^6 \theta + \sin^8 \theta. \tag{27}$$

By the use of the elementary identity $\cos^2 \theta + \sin^2 \theta = 1$, the student may reduce $\cos 8\theta$ to powers of $\sin \theta$ alone or $\cos \theta$ alone.

As a second application, let us solve the problem (done in Chapter 10) of finding the sum

$$S = \cos\theta + \cos 2\theta + \cdots + \cos N\theta = \sum_{n=1}^{N} \cos n\theta \tag{28}$$

in closed form We note that

$$\operatorname{cis}\theta = \cos\theta + i\sin\theta,$$

$$\operatorname{cis}(-\theta) = \cos\theta - i\sin\theta$$

and hence by adding and subtracting we obtain

$$\cos\theta = \frac{\operatorname{cis}\theta + \operatorname{cis}(-\theta)}{2} \tag{29}$$

and

$$\sin\theta = \frac{\operatorname{cis}\theta - \operatorname{cis}(-\theta)}{2i} \tag{30}$$

Then substituting (29) in (28) yields

$$S = \sum_{n=1}^{N} \frac{\operatorname{cis} n\theta + \operatorname{cis}(-n\theta)}{2}$$

$$= \frac{1}{2}\sum_{n=1}^{N} \operatorname{cis} n\theta + \frac{1}{2}\sum_{n=1}^{N} \frac{1}{\operatorname{cis} n\theta}$$

since $\operatorname{cis}(-n\theta) = (\operatorname{cis} n\theta)^{-1}$ But $\operatorname{cis} n\theta = (\operatorname{cis}\theta)^n$ Thus

$$S = \frac{1}{2}\sum_{n=1}^{N} (\operatorname{cis}\theta)^n + \frac{1}{2}\sum_{n=1}^{N} \frac{1}{(\operatorname{cis}\theta)^n}$$

But the sums are now both geometric series with ratios $\operatorname{cis}\theta$ and $1/\operatorname{cis}\theta$ respectively Hence

$$\sum_{n=1}^{N} (\operatorname{cis}\theta)^n = \frac{\operatorname{cis}\theta - (\operatorname{cis}\theta)^{N+1}}{1 - \operatorname{cis}\theta} \tag{31}$$

and

$$\sum_{n=1}^{N} \frac{1}{(\operatorname{cis}\theta)^n} = \frac{\dfrac{1}{\operatorname{cis}\theta} - \dfrac{1}{(\operatorname{cis}\theta)^{N+1}}}{1 - \dfrac{1}{\operatorname{cis}\theta}} \tag{32}$$

In order to compute S in the most efficient fashion, multiply and divide (31) by $\operatorname{cis}(-\theta/2)$ and multiply and divide (32) by $\operatorname{cis}\theta/2$ Then

$$S = \frac{1}{2\left[\operatorname{cis}\left(-\frac{\theta}{2}\right) - \operatorname{cis}\frac{\theta}{2}\right]} \times \left\{\operatorname{cis}\frac{\theta}{2} - \operatorname{cis}(N + \tfrac{1}{2})\theta - \operatorname{cis}\left(-\frac{\theta}{2}\right) + \operatorname{cis}\left[-(N + \tfrac{1}{2})\theta\right]\right\} \tag{33}$$

But

$$\operatorname{cis}(-\phi) - \operatorname{cis}\phi = \cos\phi - i\sin\phi - (\cos\phi + i\sin\phi)$$
$$= -2i\sin\phi$$

[see (30)]. Thus

$$S = \frac{1}{-4i\sin\dfrac{\theta}{2}}\left\{-2i\sin(N+\tfrac{1}{2})\theta + 2i\sin\frac{\theta}{2}\right\}$$

$$= \frac{1}{2}\left[\frac{\sin(N+\frac{1}{2})\theta}{\sin\dfrac{\theta}{2}} - 1\right] = \frac{\cos\dfrac{(N+1)\theta}{2}\sin\dfrac{N\theta}{2}}{\sin\dfrac{\theta}{2}}, \tag{34}$$

which is the same as (9) of Chapter 10.

11.5. Roots of Complex Numbers

The cube of what number is -8? Obviously $(-2)^3 = -8$. But from (2),

$$(1 + i\sqrt{3})^3 = -8,$$

and the student may also readily verify that

$$(1 - i\sqrt{3})^3 = -8.$$

Thus the three solutions of the equation

$$x^3 + 8 = 0$$

are

$$x_1 = -2, \qquad x_2 = 1 - i\sqrt{3}, \qquad x_3 = 1 + i\sqrt{3}.$$

Can we find the roots of any complex number by some systematic procedure? The answer is yes and is merely another application of De Moivre's formula. All the roots are again complex numbers.

Suppose we wish to find the nth roots of the complex number $\alpha = a + ib$. First we write α in polar form as

$$\alpha = r(\cos\theta + i\sin\theta). \tag{35}$$

Now by De Moivre's theorem,

$$\left[\sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)\right]^n = r\left(\cos\frac{n\theta}{n} + i\sin\frac{n\theta}{n}\right) = \alpha.$$

Thus

$$\sqrt[n]{r}\left(\cos\frac{\theta}{n} + i\sin\frac{\theta}{n}\right)$$

is one nth root of α To find the remaining roots, we note that (35) may also be written as

$$\alpha = r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$$

where k is an integer Now if we raise

$$\sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$

to the nth power, we have

$$(\sqrt[n]{r})^n\left[\cos n\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin n\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right]$$
$$= r[\cos(\theta + 2k\pi) + i\sin(\theta + 2k\pi)]$$
$$= r(\cos\theta + i\sin\theta) = \alpha$$

Thus the n nth roots of α are

$$x_k = \sqrt[n]{r}\left[\cos\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right) + i\sin\left(\frac{\theta}{n} + \frac{2k\pi}{n}\right)\right], \qquad k = 0, 1, \quad , n - 1$$

Note that if $k = n$, x_n becomes identical with x_0, if $k = n + 1$, x_{n+1} becomes identical with x_1, etc Thus we have precisely n distinct roots

For example, let us find the four fourth roots of

$$\alpha = -3 + 4i$$

We write α in polar form as

$$\alpha = 5(\cos 126° 52 + i\sin 126° 52)$$

(see Fig 11 2) Then the four fourth roots are

$$x_0 = \sqrt[4]{5}\operatorname{cis} 31° 43,$$
$$x_1 = \sqrt[4]{5}\operatorname{cis}\left(31° 43 + \frac{360°}{4}\right),$$
$$x_2 = \sqrt[4]{5}\operatorname{cis}\left(31° 43 + \frac{720°}{4}\right),$$
$$x_3 = \sqrt[4]{5}\operatorname{cis}\left(31° 43 + \frac{1080°}{4}\right),$$

or, since $\sqrt[4]{5} = 1\ 50$,

$$x_0 = 1\ 50 \operatorname{cis} 31° 43' = 1\ 27 + i0\ 79,$$
$$x_1 = 1\ 50 \operatorname{cis} 121° 43' = -0\ 79 + i1\ 27,$$
$$x_2 = 1\ 50 \operatorname{cis} 211° 43 = -1\ 27 - i0\ 79,$$
$$x_3 = 1\ 50 \operatorname{cis} 301° 43' = 0\ 79 - i1\ 27$$

These numbers may be conveniently represented graphically as the four equally spaced points on the circumference of a circle of radius $\sqrt[4]{5} = 1.50$ (see Fig. 11.3).

The graphical representation makes it very easy to find roots of positive real numbers by inspection. For example, the n nth roots of the positive real

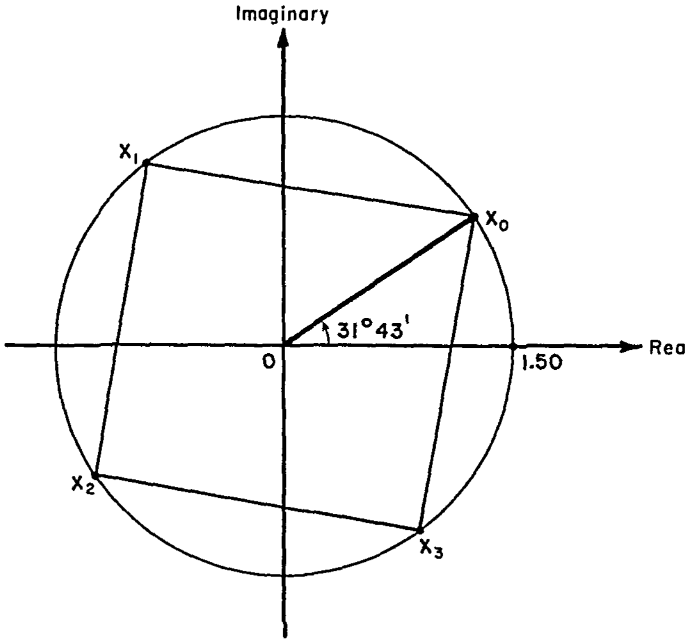


Fig. 11.3

number A may be represented by the vertices of a regular polygon inscribed in a circle of radius $\sqrt[n]{A}$ with one vertex at the point $(\sqrt[n]{A},0)$. Thus the six sixth roots of unity may be illustrated as in Fig. 11.4; and, from the diagram,

$$x_0 = \cos 0° + i \sin 0° = 1,$$

$$x_1 = \cos 60° + i \sin 60° = \frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$x_2 = \cos 120° + i \sin 120° = -\frac{1}{2} + i\frac{\sqrt{3}}{2},$$

$$x_3 = \cos 180° + i \sin 180° = -1,$$

$$x_4 = \cos 240° + i \sin 240° = -\frac{1}{2} - i\frac{\sqrt{3}}{2},$$

$$x_5 = \cos 300° + i \sin 300° = \frac{1}{2} - i\frac{\sqrt{3}}{2}.$$

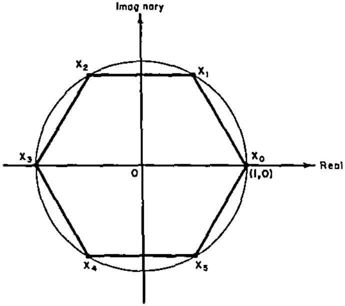


Fig 11 4

PROBLEMS

1 If α β, and γ are complex numbers, prove that

(a) $\alpha + (\beta + \gamma) = (\alpha + \beta) + \gamma$ (associativity of addition)

(b) $\alpha + \beta = \beta + \alpha$ (commutativity of addition)

(c) $\alpha(\beta\gamma) = (\alpha\beta)\gamma$ (associativity of multiplication)

(d) $\alpha\beta = \beta\alpha$ (commutativity of multiplication)

(e) $\alpha(\beta + \gamma) = \alpha\beta + \alpha\gamma$ (distributivity of multiplication with respect to addition)

2 Interpret the sum and product of complex numbers graphically

3 Express the following complex numbers in polar form

(a) $2 + 3i$

(b) $\sqrt{2} - i$

(c) $-2 + i\sqrt{3}$

(d) $a + i\sqrt{1 - a^2}$ $|a| \leq 1$

(e) $\cos 30° + i \sin 60°$

4 Express the following complex numbers in cartesian form

(a) $4 \operatorname{cis} 20°$

(b) $\cos 40° + i \sin 30°$

(c) $3 \cos 20° + i \sin 20°$

(d) $2(\cos \theta + i \sin \phi)$

(e) $\sin \theta + i \cos \theta$

5. Reduce the following expressions to the form $a + ib$.

(a) $(1 + i)^4$.

(b) $\dfrac{(2 - i)^2}{3 + 7i} - (-1 + i\sqrt{2})^{-3}$.

(c) $4 \operatorname{cis}(-10°) - (6 + i\pi)$.

(d) $\dfrac{-2 - 4i}{-6 \operatorname{cis} 40°}$.

(e) $(3 + 2i)^{1+i}$. (HINT: See Chapter 14.)

6. Express

$$\sum_{n=1}^{N} \sin n\theta$$

in closed form by the use of De Moivre's theorem.

7. Find the indicated roots of the following real numbers:

(a)	2	cube roots.
(b)	16	fourth roots.
(c)	-5	square roots.
(d)	-7	fourth roots.
(e)	1	nth roots. (See also Chapter 14.)

8. Find the indicated roots of the following complex numbers:

(a)	$1 + i$	square roots.
(b)	i	cube roots.
(c)	$4 - 3i$	fourth roots.
(d)	$\sqrt{2} \operatorname{cis} 45°$	square roots.
(e)	$a(a^2 - 3b^2) + ib(3a^2 - b^2)$	cube roots.

9. If n is a positive integer, show that the ratio

$$\frac{\theta^{2n} - 1}{\theta^2 - 1}$$

approaches n as θ approaches $+1$ or -1.

10. If n is a positive integer, show that

$$\frac{\theta^{2n} - 1}{\theta^2 - 1} = \left(\theta^2 - 2\theta \cos\frac{\pi}{n} + 1\right)\left(\theta^2 - 2\theta \cos\frac{2\pi}{n} + 1\right)\cdots \left(\theta^2 - 2\theta \cos\frac{(n-1)\pi}{n} + 1\right).$$

11. Prove the identities

$$n = 2^{n-1}\left(1 - \cos\frac{\pi}{n}\right)\left(1 - \cos\frac{2\pi}{n}\right)\cdots\left(1 - \cos\frac{(n-1)\pi}{n}\right)$$
$$= 2^{n-1}\left(1 + \cos\frac{\pi}{n}\right)\left(1 + \cos\frac{2\pi}{n}\right)\cdots\left(1 + \cos\frac{(n-1)\pi}{n}\right).$$

12. Establish the formula

$$\sin\frac{\pi}{n}\sin\frac{2\pi}{n}\cdots\sin\frac{(n-1)\pi}{n} = \frac{n}{2^{n-1}}.$$

12

INTRODUCTION TO THE THEORY OF LIMITS

In elementary arithmetic, the student first learned how to use numbers. The next significant advance in his mathematical training occurred when he studied algebra and introduced *letters* such as x, y, a, b instead of numbers such as 2, -3, $\frac{1}{2}$, -0.761. The third stage in the development of mathematics is an understanding of the notion of *limit*. Actually, we have used limits before, for example, in finding the sum of a geometric series as the number of terms increased without limit. Also, in Chapter 10 we considered an area as the *limit* of a sum and the slope was the *limit* of a certain ratio. In the present chapter, we wish to give some precise definitions and results in the theory of limits. A complete development of such analyses may be found in courses on differential and integral calculus and in courses on the theory of functions. Although this is probably the most subtle and conceptually difficult chapter of our book, the conscientious student will be amply rewarded for whatever effort he applies.

It will be noted that the contents of this chapter have nothing to do with trigonometry per se. However, one cannot deeply penetrate any subject without laying a substantial foundation. Thus it is sometimes necessary to consider difficult and perhaps seemingly diverse material before one's goal is reached. The great strides we shall be able to make in succeeding chapters in the development of advanced trigonometric theory will, we believe, more than compensate the student for struggling through the necessary preliminaries.

12.1. Sequences

Suppose that to every positive integer n, $n = 1, 2, \cdots$, a number s_n is assigned according to some rule. Then the set of numbers

$$s_1, s_2, s_3, \cdots, s_n, \cdots \tag{1}$$

is called a *sequence* Sequences are undoubtedly the most basic concept of advanced mathematics For example,

$$1, 2, 3, \quad , n, \quad , \tag{2}$$

$$1, \frac{1}{2}, \frac{1}{4}, \quad , \frac{1}{2^{n-1}}, \quad , \tag{3}$$

$$+1, -1, +1, \quad , (-1)^{n+1}, \quad , \tag{4}$$

$$(1+1)^1, \left(1+\frac{1}{2}\right)^2, \left(1+\frac{1}{3}\right)^3, \quad , \left(1+\frac{1}{n}\right)^n, \tag{5}$$

are all sequences

By the *limit of a sequence* such as (1) we mean a *number s* which has the property that for n large the terms of the sequence *get and remain* arbitrarily close to s Thus the definition of limit can be expressed in a few words When the student has really assimilated this definition, he will be well on his way toward an understanding of the calculus

Let us see if we can formulate the definition of limit in terms of mathematical symbols The formal definition follows

Definition The sequence

$$s_1, s_2 \; s_3 \qquad s_n,$$

is said to have s as a *limit* (or to *converge* to s) if for every positive number ε there exists an integer N such that

$$|s - s_n| < \varepsilon$$

for all $n > N$

In symbols, we write

$$\lim_{n\to\infty} s_n = s,$$

which is merely shorthand for the above definition

If a sequence does not have a limit, we shall say it *diverges* Thus any sequence either converges or diverges There is no middle ground For example the sequence (3) converges to zero While this is probably obvious to the student let us prove it rigorously within the framework of our formal definition

Let $\varepsilon > 0$ be assigned Then we can certainly find an integer N such that $2^N > \frac{1}{\varepsilon}$ since ε is fixed Thus for all $n > N$,

$$\left|\frac{1}{2^{n-1}} - 0\right| = \frac{1}{2^{n-1}} \le \frac{1}{2^N} < \varepsilon,$$

which completes the proof The sequence (2) *diverges* for there exists no number A such that

$$|n - A|$$

can be made small for *all* n large. Similarly, (4) diverges since there exists no number which is simultaneously close to both $+1$ and -1. The sequence (5) converges. The proof of this fact is more difficult and will be deferred to Section 12.3.

Let us consider an abstract proof based on our formal definition, namely, let us prove that the limit of a sequence, if it exists, is unique. We shall do so by assuming the contrary and showing that this leads to a contradiction. Suppose then that the sequence (1) converges to both s and s' with $s \neq s'$. Let $\varepsilon = \frac{1}{4}|s' - s| > 0$. Then since (1) converges to s, we can find an N such that

$$|s_n - s| < \varepsilon \tag{6}$$

for all $n > N$, and also such that

$$|s_n - s'| < \varepsilon \tag{7}$$

since s' is also a limit. Now consider the identity

$$s - s' = s - s_n + s_n - s',$$

obtained by adding and subtracting s_n. If we introduce parentheses, we have

$$s - s' = (s - s_n) + (s_n - s'),$$

from which we conclude that

$$|s - s'| \leqq |s - s_n| + |s_n - s'|, \tag{8}$$

since the absolute value of a sum is less than or equal to the sum of absolute values. But from (6) and (7),

$$|s - s_n| + |s_n - s'| < \varepsilon + \varepsilon = 2\varepsilon$$

and, by definition of ε, $|s' - s| = 4\varepsilon$. Thus (8) implies

$$4\varepsilon < 2\varepsilon$$

which is absurd since $\varepsilon > 0$. Thus we have a contradiction, and hence the limit of a sequence, if it exists, must be unique.

The arguments used in the above proof are typical of those that will be found in the theory of limits.

12.2. Series

Even though sequences are more fundamental than series, the student has probably had more experience with series. For example, the geometric series

$$1 + r + r^2 + \cdots + r^{n-1} + \cdots \tag{9}$$

is a familiar sight. Let us, however, consider a more general series, say

$$a_1 + a_2 + \cdots + a_n + \cdots \equiv \sum_{k=1}^{\infty} a_k \tag{10}$$

What do we mean by the "sum" of (10)? This is an easy question to answer. We construct the following numbers:

$$\begin{aligned} s_1 &= a_1, \\ s_2 &= a_1 + a_2, \\ s_3 &= a_1 + a_2 + a_3, \\ &\cdots\cdots \\ s_n &= a_1 + a_2 + \cdots + a_n, \\ &\cdots\cdots \end{aligned}$$

The numbers $s_1, s_2, s_3, \cdots$ are called the *partial sums* of the series (10) Now consider the *sequence*

$$s_1, s_2, s_3, \cdots, s_n, \cdots \tag{11}$$

Then if (11) converges, we say that the infinite series (10) converges and that the sum of (10) is the limit of (11) Thus the definitions of convergence and "sum" of an infinite series are referred to the more basic sequences. Symbolically, we write

$$s = \lim_{n\to\infty} s_n = \lim_{n\to\infty} \sum_{k=1}^{n} a_k = \sum_{k=1}^{\infty} a_k \tag{12}$$

Consider, for example, the geometric series (9) The partial sums are

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + r = \frac{1 - r^2}{1 - r}, \\ s_3 &= 1 + r + r^2 = \frac{1 - r^3}{1 - r}, \\ &\cdots \\ s_n &= 1 + r + r^2 + \cdots + r^{n-1} = \frac{1 - r^n}{1 - r}, \\ &\cdots\cdots \end{aligned}$$

Thus (9) converges if $\lim_{n\to\infty} s_n$ exists, and if this limit exists it is, by definition, the sum of the series. Now for $|r| < 1$, we know that

$$\lim_{n\to\infty} s_n = \frac{1}{1 - r}.$$

(Prove this using the definition of convergence of a sequence!) Thus the sum of (9), for $|r| < 1$ is $\dfrac{1}{1-r}$. If $|r| > 1$, the series diverges since r^n becomes arbitrarily large as n increases without limit. Finally if $r = 1$, (9) becomes

$$1 + 1 + 1 + \cdots + 1 + \cdots,$$

and the corresponding partial sums are

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 + 1 = 2, \\ s_3 &= 1 + 1 + 1 = 3, \\ &\cdots\cdots\cdots\cdots \\ s_n &= 1 + 1 + \cdots + 1 = n, \\ &\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

As we have seen before, the sequence

$$1, 2, 3, \cdots, n, \cdots$$

diverges. If $r = -1$, (9) becomes

$$1 - 1 + 1 - \cdots + (-1)^{n+1} + \cdots,$$

and the corresponding partial sums are

$$\begin{aligned} s_1 &= 1, \\ s_2 &= 1 - 1 = 0, \\ s_3 &= 1 - 1 + 1 = 1, \\ &\cdots\cdots\cdots\cdots \\ s_n &= 1 - 1 + 1 - \cdots + (-1)^{n+1} = \tfrac{1}{2}[1 + (-1)^{n+1}], \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \end{aligned}$$

Just as in the case (4), the sequence

$$1, 0, 1, 0, 1, 0, 1, 0, \cdots, 1, 0, \cdots$$

diverges.

12.3. A Special Sequence

We shall devote this section to a proof of the convergence of the sequence (5),

$$(1 + 1)^1, \left(1 + \frac{1}{2}\right)^2, \left(1 + \frac{1}{3}\right)^3, \cdots, \left(1 + \frac{1}{n}\right)^n, \cdots. \tag{5}$$

Even though this sequence, in itself, forms an interesting application of our theory, our reasons for discussing it go much deeper. The limit of this sequence is approximately 2 718, which is a very famous number in the history of mathematics It is generally denoted by the letter e As we have remarked in Part I (Section 9 3 of Chapter 9), it is the base of *natural logarithms* We shall see how this limit arises in a natural fashion and investigate some of its fascinating properties in the next chapter To whet the appetite, let us mention one such truly remarkable result which will be proved in Section 13 6 of Chapter 13, namely, that

$$\text{cis } \theta = e^{i\theta} \text{!}$$

We start the proof of the convergence of (5) by considering the elementary identity

$$\alpha^n - \beta^n = (\alpha - \beta)(\alpha^{n-1} + \alpha^{n-2}\beta + \alpha^{n-3}\beta^2 + \quad + \alpha\beta^{n-2} + \beta^{n-1}), \quad (13)$$

which is obtained by dividing $\alpha^n - \beta^n$ by $\alpha - \beta$ Now if $\alpha > \beta > 0$ and if we replace β by α in the second set of parentheses on the right of (13), we deduce that

$$\alpha^n - \beta^n < (\alpha - \beta)(\alpha^{n-1} + \alpha^{n-2}\alpha + \alpha^{n-3}\alpha^2 + \quad + \alpha\alpha^{n-2} + \alpha^{n-1}), \quad (14)$$

or

$$\alpha^n - \beta^n < (\alpha - \beta)n\alpha^{n-1},$$

since there are n of the α^{n-1} terms on the right of (14) Transposing yields

$$\beta^n > \alpha^n - (\alpha - \beta)n\alpha^{n-1},$$

and factoring out α^{n-1} on the right gives us

$$\beta^n > \alpha^{n-1}[\alpha - n(\alpha - \beta)] \quad (15)$$

Now set

$$\alpha = 1 + \frac{1}{n-1} \quad \text{and} \quad \beta = 1 + \frac{1}{n},$$

where n is a positive integer greater than one Clearly $\alpha > \beta > 0$, and hence (15) is valid with this choice of α and β If one substitutes, (15) becomes

$$\begin{aligned}\left(1 + \frac{1}{n}\right)^n &> \left(1 + \frac{1}{n-1}\right)^{n-1}\left\{\left[1 + \frac{1}{n-1}\right] - n\left[\left(1 + \frac{1}{n-1}\right)\right.\right. \\ &\qquad \left.\left. - \left(1 + \frac{1}{n}\right)\right]\right\} \\ &= \left(1 + \frac{1}{n-1}\right)^{n-1}\left\{1 + \frac{1}{n-1} - n\left[\frac{n - n + 1}{n(n-1)}\right]\right\} \\ &= \left(1 + \frac{1}{n-1}\right)^{n-1}\left[1 + \frac{1}{n-1} - \frac{1}{n-1}\right] \\ &= \left(1 + \frac{1}{n-1}\right)^{n-1},\end{aligned}$$

or

$$\left(1+\frac{1}{n}\right)^n > \left(1+\frac{1}{n-1}\right)^{n-1}$$

Thus the terms of the sequence (5) have the property that

$$(1+1)^1 < \left(1+\frac{1}{2}\right)^2 < \cdots < \left(1+\frac{1}{n-1}\right)^{n-1} < \left(1+\frac{1}{n}\right)^n < \cdots. \quad (16)$$

That is, the terms are steadily increasing. We shall next show that the terms do not get arbitrarily large. That is, we shall show that there exists a number M such that

$$\left(1+\frac{1}{n}\right)^n < M$$

for all n. Thus *if* (5) has a limit, it must certainly be less than or equal to M.

To prove this assertion, set

$$m = n - 1, \qquad \alpha = 1 + \frac{1}{2m}, \qquad \beta = 1$$

in (15). Then

$$(+1)^{m+1} > \left(1+\frac{1}{2m}\right)^m\left[\left(1+\frac{1}{2m}\right) - (m+1)\left(1+\frac{1}{2m}-1\right)\right],$$

or

$$1 > \left(1+\frac{1}{2m}\right)^m\left[1+\frac{1}{2m}-\frac{1}{2}-\frac{1}{2m}\right] = \left(1+\frac{1}{2m}\right)^m\left(\frac{1}{2}\right).$$

Thus

$$1 > \frac{1}{2}\left(1+\frac{1}{2m}\right)^m,$$

or

$$2 > \left(1+\frac{1}{2m}\right)^m.$$

Squaring both sides of this inequality, we obtain

$$4 > \left(1+\frac{1}{2m}\right)^{2m}.$$

Thus by virtue of (16), *every* term of the sequence (5) is less than 4 (the "M" of the previous paragraph).

Of course, it still remains to prove that (5) converges to some number (less than 4). However, after these lengthy preliminaries, we can finally prove our result as a special case of the following theorem.

Theorem. Let

$$s_1, s_2, s_3, \cdots, s_n, \cdots \quad (17)$$

be a sequence with the property that

$$s_1 < s_2 < s_3 < \quad < s_n <$$

and let there exist a number M such that $s_n < M$ for all n, $n = 1, 2, 3,$ Then (17) converges

Clearly [see (16)] our sequence is a special case which meets the hypotheses of the above theorem Thus if we can prove this theorem, we shall have demonstrated the convergence of (5)

To prove the theorem, let r_0 be the largest integer such that $s_n \geqq r_0$ for all n sufficiently large (that is, $s_n < r_0 + 1$ for all n, $n = 1, 2,$) Let r_1 be the largest integer between 0 and 9 inclusive such that

$$s_n \geqq r_0 + \frac{r_1}{10}$$

for n sufficiently large Let r_2 be the largest integer between 0 and 9 inclusive such that

$$s_n \geqq r_0 + \frac{r_1}{10} + \frac{r_2}{10^2}$$

for n sufficiently large Continuing this process, we find

$$s_n \geqq r_0 + \frac{r_1}{10} + \frac{r_2}{10^2} + \quad + \frac{r_N}{10^N}$$

for any N, and n sufficiently large We assert that the complete decimal expansion for the limit of the sequence is

$$s = r_0 + \frac{r_1}{10} + \frac{r_2}{10^2} + \quad ,$$

where $0 \leqq r_k \leqq 9$ for all k For, given any $\varepsilon > 0$, we can find an N such that $10^{N-1} > \frac{1}{\varepsilon}$ Then for any $n > N$,

$$\begin{aligned} |s - s_n| &= \left|\frac{r_{n+1}}{10^{n+1}} + \frac{r_{n+2}}{10^{n+2}} + \quad \right| < \left|\frac{r_{N+1}}{10^{N+1}} + \frac{r_{N+2}}{10^{N+2}} + \quad \right| \\ &< \left|\frac{10}{10^{N+1}} + \frac{10}{10^{N+2}} + \quad \right| = \left|\frac{1}{10^N} + \frac{1}{10^{N+1}} + \quad \right| \\ &= \frac{1}{10^N}\left[1 + \frac{1}{10} + \frac{1}{10^2} + \quad \right] = \frac{1}{10^N}\frac{1}{1 - \frac{1}{10}} = \frac{1}{10^N}\frac{1}{0\ 9} \\ &< \frac{2}{10^N} < \frac{10}{10^N} = \frac{1}{10^{N-1}} < \varepsilon \end{aligned}$$

As remarked before, this theorem completes the proof of the convergence of (5) We shall call the limit e Of course, it does not exhibit e numerically

All it tells us is that e exists and is a number less than 4. Since the sequence *does* converge, this number will be close to $\left(1 + \frac{1}{n}\right)^n$ when n is large. Thus a fair approximation to e may be obtained by evaluating $\left(1 + \frac{1}{n}\right)^n$ for various values of n. The larger n, the better the approximation. For example,

$$n = 1 \qquad (1 + 1)^1 = 2,$$
$$n = 2 \qquad (1 + \tfrac{1}{2})^2 = 2.25,$$
$$n = 3 \qquad (1 + \tfrac{1}{3})^3 = 2.370 \cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$n = 10 \qquad (1 + \tfrac{1}{10})^{10} = 2.5937 \cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$n = 100 \qquad (1 + \tfrac{1}{100})^{100} = 2.7048 \cdots,$$
$$\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots$$
$$n = 1000 \qquad (1 + \tfrac{1}{1000})^{1000} = 2.7169 \cdots.$$

Actually, this limit e, the base of natural logarithms, is

$$e = 2.7182818285, \tag{18}$$

correct to ten decimal places.

PROBLEMS

1. Determine whether the following sequences converge or diverge:

(a) $1, \frac{1}{2}, \frac{1}{3}, \cdots, \frac{1}{n}, \cdots.$

(b) $0, \frac{1}{2}, \frac{3}{4}, \frac{7}{8}, \cdots, 1 - \frac{1}{2^{n-1}}, \cdots.$

(c) $1, -\frac{1}{2}, +\frac{1}{3}, -\frac{1}{4}, \cdots, +\frac{1}{2n-1}, -\frac{1}{2n}, \cdots.$

(d) $s_1, s_2, \cdots, s_n, \cdots$ where

$$s_{2n} = 0, \qquad n = 1, 2, 3, \cdots$$
$$s_{2n-1} = \frac{1}{n}, \qquad n = 1, 2, 3, \cdots.$$

(e) $\sigma_1, \sigma_2, \cdots, \sigma_n, \cdots$ where

$$\sigma_n = \frac{1}{n^2} + \frac{2}{n^2} + \cdots + \frac{n}{n^2}.$$

2 If

$$s_1, s_2, \quad , s_n,$$

converges to s and

$$\sigma_1, \sigma_2, \quad , \sigma_n,$$

converges to σ, prove that

$$s_1 + \sigma_1, s_2 + \sigma_2, \quad , s_n + \sigma_n,$$

converges to $s + \sigma$

3 If

$$s_1, s_2 \quad , s_n,$$

converges to s and

$$\sigma_1, \sigma_2 \quad , \sigma_n,$$

converges to σ, prove that

$$s_1\sigma_1, s_2\sigma_2, \quad , s_n\sigma_n,$$

converges to $s\sigma$

4 If

$$s_1, s_2 \quad , s_n \qquad s_n \neq 0, n = 1, 2,$$

converges to $s \neq 0$, prove that

$$\frac{1}{s_1}, \frac{1}{s_2}, \quad , \frac{1}{s_n},$$

converges to $\frac{1}{s}$

5 Do the following series converge or diverge?

(a) $1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \quad + \frac{1}{n!} +$

(b) $1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \quad + \frac{(-1)^{n-1}}{(2n-1)!}$

(c) $1 + \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} + \quad + \frac{1}{\sqrt{n}} +$

(d) $1 - \frac{1}{\sqrt{2}} + \frac{1}{\sqrt{3}} - \quad + \frac{(-1)^{n+1}}{\sqrt{n}} +$

(e) $\frac{3}{1\ 2} - \frac{9}{2\ 2^2} + \frac{27}{3\ 2^3} - \quad + \frac{(-1)^{n+1}\ 3^n}{n\ 2^n} +$

6 If

$$s_1 > s_2 > \quad > s_n > \quad > 0$$

prove that the sequence

$$s_1, s_2, \quad , s_n,$$

converges

7. Prove that a *necessary* condition that the sequence

$$s_1, s_2, \cdots, s_n, \cdots$$

converges is: Given any $\varepsilon > 0$, there exists a positive integer N such that for all $n > N$ and all positive integers p

$$|s_n - s_{n+p}| < \varepsilon.$$

8. If

$$s_1 + s_2 + \cdots + s_n + \cdots$$

is a convergent series of nonnegative numbers and

$$s_k \geqq |\sigma_k| \qquad k = 1, 2, \cdots$$

prove that

$$\sum_{k=1}^{\infty} \sigma_k$$

is also a convergent series.

9. If

$$s_1 + s_2 + \cdots + s_n \cdots$$

is a divergent series and

$$|s_k| \leqq \sigma_k, \qquad k = 1, 2, \cdots,$$

prove that $\sum_{k=1}^{\infty} \sigma_k$ is also a divergent series.

10. Consider the series

$$S = \sum_{n=1}^{\infty} a_n.$$

(a) Prove that if S converges then $\lim_{n \to \infty} a_n = 0$.

(b) Show by example that if $\lim_{n \to \infty} a_n = 0$, the series S does not necessarily converge.

11. A familiar series in elementary mathematics is the *harmonic series*,

$$1 + \frac{1}{2} + \frac{1}{3} + \cdots + \frac{1}{n} + \cdots.$$

Prove that this series diverges. (HINT: Group the terms as

$$1 + [\tfrac{1}{2}] + [\tfrac{1}{3} + \tfrac{1}{4}] + [\tfrac{1}{5} + \tfrac{1}{6} + \tfrac{1}{7} + \tfrac{1}{8}] + \cdots.)$$

13

ANALYTIC DEFINITION OF THE TRIGONOMETRIC FUNCTIONS

We saw in Part I (Section 3 2 of Chapter 3) that when θ was a small angle (measured in radians) $\sin\theta$ was numerically equal, approximately, to θ

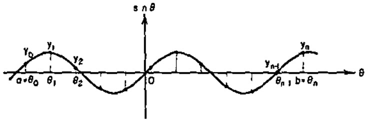


Fig 13 1

However, from a graph of the trigonometric function, we see that the relation $\sin\theta = \theta$ is certainly not valid when θ is large This raises the question Is it possible to find a polynomial

$$P(\theta) = a_0 + a_1\theta + a_2\theta^2 + \quad + a_n\theta^n \tag{1}$$

that approximates $\sin\theta$ over a range larger than that for which $\sin\theta = \theta$ is valid? Toward this end we might do the following Choose an interval, say from a to b (see Fig 13 1), and divide this interval into n equal smaller intervals by the points

$$a = \theta_0, \theta_1 \quad , \theta_n = b$$

Then we could look up the value of $\sin\theta_k$ $k = 0, 1, 2, \quad , n$ in the tables Call this value y_k Now let us determine the coefficients a_r, $r = 0, 1, \quad , n$ of the polynomial* $P(\theta)$ so that it passes through the $n + 1$ points $(\theta_0\, y_0)$,

* Recall that we may always find a polynomial of a degree not greater than n that passes through $n + 1$ prescribed points

$(\theta_1,y_1), \cdots, (\theta_n,y_n)$. Then

$$y_0 = \sum_{r=0}^{n} a_r\theta_0^r,$$

$$y_1 = \sum_{r=0}^{n} a_r\theta_1^r,$$

$$\cdots\cdots\cdots$$

$$y_n = \sum_{r=0}^{n} a_r\theta_n^r,$$

or, more compactly,

$$y_k = \sum_{r=0}^{n} a_r\theta_k^r \qquad k = 0, 1, \cdots, n,$$

represents a system of $n + 1$ equations on the $n + 1$ coefficients $a_0, a_1, \cdots, a_n$. Theoretically, we can solve these linear equations and determine a polynomial $P(\theta)$ that would approximate $\sin \theta$ between a and b.

However, this approach has a number of flaws. First, the amount of labor involved is tremendous if n is large, say $n = 100$. Second, even if we can calculate the coefficients, $P(\theta)$ is only an approximation to $\sin \theta$ (although presumably a good one if n is large). Third, even if it were an excellent approximation, it would be valid only in the range from $\theta = a$ to $\theta = b$. These latter objections lead one to think that perhaps an *infinite series* might represent $\sin \theta$ *exactly* for all θ. It is the main concern of the present chapter to show that an infinite series can be found that *does* represent $\sin \theta$ exactly for all θ and, furthermore, that it is possible to determine the coefficients of this infinite series in a reasonable fashion.

13.1. Derivatives

In Chapter 10, we defined the slope of a curve $f(x)$ at the point x by the equation

$$m = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h} \tag{2}$$

(if it existed). In the calculus, it is customary to call m the *derivative* of $f(x)$ and to use the notation $\frac{d}{dx} f(x)$. Thus

$$\frac{d}{dx} f(x) = \lim_{h\to 0} \frac{f(x + h) - f(x)}{h}$$

Of course, the derivative can be interpreted geometrically as the slope. However, one frequently uses the concept of derivative without worrying about the geometric interpretation.

We also saw in Chapter 10 that the derivative of x^2 was $2x$. In our present notation,

$$\frac{dx^2}{dx} = \lim_{h\to 0} \frac{(x+h)^2 - x^2}{h} = \lim_{h\to 0} \frac{x^2 + 2xh + h^2 - x^2}{h}$$
$$= \lim_{h\to 0} (2x + h) = 2x$$

The derivative of a constant K is zero since

$$\frac{dK}{dx} = \lim_{h\to 0} \frac{K - K}{h} = \lim_{h\to 0} \frac{0}{h} = 0 \tag{3}$$

If n is a positive integer, then

$$\frac{dx^n}{dx} = \lim_{h\to 0} \frac{(x+h)^n - x^n}{h}.$$

Using the binomial theorem, we may write

$$(x+h)^n = x^n + nhx^{n-1} + \frac{n(n-1)}{2} h^2x^{n-2} + \quad + h^n$$

Thus

$$\frac{dx^n}{dx} = \lim_{h\to 0} \left[nx^{n-1} + \frac{n(n-1)}{2} hx^{n-2} + \quad + h^{n-1} \right]$$
$$= nx^{n-1} \tag{4}$$

since every term except nx^{n-1} involves a power of h and hence approaches zero as h approaches zero

Finally, we recall from Chapter 10 [(16) and (17) of Section 10 3] that

$$\frac{d}{dx} \sin x = \cos x, \tag{5}$$

and

$$\frac{d}{dx} \cos x = -\sin x \tag{6}$$

13.2. Power Series for Sine and Cosine

Suppose now we consider the following problem Can we find coefficients $a_0, a_1, a_2, \cdots, a_n, \cdot$ such that

$$a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdot \; \cdot$$

represents $\sin x$ exactly for all x? That is, can we write $\sin x$ as an infinite series of powers of x, namely,

$$\sin x = a_0 + a_1x + a_2x^2 + a_3x^3 + \;\cdot\cdot + a_nx^n + \cdots ? \tag{7}$$

We shall solve this problem. By using the concept of derivative, it will not be difficult to evaluate the a_n. For a discussion of the validity of our procedure, we refer the student to the next section.

Since $\sin 0 = 0$, we can readily evaluate a_0 by simply replacing x by 0 in (7),

$$\begin{aligned} 0 = \sin 0 &= a_0 + a_1 0 + a_2 0^2 + a_3 0^3 + \cdots + a_n 0^n + \cdots \\ &= a_0. \end{aligned}$$

Thus $a_0 = 0$. Now let us *differentiate* (7). Then from (5), (3), and (4),

$$\cos x = a_1 + 2a_2 x + 3a_3 x^2 + \cdots + na_n x^{n-1} + \cdots, \tag{8}$$

and, if we let $x = 0$ in (8), we have

$$\begin{aligned} 1 = \cos 0 &= a_1 + 2a_2 0 + 3a_3 0^2 + \cdots + na_n 0^{n-1} + \cdots \\ &= a_1, \end{aligned}$$

and $a_1 = 1$.

Let us continue this process. Differentiate (8). Then from (6), (3), and (4),

$$-\sin x = 2a_2 + 3 \cdot 2a_3 x + \cdots + n(n-1)a_n x^{n-2} + \cdots \tag{9}$$

and, if $x = 0$,

$$0 = 2a_2 + 3 \cdot 2a_3 0 + \cdots + n(n-1)a_n 0^{n-2} + \cdots,$$

and $a_2 = 0$. Again, differentiating (9) leads to

$$-\cos x = 3 \cdot 2a_3 + \cdots + n(n-1)(n-2)a_n x^{n-3} + \cdots \tag{10}$$

and, letting $x = 0$,

$$-1 = 3 \cdot 2a_3.$$

Thus $a_3 = -\dfrac{1}{3!}$.

If this process is continued, we shall find that

$$\begin{aligned} a_n &= \frac{(-1)^{(n-1)/2}}{n!} & &\text{if } n \text{ is odd,} \\ a_n &= 0 & &\text{if } n \text{ is even.} \end{aligned}$$

Thus we have

$$\sin x = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \frac{x^9}{9!} - \cdots. \tag{11}$$

Equation (11) is known as a *power series*.

Similarly, we may write

$$\cos x = b_0 + b_1 x + b_2 x^2 + \cdots + b_n x^n + \cdots \tag{12}$$

and use exactly the same techniques as above to evaluate the b_n For example, let $x = 0$ Then (12) implies

$$1 = b_0$$

Differentiating (12) yields

$$-\sin x = b_1 + 2b_2x + \qquad + nb_nx^{n-1} +$$

and letting $x = 0$ leads to $b_1 = 0$ Continuing this process, we find that

$$\cos x = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \frac{x^8}{8!} - \qquad , \tag{13}$$

which is the power series development of $\cos x$

Power series developments of functions obtained by successive differentiation as we did to get (11) and (13), are known as *Maclaurin expansions*

13 3. Convergence of a Power Series

Let us pause for a moment and take stock of our results Just what is meant by an infinite series of functions such as (11) or (13)? In Chapter 12, we went to great pains to define the convergence of a series of numbers In harmony with earlier definitions we lay down the following definition of convergence of a power series

Definition The power series

$$a_0 + a_1x + a_2x^2 + \qquad + a_nx^n +$$

is said to *converge* at the point $x = c$ if the series

$$a_0 + a_1c + a_2c^2 + \qquad + a_nc^n +$$

of *numbers* is a convergent series

Note that a series need not converge for *all* x For example, consider the power series

$$1 + x + x^2 + \qquad + x^n + \qquad , \tag{14}$$

which is a geometric series with ratio x We saw in Section 12 2 of Chapter 12 that (14) converges if $|x| < 1$ and diverges if $|x| \geqq 1$ Thus the power series of (14) makes sense only if $|x| < 1$ By elementary long division, we see that

$$\frac{1}{1-x} = 1 + x + x^2 + \qquad + x^n + \tag{15}$$

$\left[\text{or alternatively } \frac{1}{1-x} \text{ is the sum of the infinite geometric series of (14)}\right]$ But while the left hand side of (15) is defined for all $x \neq 1$, the right hand side is defined as we have just seen, only if $|x| < 1$ Thus for example, if $x = 2$, the left-hand side of (15) is well defined:

$$\frac{1}{1-2} = -1.$$

But it is patently absurd to write

$$-1 = 1 + 2 + 2^2 + \cdots + 2^n + \cdots$$

obtained by letting $x = 2$ in (15).

The moral of the above discussion is that after we have obtained a power series by any means, we must carefully investigate it to see for what values of x it converges. If we have obtained the power series by a Maclaurin expansion, as we did in finding (11) and (13), an even more subtle question arises, namely: Even if the power series converges, does it represent the function with which we started? The answer is not necessarily. Furthermore, even if

$$f(x) = a_0 + a_1x + \cdots + a_nx^n + \cdots \equiv \sum_{k=0}^{\infty} a_kx^k$$

is a true equation for certain values of x, it is far from obvious that $\frac{d}{dx}f(x)$ will equal $\sum_{k=1}^{\infty} ka_kx^{k-1}$ (which is obtained by differentiating $\sum_{k=0}^{\infty} a_kx^k$ term by term) even at these same values of x. Thus we may well ask when term by term differentiation of an infinite series is legitimate. These difficult questions are answered in courses on the *theory of functions*. It can be shown that the power series expansions of sin x and cos x converge for *all* x and represent sin x and cos x respectively for all x.

13.4. Construction of Tables

The student has probably already asked himself the question of how the trigonometric tables appearing in the back of the book were constructed. One obvious way is to lay out an angle θ with a protractor and form a right triangle with unit hypotenuse (see Fig. 13.2). Then the numerical length of s is the sine of θ and the numerical length of c is the cosine of θ. The geometric method, of course, is limited in accuracy.

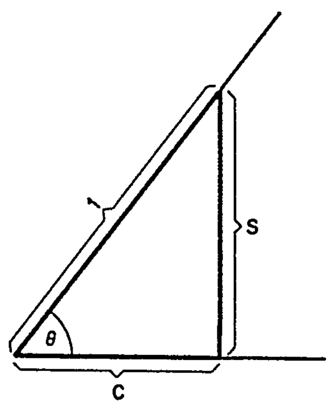


Fig. 13.2

Another approach is to use the half-angle formulas. For example, we know that cos 60° = 0.5000. Using the half-angle formula

$$\cos\frac{\theta}{2} = \sqrt{\frac{1+\cos\theta}{2}},$$

we can find cos 30° Then with this information, we can again apply the half-angle formula to obtain cos 15° Repeated applications yield cos 7° 30′, cos 3° 45′, etc We can also compute the sines of these angles as well as the sine or cosine of any multiple For example,

$$\begin{aligned}\cos 3(3^\circ\, 45') &= \cos 11^\circ\, 15' = \cos\,(7^\circ\, 30' + 3^\circ\, 45') \\ &= \cos 7^\circ\, 30' \cos 3^\circ\, 45' - \sin 7^\circ\, 30' \sin 3'\, 45'.\end{aligned}$$

Thus given any angle θ we could theoretically approximate it as closely as desired by combinations of submultiples of 60°

However, with (11) and (13) at our disposal we can evaluate the sine or cosine of any angle to any degree of accuracy This, in essence, is how tables are constructed in practice For example, the sine of one radian is

$$\begin{aligned}\sin 1 &= 1 - \frac{1}{3!} + \frac{1}{5!} - \frac{1}{7!} + \frac{1}{9!} - \\ &= 1 - 0\ 166667 + 0\ 008333 - 0\ 000198 + 0\ 000003 - \\ &= 0\ 84147\end{aligned}$$

From the tables,

$$\sin 1 = 0\ 8415$$

to four places

13.5. Power Series for the Exponential

In Section 11 3 of Chapter 11, we mentioned that cis $\theta = \cos\theta + i\sin\theta$ had many of the properties of an exponential and stated that we would show later that there was a number a such that

$$a^\theta \equiv \operatorname{cis}\theta$$

In the next section we shall do just that This number a turns out to be the complex number e^i where e is the base of natural logarithms introduced in Section 9 3 of Chapter 9 and discussed in Section 12 3 of Chapter 12 However, in order to derive this result we must first obtain the Maclaurin expansion for e^x, which in turn means we must first compute the derivative of e^x

By definition, the derivative of e^x is

$$\frac{de^x}{dx} = \lim_{h\to 0}\frac{e^{x+h} - e^x}{h} = \lim_{h\to 0} e^x\left(\frac{e^h - 1}{h}\right) \tag{16}$$

We saw in Chapter 12 that

$$e = \lim_{n\to\infty}\left(1 + \frac{1}{n}\right)^n \tag{17}$$

Now let $h = 1/n$. Then as $n \to \infty$, $h \to 0$ and (17) becomes

$$e = \lim_{h \to 0} (1 + h)^{1/h}.$$

Thus for h very small, e is approximately equal to $(1 + h)^{1/h}$, that is, $e \doteq (1 + h)^{1/h}$, and, on raising each side to the hth power,

$$e^h \doteq 1 + h.$$

Now substitute this in (16):

$$\frac{de^x}{dx} = \lim_{h \to 0} e^x \left(\frac{1 + h - 1}{h}\right) = e^x. \tag{18}$$

(We are justified in replacing e^h by $1 + h$ since we are eventually letting h approach zero.) Thus we have the remarkable result that e^x is its own

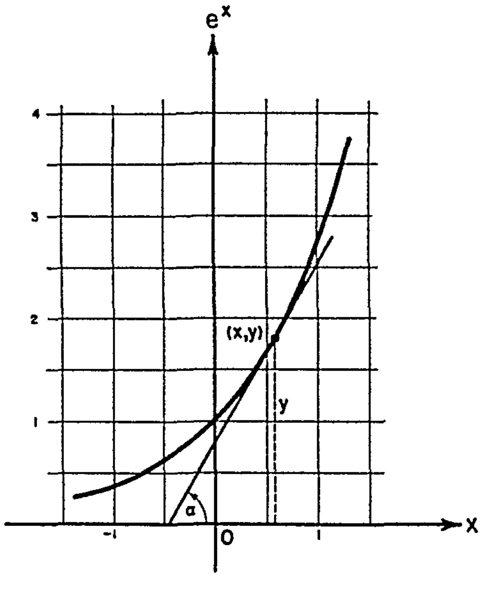


Fig. 13.3

derivative. Geometrically, this means that the numerical value of the slope at any point on the curve e^x is equal to the value of the ordinate at that point (see Fig. 13.3).

Equation (18) makes it particularly simple to compute the Maclaurin expansion of e^x. For if we write

$$e^x = a_0 + a_1x + a_2x^2 + \cdots + a_nx^n + \cdots, \tag{19}$$

then, since $e^0 = 1$,

$$e^0 = 1 = a_0.$$

Differentiating (19) leads to

$$e^x = a_1 + 2a_2x + \quad + na_nx^{n-1} + \tag{20}$$

and at $x = 0$,

$$e^0 = 1 = a_1$$

Again, differentiate (20)

$$e^x = 2a_2 + \quad + n(n-1)a_nx^{n-2} +$$

and at $x = 0$,

$$1 = 2a_2 \quad \text{or} \quad a_2 = \tfrac{1}{2}$$

Proceeding exactly as we did in the case of $\sin x$ and $\cos x$, we find that

$$e^x = 1 + x + \frac{x^2}{2!} + \frac{x^3}{3!} + \frac{x^4}{4!} + \frac{x^5}{5!} + \tag{21}$$

It can be shown that this power series converges for all x and represents e^x for all x

From (21), we can compute e to any number of significant figures For example, letting $x = 1$ in (21) and taking eleven terms yields

$$e = 1 + 1 + \frac{1}{2!} + \frac{1}{3!} + \frac{1}{4!} + \frac{1}{5!} + \frac{1}{6!} + \frac{1}{7!} + \frac{1}{8!} + \frac{1}{9!} + \frac{1}{10!}$$

$$\begin{aligned} &= 1\ 000000 \\ &\ 1\ 000000 \\ &\ 0\ 500000 \\ &\ 0\ 166667 \\ &\ 0\ 041667 \\ &\ 0\ 008333 \\ &\ 0\ 001389 \\ &\ 0\ 000198 \\ &\ 0\ 000025 \\ &\ \underline{0\ 000003} \\ e &= 2\ 718282 \end{aligned}$$

Compare this with (18) of Chapter 12

13 6 Euler's Formula

Suppose that in the power series expansion, (21), for e we set* $x = i\theta$ Then

$$e^{i\theta} = 1 + i\theta + \frac{(i\theta)^2}{2!} + \frac{(i\theta)^3}{3!} + \frac{(i\theta)^4}{4!} + \frac{(i\theta)^5}{5!} + \frac{(i\theta)^6}{6!} + \frac{(i\theta)^7}{7!} +$$

* It is shown in courses in the *theory of functions* that the power series expansion for e^x is valid even if x is complex and furthermore that the terms of the series can be rearranged at will without changing the value of the series

We recall that $i^2 = -1$, $i^3 = -i$, $i^4 = 1$, $i^5 = i(i)^4 = i$, etc. Hence the above formula may be written as

$$e^{i\theta} = 1 + i\theta - \frac{\theta^2}{2!} - i\frac{\theta^3}{3!} + \frac{\theta^4}{4!} + i\frac{\theta^5}{5!} - \frac{\theta^6}{6!} - \frac{\theta^7}{7!} + \cdots$$

and, on collecting real and imaginary parts,

$$e^{i\theta} = \left(1 - \frac{\theta^2}{2!} + \frac{\theta^4}{4!} - \frac{\theta^6}{6!} + \cdots\right) + i\left(\theta - \frac{\theta^3}{3!} + \frac{\theta^5}{5!} - \frac{\theta^7}{7!} + \cdots\right).$$

But from (11) and (13) we recognize the power series in the parentheses as $\cos\theta$ and $\sin\theta$ respectively. Thus

$$e^{i\theta} = \cos\theta + i\sin\theta. \tag{22}$$

This is the famous formula of Euler. The analogy of $\text{cis}\,\theta$ to an exponential is more than an analogy; we have, in fact,

$$\text{cis}\,\theta = e^{i\theta}.$$

If we replace θ by $-\theta$ in (22),

$$e^{-i\theta} = \cos\theta - i\sin\theta. \tag{23}$$

Adding and subtracting (22) and (23), we obtain [cf. (29) and (30) of Chapter 11]

$$\cos\theta = \frac{e^{i\theta} + e^{-i\theta}}{2}, \tag{24}$$

and

$$\sin\theta = \frac{e^{i\theta} - e^{-i\theta}}{2i}. \tag{25}$$

These formulas are sometimes very convenient for proving identities among trigonometric functions (see Chapter 6 of Part I). We need only replace $\sin\theta$ and $\cos\theta$ by the exponentials of (24) and (25). Then since there is only one type of term to consider (namely, the exponential), the manipulations become purely algebraic and do not depend on a good memory of the various trigonometric identities. Of course, one could have done exactly the same thing with $\text{cis}\,\theta$ and the rules developed for its manipulation.

These purely algebraic arguments also eliminate any question of the size of the angle. One is tempted in many proofs to draw diagrams in which all the angles appear as acute. However, most of our formulas are valid no matter what quadrant or quadrants the angles lie in. For example, we went to great pains when we derived the formula (see Section 4.1 of Chapter 4)

$$\cos(x + y) = \cos x\cos y - \sin x\sin y$$

to show that it was valid for any x and y. Now let us prove the above formula using the complex exponentials. We have

$$\cos x \cos y = \frac{e^{ix} + e^{-ix}}{2} \frac{e^{iy} + e^{-iy}}{2}$$

$$= \tfrac{1}{4}[e^{i(x+y)} + e^{i(x-y)} + e^{-i(x-y)} - e^{-i(x+y)}],$$

and

$$\sin x \sin y = \frac{e^{ix} - e^{-ix}}{2i} \frac{e^{iy} - e^{-iy}}{2i}$$

$$= -\tfrac{1}{4}[e^{i(x+y)} - e^{i(x-y)} - e^{-i(x-y)} + e^{-i(x+y)}]$$

Thus

$$\cos x \cos y - \sin x \sin y = \tfrac{1}{4}[2e^{i(x+y)} + 2e^{-i(x+y)}], \tag{26}$$

since the $e^{i(x-y)}$ and $e^{-i(x-y)}$ terms cancel But the right-hand side of (26) is

$$\tfrac{1}{2}[e^{i(x+y)} + e^{-i(x+y)}],$$

which is precisely $\cos(x + y)$

As a further example, let us prove the identity

$$\frac{1 + \tan^2\theta}{1 - \tan^2\theta} \equiv \sec 2\theta$$

In terms of the complex exponentials, we write

$$\tan\theta = \frac{\sin\theta}{\cos\theta} = \frac{1}{i}\frac{e^{i\theta} - e^{-i\theta}}{e^{i\theta} + e^{-i\theta}}$$

Thus

$$\frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{1 - \dfrac{(e^{i\theta} - e^{-i\theta})^2}{(e^{i\theta} + e^{-i\theta})^2}}{1 + \dfrac{(e^{i\theta} - e^{-i\theta})^2}{(e^{i\theta} + e^{-i\theta})^2}}$$

since $i^2 = -1$ Clearing fractions leads to

$$\frac{1 + \tan^2\theta}{1 - \tan^2\theta} = \frac{(e^{i\theta} + e^{-i\theta})^2 - (e^{i\theta} - e^{-i\theta})^2}{(e^{i\theta} + e^{-i\theta})^2 + (e^{i\theta} - e^{-i\theta})^2}$$

$$= \frac{(e^{2i\theta} + 2 + e^{-2i\theta}) - (e^{2i\theta} - 2 + e^{-2i\theta})}{(e^{2i\theta} + 2 + e^{-2i\theta}) + (e^{2i\theta} - 2 + e^{-2i\theta})}$$

$$= \frac{4}{2(e^{2i\theta} + e^{-2i\theta})} = \frac{2}{e^{2i\theta} + e^{-2i\theta}} = \frac{1}{\cos 2\theta} = \sec 2\theta$$

As another example, we prove

$$\sin 2\alpha \cos 3\alpha \equiv \sin 3\alpha \cos 2\alpha - \sin\alpha$$

Replace the trigonometric functions by exponentials:

$$\sin 2\alpha \cos 3\alpha = \frac{(e^{2i\alpha} - e^{-2i\alpha})(e^{3i\alpha} + e^{-3i\alpha})}{4i}$$

$$= \frac{e^{5i\alpha} - e^{i\alpha} + e^{-i\alpha} - e^{-5i\alpha}}{4i}, \tag{27}$$

and

$$\sin 3\alpha \cos 2\alpha - \sin \alpha = \frac{(e^{3i\alpha} - e^{-3i\alpha})(e^{2i\alpha} + e^{-2i\alpha})}{4i} - \frac{e^{i\alpha} - e^{-i\alpha}}{2i}$$

$$= \frac{e^{5i\alpha} - e^{-i\alpha} + e^{i\alpha} - e^{-5i\alpha} - 2(e^{i\alpha} - e^{-i\alpha})}{4i}$$

$$= \frac{e^{5i\alpha} + e^{-i\alpha} - e^{i\alpha} - e^{-5i\alpha}}{4i},$$

which is identical with (27).

PROBLEMS

1. Assuming

$$\cos x = a_0 + a_1 x + a_2 x^2 + \cdots + a_n x^n + \cdots,$$

use the identity

$$\cos 2x = 2\cos^2 x - 1$$

to formally deduce the power series expansion for $\cos x$. (HINT: Recall the identity below.)

$$\frac{1 - \cos x}{x^2} = \frac{1}{2}\left[\frac{\sin \frac{1}{2}x}{\frac{1}{2}x}\right]^2.$$

2. Prove that

$$\frac{d}{dx}\log_e (1 + x) = \frac{1}{1 + x}.$$

3. Prove that

$$\frac{d}{dx}\frac{1}{(1 + x)^n} = \frac{-n}{(1 + x)^{n+1}}$$

where n is a positive integer.

4. Using the results of problems 2 and 3, find the Maclaurin expansion for $\log_e (1 + x)$. (The power series converges for $-1 < x \leqq 1$ and diverges for other values of x.)

5. Find sin 10° accurate to four places without the use of tables.

6 If a is a positive number, show that

$$\log_e(-a) = \log_e a + (2n + 1)\pi i$$

where n is an integer (HINT Use Euler's formula)

7. Prove that

$$\log_e(\sqrt{1 - \theta^2} - i\theta) = -i \arcsin \theta$$

where $|\theta| < 1$ and θ is real

8 If $-1 \leqq x \leqq 1$, prove that

$$|x + (x^2 - 1)^{1/2} \cos \phi| \leqq 1$$

(HINT $(x^2 - 1)^{1/2} = i(1 - x^2)^{1/2}$)

9. Prove the identities of Exercise 6 2 of Chapter 6 using complex exponentials

14 HYPERBOLIC TRIGONOMETRY

14.1. The Hyperbolic Functions

By analogy with (24) and (25) of Chapter 13, we define

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{1}$$

and

$$\sinh x = \frac{e^x - e^{-x}}{2}. \tag{2}$$

Equation (1) defines the "hyperbolic cosine of x," which is abbreviated $\cosh x$, and (2) defines the "hyperbolic sine of x," which is abbreviated $\sinh x$. We shall see in the next section that the adjective "hyperbolic" is justified.

If we write

$$\cosh^2 x - \sinh^2 x = \left(\frac{e^x + e^{-x}}{2}\right)^2 - \left(\frac{e^x - e^{-x}}{2}\right)^2$$

$$= \frac{e^{2x} + 2 + e^{-2x}}{4} - \frac{e^{2x} - 2 + e^{-2x}}{4} = 1,$$

we obtain the identity

$$\cosh^2 x - \sinh^2 x = 1, \tag{3}$$

while

$$\sinh x \cosh y + \sinh y \cosh x$$

$$= \left(\frac{e^x - e^{-x}}{2}\right)\left(\frac{e^y + e^{-y}}{2}\right) + \left(\frac{e^y - e^{-y}}{2}\right)\left(\frac{e^x + e^{-x}}{2}\right)$$

$$= \frac{e^{x+y} + e^{x-y} - e^{-(x-y)} - e^{-(x+y)}}{4} + \frac{e^{x+y} + e^{-(x-y)} - e^{x-y} - e^{-(x+y)}}{4}$$

$$= \frac{e^{x+y} - e^{-(x+y)}}{2} = \sinh (x + y),$$

which implies that

$$\sinh(x+y) = \sinh x \cosh y + \sinh y \cosh x \tag{4}$$

Equations (3) and (4) stand in striking apposition to

$$\cos^2 x + \sin^2 x = 1$$

and

$$\sin(x+y) = \sin x \cos y + \sin y \cos x$$

respectively

For every identity involving trigonometric functions, there is an analogous formula involving hyperbolic functions As another example,

$$\cosh(x+y) = \cosh x \cosh y + \sinh x \sinh y, \tag{5}$$

which is proved in exactly the same manner as (4)

If we let $x = y$ in (4) and (5), we obtain the "double angle" formulas

$$\begin{aligned} \sinh 2x &= 2 \sinh x \cosh x, \\ \cosh 2x &= \cosh^2 x + \sinh^2 x \\ &= 1 + 2\sinh^2 x \\ &= 2\cosh^2 x - 1 \end{aligned} \tag{6}$$

Similarly, we may derive the parallels of all the basic formulas of circular trigonometry developed in Part I (see the problems at the end of this chapter)

One also defines the "hyperbolic tangent of x," written $\tanh x$, by the equation

$$\tanh x = \frac{\sinh x}{\cosh x} = \frac{e^x - e^{-x}}{e^x + e^{-x}},$$

and (although rarely used) we define the hyperbolic cotangent, secant, and cosecant in the expected fashion, namely,

$$\coth x = \frac{1}{\tanh x} = \frac{e^x + e^{-x}}{e^x - e^{-x}},$$

$$\operatorname{sech} x = \frac{1}{\cosh x} = \frac{2}{e^x + e^{-x}},$$

$$\operatorname{csch} x = \frac{1}{\sinh x} = \frac{2}{e^x - e^{-x}}$$

14 2. Graphs of the Hyperbolic Functions

We see from the definition of $\sinh x$ that

$$\begin{aligned} \sinh(-x) &= \frac{e^{-x} - e^{-(-x)}}{2} = \frac{e^{-x} - e^x}{2} = -\frac{e^x - e^{-x}}{2} \\ &= -\sinh x \end{aligned}$$

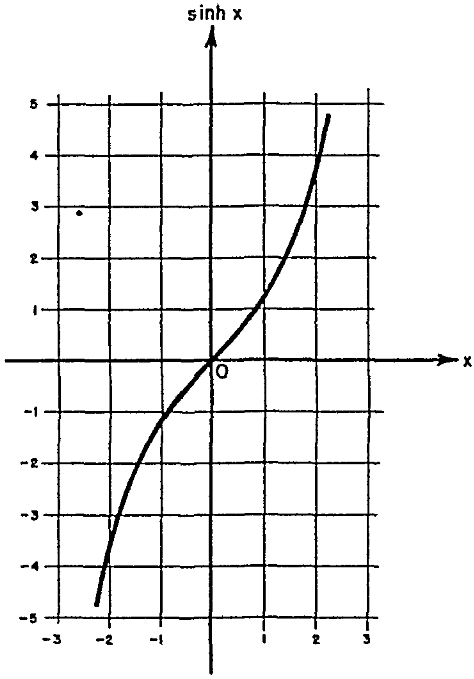


Fig. 14.1

Thus the graph of sinh x (Fig. 14.1) is symmetric about the origin and

$$\sinh 0 = \frac{e^0 - e^{-0}}{2} = \frac{1-1}{2} = 0.$$

Also, since

$$\lim_{x\to\infty} e^x = \infty$$

and

$$\lim_{x\to\infty} e^{-x} = 0,$$

we have

$$\lim_{x\to\infty} \sinh x = \infty.$$

Thus sinh x represents a nonperiodic odd function whose value increases without limit as x increases without limit and decreases without limit as x decreases without limit.

Similarly,

$$\cosh(-x) = \frac{e^{-x} + e^{-(-x)}}{2} = \frac{e^{-x} + e^x}{2} = \frac{e^x + e^{-x}}{2}$$
$$= \cosh x,$$

and the graph of cosh x (Fig. 14.2) is symmetric about the y-axis. Also

$$\cosh 0 = \frac{e^0 + e^{-0}}{2} = \frac{1+1}{2} = 1,$$

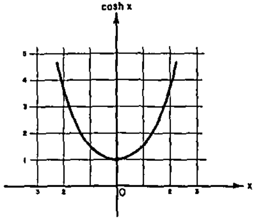


Fig 14.2

and

$$\lim_{x \to +\infty} \cosh x = +\infty$$

The graph of the hyperbolic tangent is symmetric about the origin (Fig 14 3) and the function vanishes at $x = 0$ To see what happens as x increases without limit we write

$$\tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}} = \frac{1 - e^{-2x}}{1 + e^{-2x}},$$

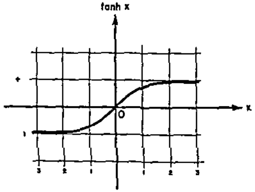


Fig 14 3

obtained by dividing numerator and denominator by e^x Since $\lim_{x \to \infty} e^{-2x} = 0$ we have

$$\lim_{x \to \infty} \tanh x = +1$$

Also, if we write

$$\tanh x = \frac{e^{2x} - 1}{e^{2x} + 1},$$

obtained by multiplying numerator and denominator of $\tanh x$ by e^x, and let x decrease without limit,

$$\lim_{x \to -\infty} \tanh x = -1$$

since $\lim_{x \to -\infty} e^{2x} = 0$.

14.3. Geometric Interpretation

As we have noted in Part I, the trigonometric functions are called "circular functions" because of their relation to a circle. We recall (see Section 1.3 of Chapter 1) that

$$x^2 + y^2 = a^2$$

is the equation of a circle with center at the origin and radius a (Fig. 14.4). If

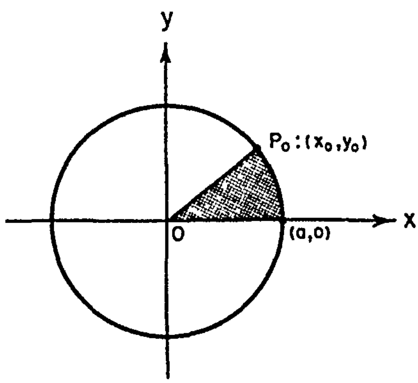


Fig. 14.4

A_0 is the shaded area in this figure, we may define θ_0 as

$$\theta_0 = \frac{2A_0}{a^2}. \tag{7}$$

Thus θ_0 is nothing more than the central angle of the shaded region bounded by OP_0 and the x-axis. The sine and cosine of θ_0 may be shown to be

$$\sin \theta_0 = \frac{y_0}{a}, \qquad \cos \theta_0 = \frac{x_0}{a}. \tag{8}$$

This appears to be a rather roundabout way of defining the circular functions. However, we give this development because of its close analogy with the way in which the hyperbolic functions may be defined geometrically.

As we have said before, $x^2 + y^2 = a^2$ is the equation of a circle Consider now the equation

$$x^2 - y^2 = a^2 \tag{9}$$

That is, we are writing x^2 *minus* y^2 rather than x^2 *plus* y^2 Equation (9) is called a *hyperbola* and is plotted in Fig 14 5 We shall briefly discuss this curve

First we see that when $y = 0$, $x = \pm a$ Also, y cannot be less than a in absolute value since, if it were, x^2 would be negative [see (9)] The dotted

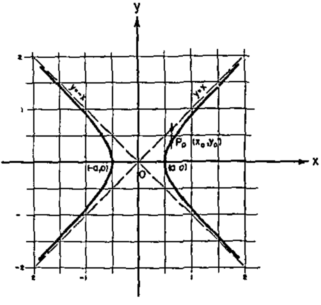


Fig 14 5

straight lines drawn on Fig 14 5 are called *asymptotes* Note that the hyperbola, although never reaching the asymptote, gets very close to it when x is large A definition of asymptote is

Definition Let C be a curve whose equation $y = f(x)$ is defined for all x Let $y = kx + b$ be a straight line with the property that $\lim_{x\to\infty} [kx + b - f(x)] = 0$ Then we call the line an *asymptote* of the curve C

If a line is parallel to the y-axis, say with equation $x = c$ and if $\lim_{y\to\infty} [c - f(x)] = 0$, then we also say that the line $x = c$ is an asymptote to the curve C (If the student plots the function $y = 1/x$, he should be able to show that the coordinate axes are asymptotes)

From Fig 14 5, it seems plausible that the asymptotes pass through the origin Let us see if we can determine a number k such that $y = kx$ is an asymptote of the hyperbola $x^2 - y^2 = a^2$ At the point P_0 on the curve, the vertical distance d between the hyperbola and the line $y = kx$ is

$$d = kx_0 - y_0$$

Thus we must try to determine k such that

$$\lim_{x_0 \to \infty} d = 0.$$

From (9), $y_0^2 = x_0^2 - a^2$. Hence

$$d = kx_0 - \sqrt{x_0^2 - a^2}.$$

Now multiply and divide the above expression by $kx_0 + \sqrt{x_0^2 - a^2}$,

$$d = \frac{k^2x_0^2 - (x_0^2 - a^2)}{kx_0 + \sqrt{x_0^2 - a^2}} = \frac{(k^2 - 1)x_0^2 + a^2}{kx_0 + \sqrt{x_0^2 - a^2}}.$$

If we choose $k = 1$, then the numerator remains constant and

$$d = \frac{a^2}{kx_0 + \sqrt{x_0^2 - a^2}}.$$

Hence

$$\lim_{x_0 \to \infty} d = 0$$

and

$$y = x$$

is the equation of one asymptote. Similarly, we find that $y = -x$ is the other asymptote.

Having disposed of the elementary properties of the hyperbola, let us turn to the definition of the hyperbolic functions. Consider the hyperbola $x^2 - y^2 = a^2$ of Fig. 14.6. Let us call the shaded area A_0 and define a quantity θ_0 by

$$\theta_0 = \frac{2A_0}{a^2}. \tag{10}$$

Then the hyperbolic sine and cosine of θ_0 may be shown to be

$$\begin{aligned} \sinh \theta_0 &= \frac{y_0}{a}, \\ \cosh \theta_0 &= \frac{x_0}{a}, \end{aligned} \tag{11}$$

which are analogous to (7) and (8) for the circular functions. Hence we see why the name "hyperbolic" is used in the definition of these functions; that is, the hyperbolic functions bear the same relation to the hyperbola as the circular functions do to the circle.

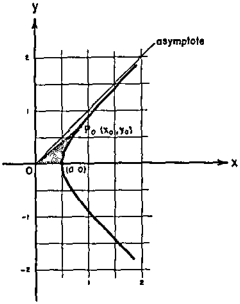


Fig 14 6

14.4. Relation to the Circular Functions

The relation

$$\cosh x = \frac{e^x + e^{-x}}{2} \tag{12}$$

suggests replacing x by the imaginary number $i\theta$ (θ real) Thus

$$\cosh i\theta = \frac{e^{i\theta} + e^{-i\theta}}{2} \tag{13}$$

But the right-hand side of (13) is precisely $\cos\theta$ Hence we have the formula

$$\cosh i\theta = \cos\theta \tag{14}$$

Similarly, letting $x = i\theta$ in

$$\sinh x = \frac{e^x - e^{-x}}{2}$$

leads to

$$\sinh i\theta = \frac{e^{i\theta} - e^{-i\theta}}{2} = i\left(\frac{e^{i\theta} - e^{-i\theta}}{2i}\right),$$

or

$$\sinh i\theta = i\sin\theta \tag{15}$$

One could now let $\theta = i\phi$ (ϕ real) in (14) and (15) to obtain the cosine and sine of an imaginary number For from (14),

$$\cos i\phi = \cosh i(i\phi) = \cosh(-\phi) = \cosh\phi, \tag{16}$$

since $\cosh \phi$ is an even function, and, from (15),

$$i \sin i\phi = \sinh i(\phi) = \sinh (-\phi) = -\sinh \phi,$$

since $\sinh \phi$ is an odd function. Dividing by i and recalling that $\frac{1}{i} = -i$, we find

$$\sin i\phi = i \sinh \phi. \tag{17}$$

Formulas (14), (15), (16), and (17) can be advantageously employed to prove identities among the hyperbolic functions from the corresponding trigonometric identity. For example, in the identity

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x + y),$$

let $x = i\theta$ and $y = i\phi$. Then it becomes

$$\sin i\theta + \sin i\phi = 2 \sin \tfrac{1}{2}i(\theta + \phi) \cos \tfrac{1}{2}i(\theta + \phi).$$

But from (17) and (16), this is the same as

$$i \sinh \theta + i \sinh \phi = 2[i \sinh \tfrac{1}{2}(\theta + \phi)][\cosh \tfrac{1}{2}(\theta - \phi)],$$

or, on canceling i,

$$\sinh \theta + \sinh \phi = 2 \sinh \tfrac{1}{2}(\theta + \phi) \cosh \tfrac{1}{2}(\theta - \phi). \tag{18}$$

Thus we see that, if we allow complex numbers, there is no distinction between circular and hyperbolic functions. (This follows from the fact that if we let $x = \xi$ and $y = i\eta$, in the equation of the circle $x^2 + y^2 = a^2$, we get $\xi^2 + (i\eta)^2 = a^2$ or $\xi^2 - \eta^2 = a^2$ which is the equation of a hyperbola.

The above operations suggest the further generalization of letting the "angle" be a *complex* number, say $a + ib$ where a and b are real. Of course, if we allow complex numbers as "angles" in computing sines and cosines, our geometric picture of a triangle must be abandoned. However, it may be shown that the analytic definitions of Chapter 13 are still valid. For example,

$$\sin (a + ib) = \sin a \cos ib + \sin ib \cos a.$$

From (16) and (17), $\cos ib = \cosh b$ and $\sin ib = i \sinh b$. Thus we may write the above formula as

$$\sin (a + ib) = \sin a \cosh b + i \sinh b \cos a \tag{19}$$

and we have expressed $\sin (a + ib)$ in the form of a complex number with real part $\sin a \cosh b$ and imaginary part $\sinh b \cos a$. We can now no longer say that $\sin \theta$ is a real number between $+1$ and -1 if we allow θ to be complex. (Of course, it *is* true if θ is *real*.)

As an application of (19), let us find $\sin \left(\frac{\pi}{2} + 2i\right)$. From (19),

$$\sin \left(\frac{\pi}{2} + 2i\right) = \sin \frac{\pi}{2} \cosh 2 + i \sinh 2 \cos \frac{\pi}{2}.$$

But $\sin \frac{\pi}{2} = 1$, $\cos \frac{\pi}{2} = 0$ and from the tables at the end of the book $\cosh 2 = 3\ 762$ Thus

$$\sin\left(\frac{\pi}{2} + 2i\right) = 3\ 762!$$

Thus we have the remarkable fact that the sine may be a real number *greater* than one if we allow the argument to be complex Hence if complex numbers are allowed in our calculations, the statement $|\sin \theta| \leqq 1$ is no longer valid!

One can, of course, similarly deduce that

$$\cos(a + ib) = \cos a \cosh b - i \sin a \sinh b, \tag{20}$$

and, for instance,

$$\begin{aligned}\cos(-0\ 5 + i) &= \cos 0\ 5 \cosh 1 + i \sin 0\ 5 \sinh 1 \\ &= (0\ 8776)(1\ 543) + i(0\ 4794)(1\ 175) \\ &= 1\ 354 + i0\ 5633\end{aligned}$$

Thus we may conclude that $\cos \theta$ as well as $\sin \theta$ may have any value (real or complex, greater or less than one) by suitably choosing a and b (where $\theta = a + ib$)

As another application, let us find what θ has its cosine equal to $2 + 3i$ Certainly such a θ must be a complex number, say $\theta = a + ib$ Then from (20),

$$2 + 3i = \cos(a + ib) = \cos a \cosh b - i \sin a \sinh b,$$

and, on equating real and imaginary parts,

$$\begin{aligned}2 &= \cos a \cosh b, \\ -3 &= \sin a \sinh b\end{aligned} \tag{21}$$

Thus we must solve (21) for a and b Now if we write these equations as

$$\cos a = \frac{2}{\cosh b},$$

$$\sin a = -\frac{3}{\sinh b},$$

and square and add, we obtain

$$1 = \cos^2 a + \sin^2 a = \frac{4}{\cosh^2 b} + \frac{9}{\sinh^2 b}, \tag{22}$$

while if we write (21) as

$$\cosh b = \frac{2}{\cos a},$$

$$\sinh b = -\frac{3}{\sin a},$$

and square and subtract, we have

$$1 = \cosh^2 b - \sinh^2 b = \frac{4}{\cos^2 a} - \frac{9}{\sin^2 a}. \tag{23}$$

Thus b is to be determined from (22) and a from (23). Equation (23) is a trigonometric *equation* such as we have treated in Part I (see Chapter 7). If we clear fractions and replace $\cos^2 a$ by $1 - \sin^2 a$,

$$(1 - \sin^2 a) \sin^2 a = 4 \sin^2 a - 9(1 - \sin^2 a).$$

For convenience, set $x = \sin^2 a$. Then the above equation becomes

$$(1 - x)x = 4x - 9(1 - x),$$

or

$$x^2 + 12x - 9 = 0$$

which is a quadratic equation in x. Its solutions are

$$x = \frac{-12 \pm \sqrt{144 + 36}}{2} = -6 \pm 3\sqrt{5}.$$

Since a is real, $\sin^2 a$ is positive and we must reject the negative root $-6 - 3\sqrt{5}$. Thus

$$x = \sin^2 a = -6 + 3\sqrt{5} = 0.7082,$$

and

$$\sin a = \pm\sqrt{0.7082} = \pm 0.842,$$

The equation for b, namely (22), may be solved in exactly the same way. Clearing fractions and replacing $\cosh^2 b$ by $1 + \sinh^2 b$ leads to

$$(1 + \sinh^2 b) \sinh^2 b = 4 \sinh^2 b + 9(1 + \sinh^2 b).$$

Setting $y = \sinh^2 b$ leads to

$$(1 + y)y = 4y + 9(1 + y),$$

or

$$y^2 - 12y - 9 = 0$$

which is a quadratic equation in y. Its solutions are

$$y = \frac{12 \pm \sqrt{144 + 36}}{2} = 6 \pm 3\sqrt{5}.$$

Since $y = \sinh^2 b > 0$ for b real, we must choose the positive root,

$$y = \sinh^2 b = 6 + 3\sqrt{5} = 12.7082,$$

and

$$\sinh b = \pm\sqrt{12.7082} = \pm 3.565.$$

Consider then the following cases

Case 1 $\quad \sin a = +0\ 842$

In this case, we deduce from the second of (21) that $\sinh b$ is negative But since $\cosh b$ is *always* positive, the first of (21) implies that $\cos a > 0$ Thus a must be in the first or fourth quadrant But $\sin a$ is also positive Thus a must be in the first quadrant Hence

$$\sin a = 0\ 842 \qquad \text{(first quadrant)},$$
$$\sinh b = -3\ 565$$

Case 2 $\quad \sin a = -0\ 842$

This information and the first of (21) implies a in the fourth quadrant Thus the second of (21) implies $\sinh b > 0$ Hence

$$\sin a = -0\ 842 \qquad \text{(fourth quadrant)},$$
$$\sinh b = 3\ 565$$

Thus

$$a + ib = \pm(1\ 00 - i1\ 98) \tag{24}$$

are except for additive multiples of 2π, the two numbers whose cosine is $2 + i3$

14 5. The Inverse Hyperbolic Functions

If $y = \sin x$, then we have considered the *inverse trigonometric function*

$$x = \arcsin y \tag{25}$$

in Chapter 5 We recall that this function is multivalued, that is, a given value of y gives rise to many values of x which satisfy (25) Analogously, if $y = \sinh x$, we write

$$x = \operatorname{argsinh} y$$

and call it the *inverse hyperbolic sine* * From the graph of $y = \sinh x$ (Fig 14 1), we see that argsinh y is single valued, that is, a given value of y gives rise to just *one* value of x (see Fig 14 7) Now an interesting result is that argsinh y may be expressed in terms of natural logarithms For if we write

$$y = \sinh x = \frac{e^x - e^{-x}}{2},$$

then we can solve for x To do this, multiply the above equation by $2e^x$ Then

$$2ye^x = e^{2x} - 1,$$

or

$$e^{2x} - 2ye^x - 1 = 0$$

* Some authors write arcsinh y or $\sinh^{-1} y$ where we have written argsinh y

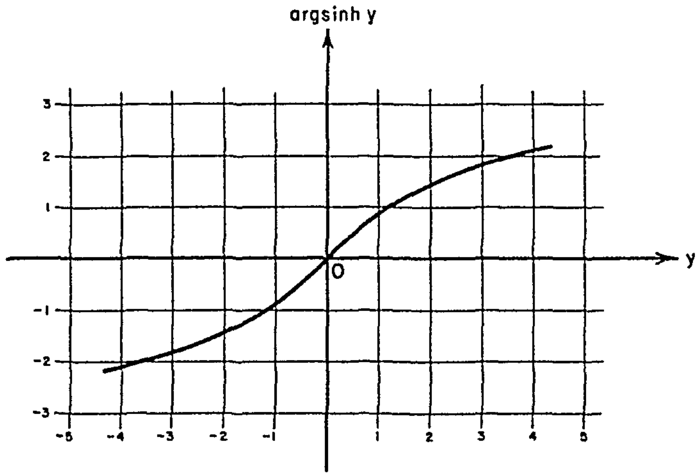


Fig. 14.7

If we let $e^x = \xi$, then the above equation becomes

$$\xi^2 - 2y\xi - 1 = 0,$$

which is a quadratic equation in ξ. Its solutions are

$$\xi = \frac{2y \pm \sqrt{4y^2 + 4}}{2} = y \pm \sqrt{y^2 + 1}.$$

But $e^x = \xi$ is always positive. Hence we discard the negative root and write

$$\xi = e^x = y + \sqrt{y^2 + 1}.$$

Taking logarithms, we obtain

$$x = \log_e (y + \sqrt{y^2 + 1}).$$

Recalling that $x = \operatorname{argsinh} y$, we have the formula

$$\operatorname{argsinh} y = \log_e (y + \sqrt{y^2 + 1}).$$

Similarly, we define

$$x = \operatorname{argcosh} y$$

as the inverse of $y = \cosh x$. From the graph of $y = \cosh x$ (Fig. 14.2), we infer that one value of y gives rise to just two values of x (Fig. 14.8). Furthermore, these two values are negatives of each other.

To express argcosh y in terms of natural logarithms, we proceed as before to write

$$y = \cosh x = \frac{e^x + e^{-x}}{2}.$$

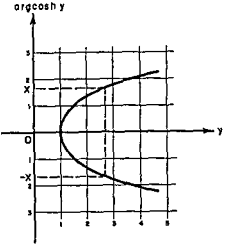


Fig 14 8

This leads to the quadratic equation

$$\zeta^2 - 2y\zeta + 1 = 0$$

where $\zeta = e^x$ Hence

$$\zeta = \frac{2y \pm \sqrt{4y^2 - 4}}{2} = y \pm \sqrt{y^2 - 1}$$

(Note from Fig 14 8 that y is always greater than or equal to one Hence for y real, $\sqrt{y^2 - 1}$ is always real) Since both values of ζ are positive,

$$\log_e \zeta = x = \log_e (y \pm \sqrt{y^2 - 1})$$

and, since $x = \operatorname{argcosh} y$,

$$\operatorname{argcosh} y = \log_e (y \pm \sqrt{y^2 - 1}), \qquad y \geqq 1$$

As we mentioned earlier, the two values of x are negatives of each other We must therefore have

$$\log_e (y + \sqrt{y^2 - 1}) = -\log_e (y - \sqrt{y^2 - 1})$$

This is easy to verify directly For,

$$y + \sqrt{y^2 - 1} = \left(y + \sqrt{y^2 - 1}\right)\frac{y - \sqrt{y^2 - 1}}{y - \sqrt{y^2 - 1}} = \frac{y^2 - (y^2 - 1)}{y - \sqrt{y^2 - 1}}$$
$$= \frac{1}{y - \sqrt{y^2 - 1}},$$

from which follows

$$\log_e (y + \sqrt{y^2 - 1}) = \log_e \frac{1}{y - \sqrt{y^2 - 1}} = -\log_e (y - \sqrt{y^2 - 1}),$$

since $\log u = -\log \frac{1}{u}$.

PROBLEMS

1. Prove that

(a) $1 - \tanh^2 x = \operatorname{sech}^2 x$.
(b) $1 - \coth^2 x = -\operatorname{csch}^2 x$.
(c) $\tanh (x + y) = \dfrac{\tanh x + \tanh y}{1 + \tanh x \tanh y}$.
(d) $\tanh 2x = \dfrac{2 \tanh x}{1 + \tanh^2 x}$

2. Establish the following formulas:

(a) $\cosh x + \cosh y = 2 \cosh \frac{1}{2}(x + y) \cosh \frac{1}{2}(x - y)$.
(b) $\cosh x - \cosh y = 2 \sinh \frac{1}{2}(x + y) \operatorname{sing} \frac{1}{2}(x - y)$.
(c) $\sinh x - \sinh y = 2 \cosh \frac{1}{2}(x + y) \sinh \frac{1}{2}(x - y)$.
(d) $\sinh x \sinh y = \frac{1}{2}[\cosh (x + y) - \cosh (x - y)]$.
(e) $\sinh x \cosh y = \frac{1}{2}[\sinh (x + y) + \sinh (x - y)]$.
(f) $\cosh x \cosh y = \frac{1}{2}[\cosh (x + y) + \cosh (x - y)]$.

3. Show that

(a) $\sinh \dfrac{x}{2} = \pm \sqrt{\dfrac{\cosh x - 1}{2}}$ $\quad$ $+$ if $x > 0$, $-$ if $x < 0$.

(b) $\cosh \dfrac{x}{2} = \sqrt{\dfrac{\cosh x + 1}{2}}$.

(c) $\tanh \dfrac{x}{2} = \dfrac{\cosh x - 1}{\sinh x} = \dfrac{\sinh x}{\cosh x + 1}$.

4. Prove "De Moivre's theorem" for hyperbolic functions; namely, show that

$$(\sinh x + \cosh x)^n = \sinh nx + \cosh nx.$$

5. Find the asymptotes of the hyperbola

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1.$$

6 Show that

(a) $\frac{d}{dx} \sinh x = \cosh x$

(b) $\frac{d}{dx} \cosh x = \sinh x$

(c) $\frac{d}{dx} \tanh x = \frac{1}{\cosh^2 x}$

7. Find the Maclaurin series expansion for $\sinh x$ and $\cosh x$ (These series converge for all x and represent $\sinh x$ and $\cosh x$ respectively for all x)

8 Show that $\sinh x$ and $\cosh x$ are periodic with the imaginary period $2\pi i$

9. (a) If $\sinh x = \sqrt{3}$, find $\cosh x$ and $\tanh x$
(b) If $\tanh x = 0.8$, find $\sinh x$ and $\cosh x$

10 Express the following in the form $a + ib$

(a) $\sinh i$
(b) $\sinh(-2 + 3i)$
(c) $\cosh(-3i)$
(d) $\cosh\left(\frac{\pi}{2} - 4i\right)$
(e) $\cos(\sqrt{2} + i)$
(f) $\sin(-3 + i)$

11 Find the complex numbers $x = a + ib$ such that

(a) $\sin x = 2$
(b) $\cos x = -3$
(c) $\cosh x = 0.5$
(d) $\cosh x = 3$
(e) $\sinh x = -2.6$
(f) $\cos x = 1 + i$
(g) $\sin x = 2i$
(h) $\cos x = 2 - i$
(i) $\sin x = 1 + 2i$
(j) $\tan x = i0.8$

12. Prove that

$$\text{argtanh}\, x = \tfrac{1}{2}\log_e \frac{1+x}{1-x}, \qquad |x| < 1$$

15 FOURIER SERIES

In Chapter 12 we considered infinite series of powers of x. Why not consider infinite series of sines and cosines? We shall do just that in the present chapter. Special types of series of trigonometric functions are called *Fourier series.* Fourier series are extensively used in electrical engineering to determine steady-state response of networks and are used generally in applied mathematics in the solution of boundary value problems in the theory of partial differential equations.

15.1. Periodic Functions

One could consider a series such as

$$a_0 + a_1 \cos \omega_1 x + a_2 \cos \omega_2 x + \cdots + a_n \cos \omega_n x + \cdots, \tag{1}$$

where the ω_i are arbitrary angular frequencies and the a_i are constants. Such a series is not as useful as a series in which every ω_n is an integral multiple of some *fundamental* frequency ω_0, say $\omega_n = n\omega_0$. For example,

$$a_0 + a_1 \cos \omega_0 x + a_2 \cos 2\omega_0 x + \cdots + a_n \cos n\omega_0 x + \cdots \tag{2}$$

is such a series. Thus in this chapter we shall consider series of the form (2) or of the form

$$b_1 \sin \omega_0 x + b_1 \sin 2\omega_0 x + \cdots + b_n \sin n\omega_0 x + \cdots, \tag{3}$$

or, quite generally, series of the form

$$a_0 + \sum_{n=1}^{\infty} (a_n \cos n\omega_0 x + b_n \sin n\omega_0 x) \tag{4}$$

—which represents the sum of (2) and (3). We see that in these three cases the function represented is periodic of period $2\pi/\omega_0$, while (1) may not necessarily represent a periodic function (For example, $\cos x$ and $\cos \pi x$ are each periodic functions, but

$$\cos x + \cos \pi x$$

is *not* periodic)

As an illustration, let us consider the function

$$f(x) = \sin x - \frac{1}{3^2} \sin 3x + \frac{1}{5^2} \sin 5x - \frac{1}{7^2} \sin 7x + \qquad (5)$$

From our discussion of infinite series in Chapter 12, we know that if the infinite series converges the nth partial sum, for n large, will be a good

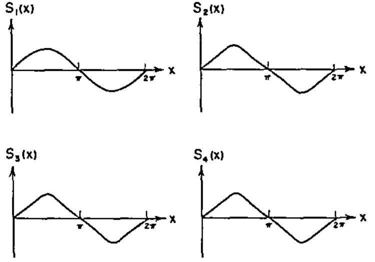


Fig. 15 1

approximation to $f(x)$ Let us therefore plot the partial sums of (5) By definition, they are

$$\begin{aligned} s_1(x) &= \sin x, \\ s_2(x) &= \sin x - \tfrac{1}{9} \sin 3x, \\ s_3(x) &= \sin x - \tfrac{1}{9} \sin 3x + \tfrac{1}{25} \sin 5x, \\ s_4(x) &= \sin x - \tfrac{1}{9} \sin 3x + \tfrac{1}{25} \sin 5x - \tfrac{1}{49} \sin 7x, \end{aligned} \qquad (6)$$

We need only plot $s_1(x)$, $s_2(x)$, $s_3(x)$, $s_4(x)$, from 0 to 2π since, because of the periodicity of the sine, the same function is repeated in every interval of length 2π from $-\infty$ to $+\infty$ An examination of Fig 15 1 reveals an astonishing fact As n gets larger, the partial sums seem to be approaching a "saw-tooth wave " If n is increased without limit, will (5) actually represent the function of Fig 15 2? The answer is *yes*! It can be shown that the infinite

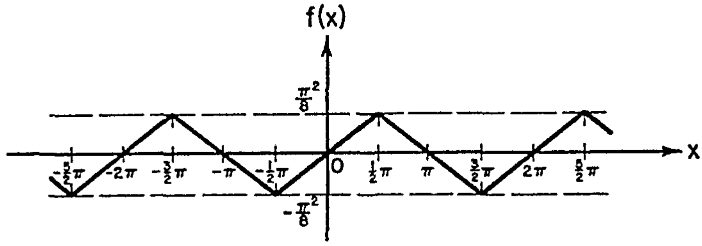


Fig. 15.2

series converges at *every* value of x to the function illustrated in Fig. 15.2! Truly a remarkable result. Equation (5) is called the *Fourier series expansion* of $f(x)$ or simply the *Fourier series* of $f(x)$. We call the coefficients a_n and b_n of (4) the *Fourier coefficients.*

15.2. Fourier Series

The analysis of the last section raises a very interesting question: Given an arbitrary periodic function, is it possible to find an infinite series such as given by (4) which actually converges to this function? In other words, is there a systematic procedure for finding the coefficients a_n and b_n of the expansion of (4) and are there rules for determining the convergence of the series? Again, the answer to this question is yes. It is shown in the theory of Fourier series that, subject to very broad mathematical conditions, periodic functions *do* have expansions in terms of infinite series of sines and cosines, and, furthermore, these Fourier series *do* converge to the given function. However, in order to calculate the coefficients [such as in deducing (5) from Fig. 15.2], a knowledge of the *integral calculus* is required and hence is beyond the scope of this book. Thus we can only tempt the student to study mathematics further.

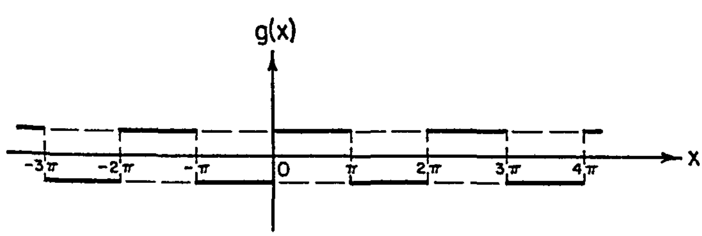


Fig. 15.3

As we have just said, we do not have the mathematical background to systematically derive Fourier series. However, we have often been able in past chapters to deduce interesting and useful results in certain special cases that actually belonged to the domain of higher mathematics. Let us test our ingenuity in the present situation.

Suppose, then, we have a "square wave," as illustrated in Fig 15 3 We note first that $g(x)$ is an *odd* function, that is, $g(-x) = -g(x)$ Hence if it has a Fourier series expansion, it must be of the form of (3) (with $\omega_0 = 1$), that is, it must be the sum only of sines with fundamental frequency 2π Let us therefore write

$$g(x) = \sum_{n=1}^{\infty} b_n \sin nx \tag{7}$$

and see if we can determine the b_n coefficients

We recall that in Chapter 13 we *differentiated* in order to obtain the coefficients of the Maclaurin series expansion Differentiation of (7) would get us nowhere since the derivative of plus or minus one is zero However, consider the following line of reasoning If (7) is to be true, then certainly

$$g(x) \sin mx = \sum_{n=1}^{\infty} b_n \sin nx \sin mx \tag{8}$$

must be a true equation where m is a fixed integer Also, the *areas* under the functions must be equal Let us introduce the notation

$$\mathscr{A}_a^b[f(x)] = \text{area under the curve } f(x) \text{ from } x = a \text{ to } x = b$$

In this notation, we may formally obtain

$$\mathscr{A}_0^{2\pi}[g(x) \sin mx] = \sum_{n=1}^{\infty} b_n \mathscr{A}_0^{2\pi}[\sin nx \sin mx] \tag{9}$$

from (8) But

$$\mathscr{A}_0^{2\pi}[g(x) \sin mx] = \mathscr{A}_0^{\pi}[\sin mx] + \mathscr{A}_{\pi}^{2\pi}[-\sin mx]$$

since, from Fig 15 3,

$$\begin{aligned} g(x) &= +1, \quad 0 < x < \pi \\ g(x) &= -1, \quad \pi < x < 2\pi \end{aligned} \tag{10}$$

A simple extension of (15) of Chapter 10 (see also Problem 8 of Chapter 10) yields

$$\mathscr{A}_0^{\pi}[\sin mx] = \frac{1}{m}[1 - (-1)^m],$$

and

$$\mathscr{A}_{\pi}^{2\pi}[\sin mx] = -\frac{1}{m}[1 - (-1)^m]$$

Thus

$$\mathscr{A}_0^{2\pi}[g(x) \sin mx] = \frac{2}{m}[1 - (-1)^m] \tag{11}$$

Similarly, we see from Chapter 10, Problem 9, that

$$\mathscr{A}_0^{2\pi}[\sin nx \sin mx] = \begin{cases} 0 & n \neq m, \\ \pi & n = m \end{cases}$$

Thus every term of the series on the right-hand side of (9) is zero, except the one corresponding to $n = m$. The value of this term is $b_m\pi$. Thus from (9) and (11),

$$\frac{2}{m}[1 - (-1)^m] = \pi b_m$$

and

$$b_m = \frac{4}{m\pi}, \qquad m \text{ odd}$$

$$b_m = 0, \qquad m \text{ even.}$$

Hence

$$g(x) = \frac{4}{\pi}[\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x + \ldots] \tag{12}$$

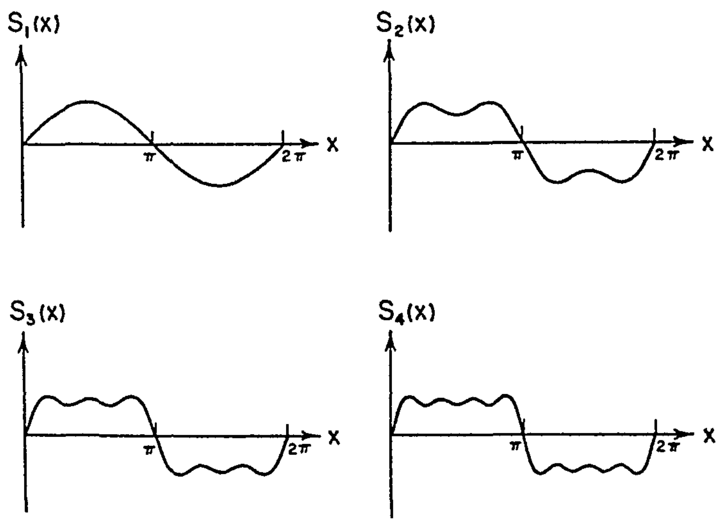


Fig. 15.4

is the Fourier series expansion of $g(x)$. The partial sums

$$s_1(x) = \frac{4}{\pi}\sin x,$$

$$s_2(x) = \frac{4}{\pi}[\sin x + \tfrac{1}{3}\sin 3x],$$

$$s_3(x) = \frac{4}{\pi}[\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x],$$

$$s_4(x) = \frac{4}{\pi}[\sin x + \tfrac{1}{3}\sin 3x + \tfrac{1}{5}\sin 5x + \tfrac{1}{7}\sin 7x]$$

are plotted from 0 to 2π in Fig. 15.4.

It is shown in the mathematical theory of Fourier series that the operation of "taking the area" [see (9)] is a valid operation on this infinite series and that the series on the right-hand side of (12) actually converges to $g(x)$

15.3. Numerical Analysis of Periodic Functions

In many applied problems, especially in electrical engineering, we experimentally determine a periodic function either graphically or in tabular form For example, the record of an alternating current machine or the wave form of an oscilloscope fall into this category Although these wave forms actually have a Fourier series representation, it is not practical to determine an infinite number of coefficients from an empirical function Suppose, then, we agree to *approximate* this record, say $f(x)$, by a *finite* Fourier series of sines and cosines, that is, by a series of the form

$$a_0 + \sum_{n=1}^{N} (a_n \cos \omega_n x + b_n \sin \omega_n x)$$

where $\omega_n = n\omega_0$ and N is fixed and finite In principle, we may solve this problem as follows From an examination of the experimental record $f(x)$ we

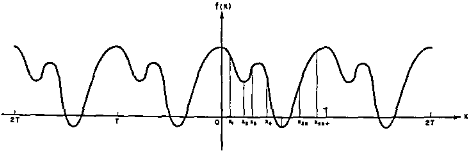


Fig 15 5

determine that the fundamental frequency is, say, ω_0 Then we choose an integer N which we feel is so large that terms of frequency greater than $\omega_N = N\omega_0$ will have but a negligible effect on the wave form Now take any $2N + 1$ points x_k, $k = 1, 2, \dots, 2N + 1$ in some period $[0,T]$ where $T = 2\pi/\omega_0$ From the graph of $f(x)$ (Fig 15 5), we can numerically evaluate $f(x_k)$ Then

$$f(x_k) = a_0 + \sum_{n=1}^{N} (a_n \cos n\omega_0 x_k + b_n \sin n\omega_0 x_k), \quad k = 1, 2, \dots, 2N + 1 \quad (13)$$

becomes a system of $2N + 1$ *linear algebraic* equations on the $2N + 1$ unknowns $a_0, a_1, \dots, a_N, b_1, b_2, \dots, b_N$ The solution of these algebraic equations will yield the approximate values of the Fourier coefficients a_n and b_n

However, there are more systematic and easily applicable rules for determining the coefficients than the above brute force method. One such is the *Runge 6-ordinate scheme* which we shall now expound. This method assumes that

$$E(x) = A_0 + A_1 \cos\frac{2\pi x}{T} + A_2 \cos\frac{4\pi x}{T} + A_3 \cos\frac{6\pi x}{T} + B_1 \sin\frac{2\pi x}{T} + B_2 \sin\frac{4\pi x}{T} \tag{14}$$

is an adequate approximation to $f(x)$. If we divide the interval $[0,T]$ into six equal parts by the points of subdivision

$$\begin{aligned} x_0 &= 0, \\ x_1 &= \frac{T}{6}, \\ x_2 &= \frac{T}{3}, \\ x_3 &= \frac{T}{2}, \\ x_4 &= \frac{2T}{3}, \\ x_5 &= \frac{5T}{6}, \\ x_6 &= T, \end{aligned} \tag{15}$$

then the arguments of the trigonometric functions appearing in (14) are all multiples of $\pi/3$ radians $= 60°$. Thus these trigonometric functions are ± 1, $\pm\frac{1}{2}\sqrt{3}$, $\pm\frac{1}{2}$. Hence we have the six algebraic equations

$$\begin{aligned} \tfrac{1}{2}[E(x_0) + E(x_6)] &= A_0 + A_1 + A_2 + A_3, \\ E(x_1) &= A_0 + \tfrac{1}{2}A_1 - \tfrac{1}{2}A_2 - A_3 + \frac{\sqrt{3}}{2}B_1 + \frac{\sqrt{3}}{2}B_2, \\ E(x_2) &= A_0 - \tfrac{1}{2}A_1 - \tfrac{1}{2}A_2 + A_3 + \frac{\sqrt{3}}{2}B_1 - \frac{\sqrt{3}}{2}B_2 \\ E(x_3) &= A_0 - A_1 + A_2 - A_3, \\ E(x_4) &= A_0 - \tfrac{1}{2}A_1 - \tfrac{1}{2}A_2 + A_3 - \frac{\sqrt{3}}{2}B_1 + \frac{\sqrt{3}}{2}B_2, \\ E(x_5) &= A_0 + \tfrac{1}{2}A_1 - \tfrac{1}{2}A_2 - A_3 - \frac{\sqrt{3}}{2}B_1 - \frac{\sqrt{3}}{2}B_2. \end{aligned} \tag{16}$$

(We use the average of $E(x_0)$ and $E(x_6)$ since they may not be equal) The heart of the Runge scheme is illustrated in the following table which we shall explain below

TABLE 1

$$
\begin{array}{rccc}
 & E_0 & E_1 & E_2 \\
 & E_3 & E_4 & E_5 \\ \hline
\Sigma(F) & F_0 & F_1 & F_2 \\
\Delta(G) & G_0 & G_1 & G_2
\end{array}
$$

$$
\begin{array}{rcc}
 & F_0 & F_1 \\
 & & F_2 \\ \hline
\Sigma(H) & H_0 & H_1 \\
\Delta(K) & & K_1
\end{array}
\qquad\qquad
\begin{array}{rcc}
 & G_0 & G_1 \\
 & & G_2 \\ \hline
\Sigma(L) & L_0 & L_1 \\
\Delta(M) & & M_1
\end{array}
$$

$$
\begin{array}{rc}
 & H_0 \\
 & H_1 \\ \hline
\Sigma(N) & N_0
\end{array}
\qquad\qquad
\begin{array}{rc}
 & L_0 \\
 & M_1 \\ \hline
\Delta(P) & P_0
\end{array}
$$

The explanation of the table is as follows First for simplicity in notation, we have set E_k equal to $E(x_k)$, $k = 0, 1, 2, 3, 4, 5, 6$ and $E_0 = \frac{1}{2}(E_0 + E_6)$ The F_0, F_1, F_2 line represents the sum of the corresponding E_k's That is, $F_0 = E_0' + E_3$, $F_1 = E_1 + E_4$, $F_2 = E_2 + E_5$ while the G_0, G_1, G_2 line represents the difference of the corresponding E_k's That is, $G_0 = E_0 - E_3$, $G_1 = E_1 - E_4$, $G_2 = E_2 - E_5$ The remainder of the table is self-explanatory To show the value of the above table, let us calculate the indicated quantities from (16) We have

$$F_0 = 2A_0 + 2A_2,$$

$$F_1 = 2A_0 - A_2 + \sqrt{3}B_2,$$

$$F_2 = 2A_0 - A_2 - \sqrt{3}B_2,$$

and thus

$$H_0 = 2A_0 + 2A_2,$$

$$H_1 = 4A_0 - 2A_2,$$

while

$$N_0 = 6A_0$$

Hence the first coefficient A_0 is simply $\frac{1}{6}$ of N_0

To calculate A_1

$$G_0 = 2A_1 + 2A_3,$$

$$G_1 = A_1 - 2A_3 + \sqrt{3}B_1,$$

$$G_2 = -A_1 + 2A_3 + \sqrt{3}B_1,$$

and thus

$$L_0 = 2A_1 + 2A_3,$$
$$L_1 = \qquad\qquad 2\sqrt{3}B_1,.$$
$$M_1 = 2A_1 - 4A_3.$$

Hence

$$3A_1 = \tfrac{1}{2}(M_1) + 1 \cdot (L_0).$$

Proceeding in a similar fashion, we can calculate all the coefficients A_0, A_1, A_2, B_1, B_2 as simple linear combinations of H_0, H_1, K_1, L_0, L_1, M_1, N_0, P_0. We summarize our findings in the table below.

TABLE 2.

Multiplier						
$\frac{1}{2}$		M_1	$-H_1$			
$\frac{1}{2}\sqrt{3}$					L_1	K_1
1	N_0	L_0	H_0	P_0		
	$6A_0$	$3A_1$	$3A_2$	$6A_3$	$3B_1$	$3B_2$

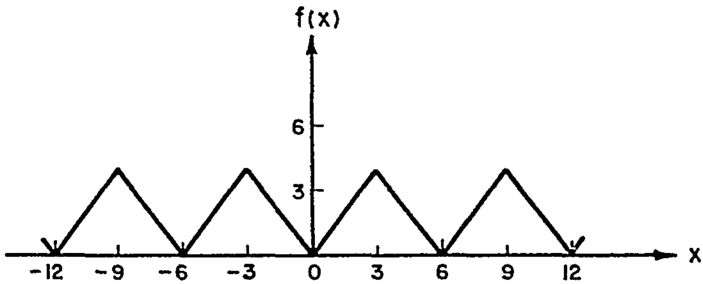


Fig. 15.6

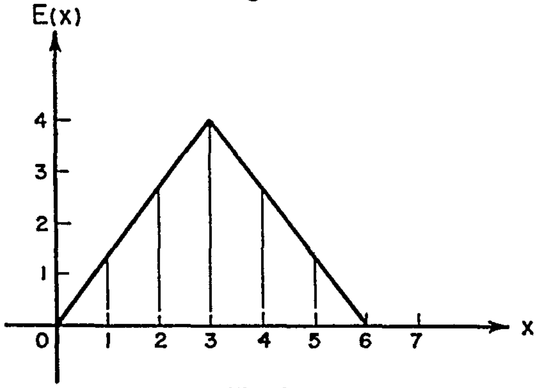


Fig. 15.7

To illustrate the technique, let us apply it to the saw-tooth wave of Fig. 15.6. Then one period of the wave form appears as in Fig. 15.7. The points x_k are therefore

$$x_0 = 0, \; x_1 = 1, \; x_2 = 2, \; x_3 = 3, \; x_4 = 4, \; x_5 = 5, \; x_6 = 6$$

and the corresponding values of E_k are

$$E_0 = 0, E_1 = \tfrac{4}{3}, E_2 = \tfrac{8}{3}, E_3 = 4, E_4 = \tfrac{8}{3}, E_5 = \tfrac{4}{3}$$

Table 1 then becomes, numerically,

	0	$\frac{4}{3}$	$\frac{8}{3}$
	4	$\frac{8}{3}$	$\frac{4}{3}$
$\Sigma(F)$	4	4	4
$\Delta(G)$	-4	$-\frac{4}{3}$	$+\frac{4}{3}$

	4	4
		4
$\Sigma(H)$	4	8
$\Delta(K)$		0

	-4	$-\frac{4}{3}$
		$\frac{4}{3}$
$\Sigma(L)$	-4	0
$\Delta(M)$		$-\frac{8}{3}$

	4		-4
	8		$-\frac{8}{3}$
$\Sigma(N)$	12	$\Delta(P)$	$-\frac{4}{3}$

and Table 2 becomes

0 500		$-\frac{8}{3}$	-8			
0 866					0	0
1 000	12	-4	4	$-\frac{4}{3}$		
	12	$-\frac{16}{3}$	0	$-\frac{4}{3}$	0	0

Thus we obtain

$$A_0 = 2,$$
$$A_1 = -\tfrac{16}{9},$$
$$A_2 = 0,$$
$$A_3 = -\tfrac{2}{9},$$
$$B_1 = 0,$$
$$B_2 = 0,$$

and our Fourier analysis of the saw-tooth wave is

$$E(x) = 2 - 1\ 78 \cos \frac{\pi x}{3} - 0\ 222 \cos \pi x \qquad (17)$$

The corresponding terms of the actual Fourier series expansion of $f(x)$ are

$$2 - \frac{16}{\pi^2} \cos \frac{\pi x}{3} - \frac{16}{9\pi^2} \cos \pi x = 2 - 1\ 621 \cos \frac{\pi x}{3} - 0\ 1801 \cos \pi x, \qquad (18)$$

which does not compare too unfavorably with (17)

There are also 12-ordinate, 24-ordinate, 48-ordinate, and 72-ordinate Runge schemes which are similar in form to the 6-ordinate method determined above. Beside the Runge schemes, we also have the selected-ordinate method of Fischer-Hinnen. It is not our intention to make a thorough analysis of the techniques of this method. The interested student may find some of the details in M. G. Salvadori and K. S. Miller, *The mathematical solution of engineering problems*, Columbia University Press, (1953), p. 210 *ff*.

PROBLEMS

1. Why do you think the square wave of Fig. 15.3 and the saw-tooth wave of Fig. 15.2 cannot be expanded in a Maclaurin series?

2. Find the Fourier series expansion of the periodic function illustrated in Fig. 15.8.

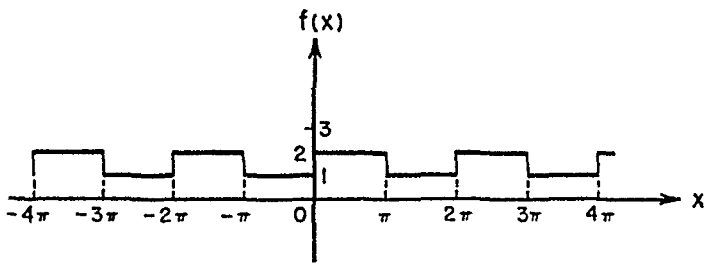


Fig. 15.8

3. Find the Fourier series expansion for the periodic function

$$f(x) = \cos^2 x + 6 \sin^3 x.$$

4. Use the Runge 6-ordinate scheme to approximate the function of Fig. 15.9.

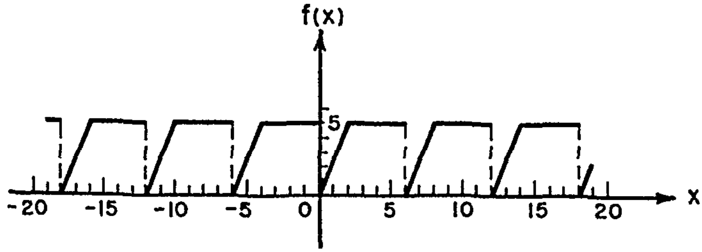


Fig. 15.9

5. Let $g(x) = 1 - x^2$ between $x = -1$ and $x = +1$. If $f(x)$ is a periodic function of period two which is identical with $g(x)$ for $-1 \leqq x \leqq 1$, find the Fourier series approximation to $f(x)$ using the Runge scheme.

6. A function $f(t)$ of period 12 seconds has been tabulated for $t = 0, 1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11, 12$ seconds Use the Runge scheme to determine a trigonometric approximation to $f(t)$

t	$f(t)$
0	0
1	1
2	4
3	9
4	16
5	16
6	16
7	16
8	16
9	12
10	8
11	4
12	0

16
THE TSCHEBYSCHEFF POLYNOMIALS

In various branches of higher mathematics, certain special sets of polynomials play important roles. For example, the *Legendre polynomials* are used to find the solutions of certain boundary value problems in the theory of partial differential equations. The *Hermite polynomials* are used in the theory of probability to obtain Gram-Charlier expansions of density functions. In the theory of noise and stochastic processes, one has occasion to use the *Laguerre polynomials*. *Tschebyscheff polynomials* are used in network and antenna beam synthesis as well as in the design of control systems.

The Tschebyscheff polynomials can be defined in terms of the inverse trigonometric functions. Hence they are appropriate grist for our mill. We shall define them, deduce some of their elementary properties, and then prove a remarkable theorem in the theory of approximations.

16.1. Definition of the Tschebyscheff Polynomials

The Tschebyscheff polynomials may be defined by the equation

$$T_n(x) = \cos (n \arccos x), \qquad n = 0, 1, 2, \cdots. \tag{1}$$

Thus

$$\begin{aligned} T_0(x) &= \cos 0 = 1, \\ T_1(x) &= \cos (\arccos x) = x, \\ T_2(x) &= \cos (2 \arccos x) = 2 \cos^2 (\arccos x) - 1 \\ &= 2x^2 - 1. \end{aligned}$$

In calculating $T_2(x)$, we have used the identity $\cos 2\theta = 2 \cos^2 \theta - 1$ with $\theta = \arccos x$.

If we write

$$u = \arccos x,$$

then

$$T_n(x) = \cos nu \tag{2}$$

Now from (25) of Chapter 11, we have

$$\cos nu = \cos^n u - \binom{n}{2} \cos^{n-2} u \sin^2 u + \binom{n}{4} \cos^{n-4} u \sin^4 u$$
$$- \quad + (-1)^{n/2} \sin^n u, \qquad n = 0, 2, 4, \quad ,$$

and

$$\cos nu = \cos^n u - \binom{n}{2} \cos^{n-2} u \sin^2 u + \binom{n}{4} \cos^{n-4} u \sin^4 u$$
$$- \quad + (-1)^{(n-1)/2} \binom{n}{n-1} \cos u \sin^{n-1} u, \qquad n = 1, 3, 5,$$

Thus in terms of x,

$$T_{2N}(x) = x^{2N} - \binom{2N}{2} x^{2N-2}(1 - x^2) + \binom{2N}{4} x^{2N-4}(1 - x^2)^2$$
$$- \quad + (-1)^N(1 - x^2)^N, \tag{3}$$

and

$$T_{2N+1}(x) = x^{2N+1} - \binom{2N+1}{2} x^{2N-1}(1 - x^2) + \binom{2N+1}{4} x^{2N-3}(1 - x^2)^2$$
$$- \quad + (-1)^N \binom{2N+1}{2N} x(1 - x^2)^N \tag{4}$$

Note that we must have two formulas for $T_n(x)$ corresponding to n even even ($= 2N$) and n odd ($= 2N + 1$)

From (3) and (4), we readily compute explicitly the first few Tschebyscheff polynomials They are

$$\begin{aligned} T_0(x) &= 1, \\ T_1(x) &= x, \\ T_2(x) &= 2x^2 - 1, \\ T_3(x) &= 4x^3 - 3x, \\ T_4(x) &= 8x^4 - 8x^2 + 1, \\ T_5(x) &= 16x^5 - 20x^3 + 5x, \\ T_6(x) &= 32x^6 - 48x^4 + 18x^2 - 1, \\ T_7(x) &= 64x^7 - 112x^5 + 56x^3 - 7x \end{aligned} \tag{5}$$

16 2. Properties of the Tschebyscheff Polynomials

Our first conclusion is that $T_n(x)$ is a polynomial of degree n in x This is clear by an inspection of (3) and (4) Note that every term in these expansions gives rise to a factor involving x^n. As our first nonobvious result, let us calculate the coefficient of x^n in $T_n(x)$.

If we call C_{2N} the coefficient of x^{2N} in $T_{2N}(x)$ for $N > 0$, then from (3),

$$C_{2N} = 1 + \binom{2N}{2} + \binom{2N}{4} + \cdots + \binom{2N}{2N}, \tag{6}$$

and if we call C_{2N+1} the coefficient of x^{2N+1} in $T_{2N+1}(x)$, then from (4),

$$C_{2N+1} = 1 + \binom{2N+1}{2} + \binom{2N+1}{4} + \cdots + \binom{2N+1}{2N}. \tag{7}$$

By the use of the binomial theorem, the sum of (6) and (7) may be explicitly evaluated. First we note that for any p and q

$$(p+q)^{2N} = \sum_{k=0}^{2N} \binom{2N}{k} p^k q^{2N-k}. \tag{8}$$

Thus if we let $p = q = 1$, we obtain

$$2^{2N} = 1 + \binom{2N}{1} + \binom{2N}{2} + \cdots + \binom{2N}{2N}, \tag{9}$$

while if we set $p = -q = 1$ in (8),

$$0 = 1 - \binom{2N}{1} + \binom{2N}{2} - \cdots + \binom{2N}{2N} \tag{10}$$

Adding (9) and (10) yields

$$2^{2N} = 2\left[1 + \binom{2N}{2} + \binom{2N}{4} + \cdots + \binom{2N}{2N}\right],$$

from which it follows that

$$C_{2N} = 2^{2N-1}, \qquad N = 1, 2, \cdots. \tag{11}$$

A similar analysis establishes the fact that

$$C_{2N+1} = 2^{2N}, \qquad N = 0, 1, 2, \cdots. \tag{12}$$

Clearly $C_0 = 1$, and we have evaluated the leading coefficient of every Tschebyscheff polynomial.

Other properties of the Tschebyscheff polynomials that follow from the definition are

$$T_n(1) = \cos(n \arccos 1) = \cos n(2k\pi) = (-1)2^{nk} = +1,$$

and

$$T_n(-1) = \cos(n \arccos -1) = \cos n(2k+1)\pi = (-1)^{n(2k+1)} = (-1)^n.$$

For $\arccos 1 = 2k\pi$ where k is an integer. Thus $n \arccos 1 = n(2k\pi)$ is an even multiple of π and its cosine is $+1$. Similarly, $\arccos(-1) = (2k+1)\pi$

where k is an integer Thus n arccos $(-1) = n(2k + 1)\pi$ which is an even multiple of π if n is even and an odd multiple of π if n is odd In a like manner,

$$T_{2N}(0) = \cos(2N \arccos 0) = \cos 2N\left(\frac{\pi}{2} + k\pi\right) = (-1)^N,$$

and

$$T_{2N+1}(0) = \cos[(2N + 1)\arccos 0] = \cos(2N + 1)\left(\frac{\pi}{2} + k\pi\right) = 0$$

Actually, the above analysis is somewhat redundant since the Tschebyscheff polynomials are uniquely defined For consider the definition

$$T_n(x) = \cos(n \arccos x)$$

Choose an x Then there are many values of u such that

$$u = \arccos x$$

Let u^* be the principal value Then any value of u which satisfies the above equation must be of the form $\pm u^* + 2k\pi$ where k is an integer, positive, negative or zero Thus

$$\begin{aligned} T_n(x) &= \cos(n \arccos x) = \cos nu = \cos n(\pm u^* + 2k\pi) \\ &= \cos(\pm nu^* + 2kn\pi) \end{aligned}$$

But $2nk$ is an even integer and $\cos \xi$ is an even function Thus

$$T_n(x) = \cos nu^*$$

Hence it would have amounted to the same thing if we had simply used the principal value of u to begin with

We now continue with our discussion of the properties of the Tschebyscheff polynomials Since

$$-1 \leqq \cos nu \leqq 1$$

for all real u, we infer that $-1 \leqq x \leqq 1$ and that

$$|T_n(x)| \leqq 1$$

Thus $T_n(x)$ is always less than or equal to one in absolute value for all x which also have this property

It is also clear that the distinct roots of $T_n(x) = 0$ occur when

$$n \arccos x = \frac{\pi}{2} + k\pi, \qquad k = 0\ 1, 2, \quad , n - 1 \tag{13}$$

since

$$T_n(x) = \cos(n \arccos x) = \cos\left(\frac{\pi}{2} + k\pi\right) = 0$$

From (13),

$$x_k = \cos\frac{(2k+1)\pi}{2n} \qquad k = 0, 1, \cdots, n-1 \tag{14}$$

are the n roots of $T_n(x) = 0$. Note that all the roots lie between -1 and $+1$, that is, $-1 < x_k < 1$ for $k = 0, 1, \cdots, n-1$. Clearly the x_k are distinct. But $T_n(x)$ is a polynomial of degree n. Thus the above remarks imply that *all* the roots of $T_n(x)$ are *real* and lie in the interval $-1 < x < 1$.

16.3. An Approximation Problem

In Fig. 16.1, $P(x)$ is a polynomial plotted from -1 to $+1$. An interesting problem is to minimize the "overshoot" and "undershoot," that is, the deviation of $P(x)$ from zero over this interval. In Fig. 16.1, a_1, a_3, a_5 are "overshoots" and a_2, a_4, a_6 are "undershoots." The problem is to determine

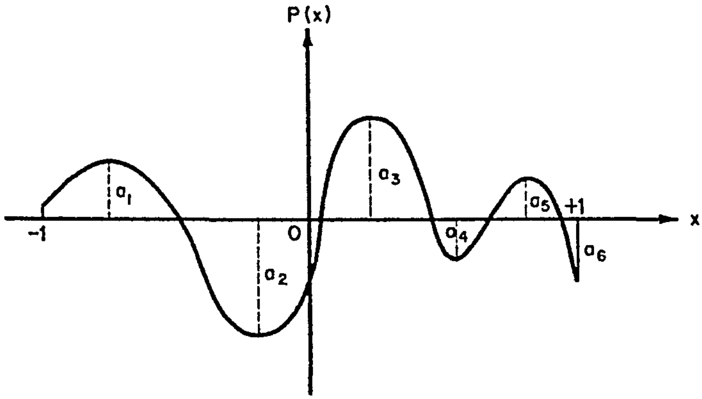


Fig. 16.1

a polynomial of degree n whose maximum deviation from zero (that is, the largest in absolute of the overshoots and undershoots) is as small as possible. Thus if $P_n(x)$ is a polynomial of degree n and we define α_P as the maximum absolute deviation from zero,

$$\alpha_P = \max |P_n(x)|,$$

then the problem is to find a polynomial $P_n^*(x)$ with the property that

$$\alpha_{P*} \leqq \alpha_P$$

for any other polynomial $P_n(x)$ of degree n.

Of course, the above problem is meaningless unless we *normalize* the polynomials. For example, if $P_n(x)$ is any polynomial of degree n, then we can always find another polynomial $Q_n(x)$ of degree n such that

$$\alpha_Q < \alpha_P$$

by the simple expedient of letting $Q_n(x) = \lambda P_n(x)$ with $|\lambda| < 1$ To avoid this situation, let us agree that we shall consider only polynomials whose leading coefficient is one Thus if

$$P_n(x) = ax^n + \quad ,$$

the polynomial

$$p_n(x) = \frac{1}{a} P_n(x) = x^n + \quad \cdot$$

will be a polynomial with leading coefficient one We call $P_n(x)$ the *normalized polynomial* associated with $P_n(x)$

We may now clearly state the approximation problem

Theorem The normalized Tschebyscheff polynomial $t_n(x)$ is the polynomial of degree n with leading coefficient one whose maximum absolute value is smallest in the interval $-1 \leq x \leq 1$

Proof The normalized Tschebyscheff polynomials $t_n(x)$ are defined by the equations

$$t_0(x) = 1,$$

$$t_n(x) = \frac{1}{2^{n-1}} T_n(x), \qquad n = 1, 2,$$

From (11) and (12), it is clear that $t_n(x)$ has leading coefficient one Let $u =$ arccos x, as before Then

$$t_n(x) = \frac{1}{2^{n-1}} \cos nu$$

The maximum deviation from zero occurs when $\cos nu = \pm 1$ That is, when nu is an integral multiple of π,

$$u = \frac{k\pi}{n}, \qquad k = 0, 1, \quad , n$$

If we let

$$\xi_k = \cos \frac{k\pi}{n},$$

then

$$t_n(\xi_k) = \frac{(-1)^k}{2^{n-1}},$$

while for $\xi \neq \xi_k$,

$$|t_n(\xi)| < |t_n(\xi_k)|$$

Now suppose $r_n(x)$ is a polynomial of degree n with leading coefficient one whose deviation from zero in the interval $-1 \leq x < 1$ is less than that of $t_n(x)$ Let

$$q_n(x) = t_n(x) - r_n(x)$$

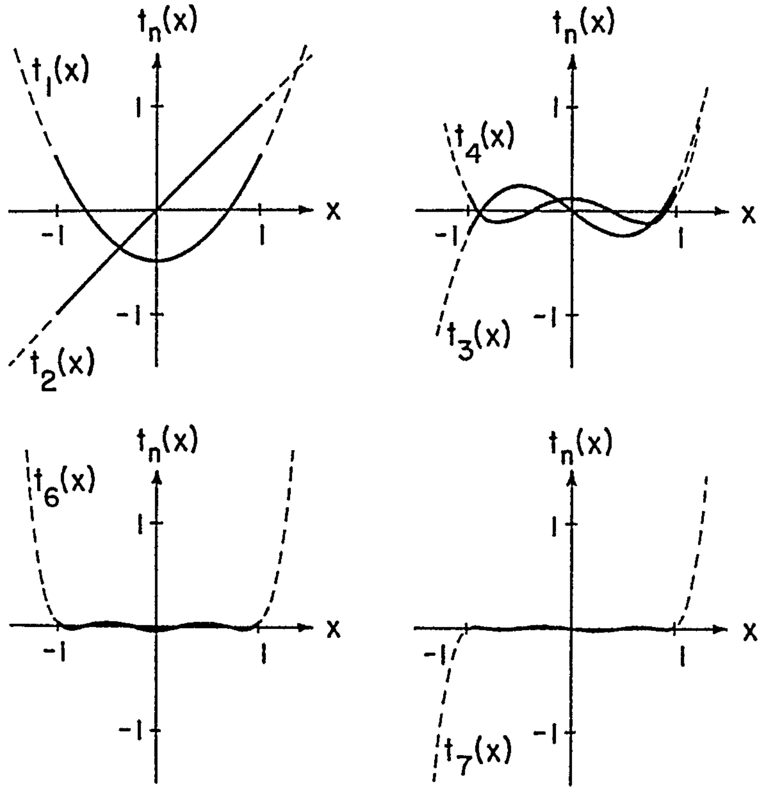


Fig. 16.2

Then $q_n(x)$ is of degree at most $n - 1$ and

$$q_n(\xi_k) > 0, \qquad k \text{ even},$$
$$q_n(\xi_k) < 0, \qquad k \text{ odd}.$$

Since $q_n(x)$ is alternately positive and negative at the successive points ξ_0, $\xi_1, \cdots, \xi_n$, it must have at least n roots in the interval $-1 \leqq x \leqq 1$. But this is impossible since $q_n(x)$ is of degree at most $n - 1$.

The behavior of the normalized Tschebyscheff polynomials in the interval $[-1, 1]$ is seen in Fig. 16.2.

16.4. A Recursion Formula

We may write

$$T_n(x) = \cos nu$$

as before where $u = \arccos x$. Now

$$T_{n+1}(x) = \cos (n + 1)u = \cos nu \cos u - \sin nu \sin u, \tag{15}$$

and

$$T_{n-1}(x) = \cos(n-1)u = \cos nu \cos u + \sin nu \sin u \tag{16}$$

Adding (15) and (16) we have

$$T_{n+1}(x) + T_{n-1}(x) = 2\cos nu \cos u = 2T_n(x)T_1(x)$$

But $T_1(x) = x$. Thus we have the identity

$$T_{n+1}(x) = 2xT_n(x) - T_{n-1}(x) \tag{17}$$

Equation (17) is known as a *recursion* formula.

One use of the recursion formula is to enable us to compute higher order Tschebyscheff polynomials in terms of those of lower order. For example, we have $T_k(x)$ for $k = 0, 1, 2, 3, 4, 5, 6, 7$ from (5). Now $T_8(x)$ could be calculated directly from (3) [see also (27) of Chapter 11] but it is easier to use (17). We do this

$$\begin{aligned} T_8(x) &= 2xT_7(x) - T_6(x) \\ &= 128x^8 - 224x^6 + 112x^4 - 14x^2 - 32x^6 + 48x^4 - 18x^2 + 1 \\ &= 128x^8 - 256x^6 + 160x^4 - 32x^2 + 1 \end{aligned}$$

One could now readily compute $T_9(x)$, etc.

PROBLEMS

1 Explicitly compute $T_9(x)$, $T_{10}(x)$, $T_{11}(x)$ and $T_{12}(x)$.

2 Show that the Tschebyscheff polynomials satisfy the recursion formula

$$T_{n+k}(x) + T_{n-k}(x) = 2T_n(x)T_k(x), \qquad n > k \geq 0$$

3 Prove that

$$\sum_{k=0}^{n} (2x)^k T_k(x) = (2x)^{n+1}T_{n-1}(x) - T_0(x)$$

4 Show that $T_n(x)$ may be factored as

$$T_n(x) = 2^{n-1}\left(x - \cos\frac{\pi}{2n}\right)\left(x - \cos\frac{3\pi}{2n}\right)\cdots\left(x - \cos\frac{(2n-1)\pi}{2n}\right)$$

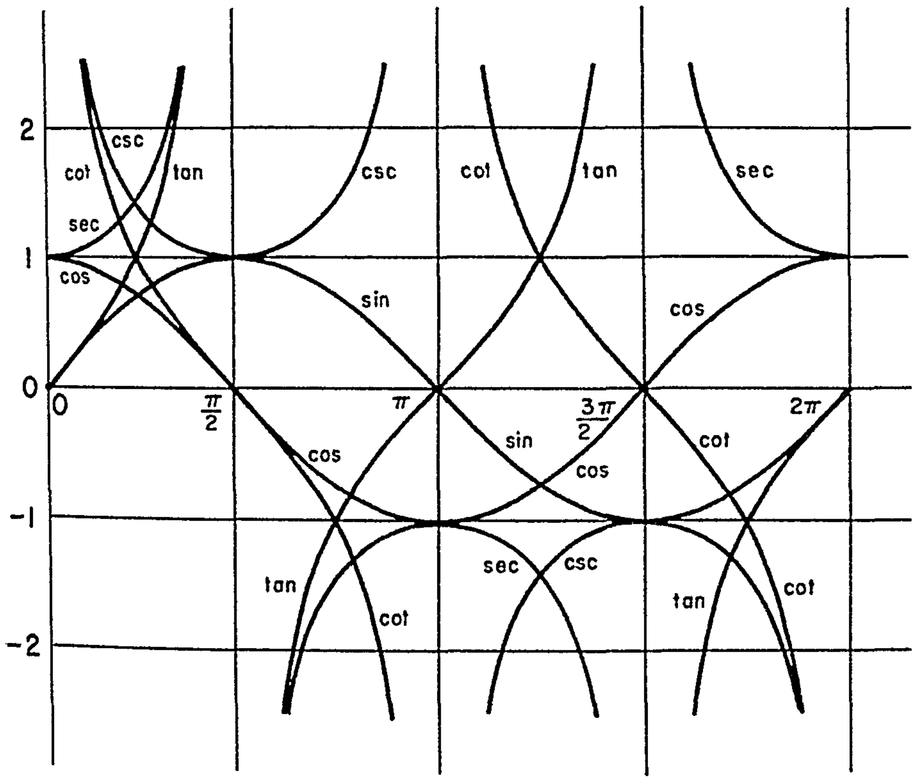
5 Establish the identity

$$\sin\frac{\pi}{2n}\sin\frac{3\pi}{2n}\cdots\sin\frac{(2n-1)\pi}{2n} = \frac{1}{2^{n-1}}$$

6 Prove that

$$\begin{aligned} T_{2n}\left(\frac{\sqrt{2}}{2}\right) &= 0 \qquad \text{if } n \text{ is odd} \\ &= (-1)^{n/2} \qquad \text{if } n \text{ is even} \end{aligned}$$

APPENDIX


2
1
0
-1
-2
0
π/2
π
3π/2
2π
csc
cot
tan
sec
cos
csc
cot
tan
sec
sin
cos
cos
sin
cot
cos
tan
cot
sec
csc
tan
cot

TRIGONOMETRIC IDENTITIES

$$\sin^2 x + \cos^2 x = 1$$
$$1 + \tan^2 x = \sec^2 x$$
$$1 + \cot^2 x = \csc^2 x$$

$$\sin(x + y) = \sin x \cos y + \cos x \sin y, \qquad \sin 2x = 2 \sin x \cos x$$
$$\cos(x + y) = \cos x \cos y - \sin x \sin y, \qquad \cos 2x = \cos^2 x - \sin^2 x = 2\cos^2 x - 1 = 1 - 2\sin^2 x$$

$$\tan(x + y) = \frac{\tan x + \tan y}{1 - \tan x \tan y}, \qquad \tan 2x = \frac{2 \tan x}{1 - \tan^2 x}$$

$$\sin 3x = 3 \sin x - 4 \sin^3 x, \qquad \cos 3x = 4\cos^3 x - 3 \cos x$$

$$\sin\frac{x}{2} = +\sqrt{\frac{1 - \cos x}{2}} \quad \text{if } \frac{x}{2} \text{ is in I or II quadrant}$$
$$= -\sqrt{\frac{1 - \cos x}{2}} \quad \text{if } \frac{x}{2} \text{ is in III or IV quadrant}$$

$$\cos\frac{x}{2} = +\sqrt{\frac{1 + \cos x}{2}} \quad \text{if } \frac{x}{2} \text{ is in I or IV quadrant}$$
$$= -\sqrt{\frac{1 + \cos x}{2}} \quad \text{if } \frac{x}{2} \text{ is in II or III quadrant}$$

$$\tan\frac{x}{2} = \frac{1 - \cos x}{\sin x} = \frac{\sin x}{1 + \cos x}$$
$$= +\sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad \text{if } \frac{x}{2} \text{ is in I or III quadrant}$$
$$= -\sqrt{\frac{1 - \cos x}{1 + \cos x}} \quad \text{if } \frac{x}{2} \text{ is in II or IV quadrant}$$

$$\sin x + \sin y = 2 \sin \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$
$$\cos x + \cos y = 2 \cos \tfrac{1}{2}(x + y) \cos \tfrac{1}{2}(x - y)$$
$$\cos x - \cos y = -2 \sin \tfrac{1}{2}(x + y) \sin \tfrac{1}{2}(x - y)$$
$$\sin x \sin y = \tfrac{1}{2}[\cos(x - y) - \cos(x + y)]$$
$$\sin x \cos y = \tfrac{1}{2}[\sin(x - y) + \sin(x + y)]$$
$$\cos x \cos y = \tfrac{1}{2}[\cos(x - y) + \cos(x + y)]$$

EXPONENTIAL DEFINITIONS OF FUNCTIONS

$$\sin x = \frac{e^{ix} - e^{-ix}}{2i} \qquad \sinh x = \frac{e^x - e^{-x}}{2}$$

$$\cos x = \frac{e^{ix} + e^{-ix}}{2} \qquad \cosh x = \frac{e^x + e^{-x}}{2}$$

$$\tan x = \frac{e^{ix} - e^{-ix}}{i(e^{ix} + e^{-ix})} \qquad \tanh x = \frac{e^x - e^{-x}}{e^x + e^{-x}}$$

HYPERBOLIC FUNCTIONS

$$\cosh^2 x - \sinh^2 x = 1$$

$$1 - \tanh^2 x = \operatorname{sech}^2 x$$

$$1 - \coth^2 x = -\operatorname{csch}^2 x$$

$$\sinh (x + y) = \sinh x \cosh y + \cosh x \sinh y, \qquad \sinh 2x = 2 \sinh x \cosh x$$

$$\cosh (x + y) = \cosh x \cosh y + \sinh x \sinh y, \qquad \cosh 2x = \cosh^2 x + \sinh^2 x = 2 \cosh^2 x - 1 = 1 + 2 \sinh^2 x$$

$$\tanh (x + y) = \frac{\tanh x + \tanh y}{1 + \tanh x \tanh y}, \qquad \tanh 2x = \frac{2 \tanh x}{1 + \tanh^2 x}$$

$$\sinh \frac{x}{2} = +\sqrt{\frac{\cosh x - 1}{2}} \quad \text{if } x > 0$$

$$= -\sqrt{\frac{\cosh x - 1}{2}} \quad \text{if } x < 0$$

$$\cosh \frac{x}{2} = \sqrt{\frac{\cosh x + 1}{2}}$$

$$\tanh \frac{x}{2} = \frac{\cosh x - 1}{\sinh x} = \frac{\sinh x}{\cosh x + 1}$$

$$= +\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \quad \text{if } x > 0$$

$$= -\sqrt{\frac{\cosh x - 1}{\cosh x + 1}} \quad \text{if } x < 0$$

$$\sinh x + \sinh y = 2 \sinh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y)$$

$$\cosh x + \cosh y = 2 \cosh \tfrac{1}{2}(x + y) \cosh \tfrac{1}{2}(x - y)$$

$$\cosh x - \cosh y = 2 \sinh \tfrac{1}{2} (x + y) \sinh \tfrac{1}{2}(x - y)$$

$$\sinh x \sinh y = \tfrac{1}{2}[\cosh (x + y) - \cosh (x - y)]$$

$$\sinh x \cosh y = \tfrac{1}{2}[\sinh (x + y) + \sinh (x - y)]$$

$$\cosh x \cosh y = \tfrac{1}{2}[\cosh (x + y) + \cosh (x - y)]$$

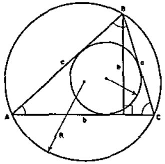
RELATIONS OF CIRCULAR AND HYPERBOLIC FUNCTIONS

$$\sin x = -i \sinh ix \qquad \sinh x = -i \sin ix$$

$$\cos x = \cosh ix \qquad \cosh x = \cos ix$$

$$\tan x = -i \tanh ix \qquad \tanh x = -i \tan ix$$

SOLUTION OF OBLIQUE TRIANGLES



General Relations

r = radius of inscribed circle

R = radius of circumscribed circle

S = area of triangle

$A + B + C = 180°$

$s = \frac{1}{2}(a + b + c)$

$S = rs$

$$r = \sqrt{\frac{(s-a)(s-b)(s-c)}{s}} \qquad R = \frac{abc}{4S} = \frac{abc}{4rs}$$

Law of sines $\quad \dfrac{a}{\sin A} = \dfrac{b}{\sin B} = \dfrac{c}{\sin C}$

Law of cosines $\quad a^2 = b^2 + c^2 - 2bc \cos A$

Law of tangents $\quad \dfrac{a+b}{a-b} = \dfrac{\tan \frac{1}{2}(A+B)}{\tan \frac{1}{2}(A-B)} = \dfrac{\cot \frac{1}{2}C}{\tan \frac{1}{2}(A-B)}$

Solution of Triangles

1 SAS Given two sides (b and c) and included angle (A) Use law of cosines or

$$\tan \tfrac{1}{2}(B - C) = \frac{b-c}{b+c} \cot \tfrac{1}{2}C, \quad \tfrac{1}{2}(B + C) = 90° - \tfrac{1}{2}A, \quad B = \tfrac{1}{2}(B + C) + \tfrac{1}{2}(B - C),$$

$$C = \tfrac{1}{2}(B + C) - \tfrac{1}{2}(B - C), \quad a = \frac{b \sin A}{\sin B} \quad S = \tfrac{1}{2}bc \sin A$$

2 ASA, SAA Given two angles (A and B) and one side (say c)

$$C = 180° - (A + B), \quad a = c\frac{\sin A}{\sin C}, \quad b = c\frac{\sin B}{\sin C}, \quad S = \frac{c^2 \sin A \sin B}{2 \sin C}$$

3 SSS Given three sides Use law of cosines or

$$\tan \tfrac{1}{2}A = \frac{r}{s-a}, \quad \tan \tfrac{1}{2}B = \frac{r}{s-b}, \quad \tan \tfrac{1}{2}C = \frac{r}{s-c}, \quad S = rs$$

4 SSA Given two sides (a and b) and an angle opposite one (say A) This case may have no solution, one solution, or two solutions corresponding to $a < b \sin A$ $a = b \sin A$, and $a > b \sin A$

$$\sin B = \frac{b \sin A}{a}, \quad C = 180° - (A + B), \quad c = \frac{a \sin C}{\sin A}, \quad S = \tfrac{1}{2}ab \sin C$$

CHECKS ON NUMERICAL WORK

$$\frac{a+b}{c} = \frac{\cos \frac{1}{2}(A-B)}{\sin \frac{1}{2}C}, \quad \frac{a-b}{c} = \frac{\sin \frac{1}{2}(A-B)}{\cos \frac{1}{2}C}$$

SOME FUNDAMENTAL CONSTANTS

$\pi = 3.141\ 592\ 65$	$\log \pi = 0.497\ 149\ 9$
$\frac{\pi}{4} = 0.785\ 398\ 16$	$\log \frac{\pi}{4} = 9.895\ 089\ 9\text{–}10$
$\frac{1}{\pi} = 0.318\ 309\ 89$	$\log \frac{1}{\pi} = 9.502\ 850\ 1\text{–}10$
$\sqrt{\pi} = 1.772\ 453\ 85$	$\log \sqrt{\pi} = 0.248\ 574\ 9$
$\frac{1}{\sqrt{\pi}} = 0.564\ 189\ 58$	$\log \frac{1}{\sqrt{\pi}} = 9.751\ 425\ 1\text{–}10$
$e = 2.718\ 281\ 83$	$\log e = 0.434\ 294\ 5$

$$1 \text{ radian} = 57.295\ 779\ 5 \text{ degrees}$$
$$= 57^\circ\ 17'\ 44.80625''$$
$$1 \text{ degree} = 0.017\ 453\ 292\ 5 \text{ radian}$$

THE GREEK ALPHABET

Α	α	Alpha	Ν	ν	Nu
Β	β	Beta	Ξ	ξ	Xi
Γ	γ	Gamma	Ο	ο	Omicron
Δ	δ	Delta	Π	π	Pi
Ε	ε	Epsilon	Ρ	ρ	Rho
Ζ	ζ	Zeta	Σ	σ	Sigma
Η	η	Eta	Τ	τ	Tau
Θ	θ	Theta	Υ	υ	Upsilon
Ι	ι	Iota	Φ	φ	Phi
Κ	κ	Kappa	Χ	χ	Chi
Λ	λ	Lambda	Ψ	ψ	Psi
Μ	μ	Mu	Ω	ω	Omega

Table I. NATURAL TRIGONOMETRIC FUNCTIONS

Degrees	sin	csc	tan	cot	sec	cos	
0° 0′	0000	——	0000	——	1 000	1 0000	90° 0
10′	029	343 8	029	343 77	000	000	50′
20′	058	171 9	058	171 89	000	000	40′
30′	087	114 6	087	114 59	000	1 0000	30′
40′	116	85 95	116	85 940	000	0 9999	20′
50′	145	68 76	145	68 750	000	999	10′
1° 0′	0175	57 30	0175	57 290	1 000	9998	89° 0′
10′	204	49 11	204	49 104	000	998	50′
20′	233	42 98	233	42 964	000	997	40′
30′	262	38 20	262	38 188	000	997	30′
40′	291	34 38	291	34 368	000	996	20′
50′	320	31 26	320	31 242	001	995	10′
2° 0	0349	28 65	0349	28 636	1 001	9994	88° 0
10	378	26 45	378	26 432	001	993	50′
20′	407	24 56	407	24 542	001	992	40′
30′	436	22 93	437	22 904	001	990	30′
40′	465	21 49	466	21 470	001	989	20′
50′	494	20 23	495	20 206	001	988	10′
3° 0	0523	19 11	0524	19 081	1 001	9986	87 0
10′	552	18 10	553	18 075	002	985	50′
20	581	17 20	582	17 169	002	983	40
30′	601	16 38	612	16 350	002	981	30′
40′	640	15 64	641	15 605	002	980	20′
50′	669	14 96	670	14 924	002	978	10
4° 0	0698	14 34	0699	14 301	1 002	9976	86° 0
10	727	13 76	729	13 727	003	974	50′
20	756	13 23	758	13 197	003	971	40′
30′	785	12 75	787	12 706	003	969	30′
40′	814	12 29	816	12 251	003	967	20′
50′	843	11 87	846	11 826	004	964	10′
5° 0	0872	11 47	0875	11 430	1 004	9962	85° 0
10′	901	11 10	904	11 059	004	959	50
20′	929	10 76	934	10 712	004	957	40′
30′	958	10 43	963	10 385	005	954	30′
40′	0987	10 13	0992	10 078	005	951	20′
50′	1016	9 839	1022	9 7882	005	948	10′
6° 0	1045	9 567	1051	9 5144	1 006	9945	84° 0′
10′	074	9 309	080	9 2553	006	942	50
20′	103	9 065	110	9 0098	006	939	40′
30′	132	8 834	139	8 7769	006	936	30′
40′	161	8 614	169	8 5555	007	932	20′
50′	190	8 405	198	8 3450	007	929	10′
7° 0′	1219	8 206	1228	8 1443	1 008	9925	83° 0
10′	248	8 016	257	7 9530	008	922	50′
20′	276	7 834	287	7 7704	008	918	40′
30′	305	7 661	317	7 5958	009	914	30′
40′	334	7 496	346	7 4287	009	911	20′
50′	363	7 337	376	7 2687	009	907	10′
8° 0′	1392	7 185	1405	7 1154	1 010	9903	82° 0
10	421	7 040	435	6 9682	010	899	50
20′	449	6 900	465	6 8269	011	894	40′
30′	478	6 765	495	6 6912	011	890	30′
40′	507	6 636	524	6 5606	012	886	20′
50′	536	6 512	554	6 4348	012	881	10′
9° 0	1564	6 392	1584	6 3138	1 012	9877	81° 0
	cos	sec	cot	tan	csc	sin	Degrees

Table I (Continued). NATURAL TRIGONOMETRIC FUNCTIONS

Degrees	sin	csc	tan	cot	sec	cos	
9° 0′	.1564	6.392	.1584	6.3138	1.012	.9877	81° 0′
10′	593	277	614	1970	013	872	50′
20′	622	166	644	6.0844	013	868	40′
30′	650	6.059	673	5.9758	014	863	30′
40′	679	5.955	703	8708	014	858	20′
50′	708	855	733	7694	015	853	10′
10° 0′	.1736	5.759	.1763	5.6713	1.015	.9848	80° 0′
10′	765	665	793	5764	016	843	50′
20′	794	575	823	4845	016	838	40′
30′	822	487	853	3955	017	833	30′
40′	851	403	883	3093	018	827	20′
50′	880	320	914	2257	018	822	10′
11° 0′	.1908	5.241	.1944	5.1446	1.019	.9816	79° 0′
10′	937	164	.1974	5.0658	019	811	50′
20′	965	089	.2004	4.9894	020	805	40′
30′	.1994	5.016	035	9152	020	799	30′
40′	.2022	4.945	065	8430	021	793	20′
50′	051	876	095	7729	022	787	10′
12° 0′	.2079	4.810	.2126	4.7046	1.022	.9781	78° 0′
10′	108	745	156	6382	023	775	50′
20′	136	682	186	5736	024	769	40′
30′	164	620	217	5107	024	9763	30′
40′	193	560	247	4494	025	757	20′
50′	221	502	278	3897	026	750	10′
13° 0′	.2250	4.445	.2309	4.3315	1.026	.9744	77° 0′
10′	278	390	339	2747	027	737	50′
20′	306	336	370	2193	028	730	40′
30′	334	284	401	1653	028	724	30′
40′	363	232	432	1126	029	717	20′
50′	391	182	462	0611	030	710	10′
14° 0′	.2419	4.134	.2493	4.0108	1.031	.9703	76° 0′
10′	447	086	524	3.9617	031	696	50′
20′	476	4.039	555	9136	032	689	40′
30′	504	3.994	586	8667	033	681	30′
40′	532	950	617	8208	034	674	20′
50′	560	906	648	7760	034	667	10′
15° 0′	.2588	3.864	.2679	3.7321	1.035	.9659	75° 0′
10′	616	822	711	6891	036	652	50′
20′	644	782	742	6470	037	644	40′
30′	672	742	773	6059	038	636	30′
40′	700	703	805	5656	039	628	20′
50′	728	665	836	5261	039	621	10′
16° 0′	.2756	3.628	.2867	3.4874	1.040	.9613	74° 0′
10′	784	592	899	4495	041	605	50′
20′	812	556	931	4124	042	596	40′
30′	840	521	962	3759	043	588	30′
40′	868	487	.2994	3402	044	580	20′
50′	896	453	.3026	3052	045	572	10′
17° 0′	.2924	3.420	.3057	3.2709	1.046	.9563	73° 0′
10′	952	388	089	2371	047	555	50′
20′	.2979	357	121	2041	048	546	40′
30′	.3007	326	153	1716	049	537	30′
40′	035	295	185	1397	049	528	20′
50′	062	265	217	1084	050	520	10′
18° 0′	.3090	3.236	.3249	3.0777	1.051	.9511	72° 0′
	cos	sec	cot	tan	csc	sin	Degrees

Table I (Continued). NATURAL TRIGONOMETRIC FUNCTIONS

Degrees	sin	csc	tan	cot	sec	cos	
18° 0′	3090	3 236	.3249	3 0777	1 051	9511	72° 0
10′	118	207	281	0475	052	502	50′
20′	145	179	314	3 0178	053	492	40′
30′	173	152	346	2 9887	054	483	30′
40′	201	124	378	9600	056	474	20′
50′	228	098	411	9319	057	465	10′
19° 0′	3256	3 072	3443	2 9042	1 058	9455	71° 0′
10	283	046	476	8770	059	446	50′
20′	311	3 021	508	8502	060	436	40′
30′	338	2 996	541	8239	061	426	30′
40′	365	971	574	7980	062	417	20′
50′	393	947	607	7725	063	407	10′
20° 0	3420	2 924	3640	2 7475	1 064	9397	70° 0′
10′	448	901	673	7228	065	387	50′
20′	475	878	706	6985	066	377	40′
30′	502	855	739	6746	068	367	30′
40′	529	833	772	6511	069	356	20′
50	557	812	805	6279	070	346	10′
21° 0	3584	2 790	3839	2 6051	1 071	9336	69° 0
10′	611	769	872	5826	072	325	50′
20′	638	749	906	5605	074	315	40′
30′	665	729	939	5386	075	304	30′
40′	692	709	3973	5172	076	293	20′
50′	719	689	4006	4960	077	283	10′
22° 0′	3746	2 669	4040	2 4751	1 079	9272	68° 0
10	773	650	074	4545	080	261	50
20′	800	632	108	4342	081	250	40′
30′	827	613	142	4142	082	239	30′
40′	854	595	176	3945	084	228	20′
50	881	577	210	3750	085	216	10′
23° 0	3907	2 559	4245	2 3559	1 086	9205	67° 0
10	934	542	279	3369	088	194	50′
20′	961	525	314	3183	089	182	40′
30′	3987	508	348	2998	090	171	30′
40′	4014	491	383	2817	092	159	20′
50′	041	475	417	2637	093	147	10′
24° 0′	4067	2 459	4452	2 2460	1 095	9135	66° 0
10′	094	443	487	2286	096	124	50′
20	120	427	522	2113	097	112	40′
30′	147	411	557	1943	099	100	30′
40′	173	396	592	1775	100	088	20′
50′	200	381	628	1609	102	075	10′
25° 0	4226	2 366	4663	2 1445	1 103	9063	65° 0′
10′	253	352	699	1283	105	051	50′
20′	279	337	734	1123	106	038	40′
30′	305	323	770	0965	108	026	30′
40′	331	309	806	0809	109	013	20′
50′	358	295	841	0655	111	9001	10′
26° 0′	4384	2 281	4877	2 0503	1 113	.8988	64° 0′
10′	410	268	913	0353	114	975	50′
20′	436	254	950	0204	116	962	40′
30′	462	241	4986	2 0057	117	949	30′
40′	488	228	5022	1 9912	119	936	20′
50′	514	215	059	9768	121	923	10′
27° 0	4540	2 203	5095	1 9626	1 122	8910	63° 0
	cos	sec	cot	tan	csc	sin	Degrees

Degrees	sin	csc	tan	cot	sec	cos	
27° 0′	.4540	2.203	.5095	1.9626	1.122	.8910	63° 0′
10′	566	190	132	9486	124	897	50′
20′	592	178	169	9347	126	884	40′
30′	617	166	206	9210	127	870	30′
40′	643	154	243	9074	129	857	20′
50′	669	142	280	8940	131	843	10′
28° 0′	.4695	2.130	.5317	1.8807	1.133	.8829	62° 0′
10′	720	118	354	8676	134	816	50′
20′	746	107	392	8546	136	802	40′
30′	772	096	430	8418	138	788	30′
40′	797	085	467	8291	140	774	20′
50′	823	074	505	8165	142	760	10′
29° 0′	.4848	2.063	.5543	1.8040	1.143	.8746	61° 0′
10′	874	052	581	7917	145	732	50′
20′	899	041	619	7796	147	718	40′
30′	924	031	658	7675	149	704	30′
40′	950	020	696	7556	151	689	20′
50′	.4975	010	735	7437	153	675	10′
30° 0′	.5000	2.000	.5774	1.7321	1.155	.8660	60° 0′
10′	025	1.990	812	7205	157	646	50′
20′	050	980	851	7090	159	631	40′
30′	075	970	890	6977	161	616	30′
40′	100	961	930	6864	163	601	20′
50′	125	951	.5969	6753	165	587	10′
31° 0′	.5150	1.942	.6009	1.6643	1.167	.8572	59° 0′
10′	175	932	048	6534	169	557	50′
20′	200	923	088	6426	171	542	40′
30′	225	914	128	6319	173	526	30′
40′	250	905	168	6212	175	511	20′
50′	275	896	208	6107	177	496	10′
32° 0′	.5299	1.887	.6249	1.6003	1.179	.8480	58° 0′
10′	324	878	289	5900	181	465	50′
20′	348	870	330	5798	184	450	40′
30′	373	861	371	5697	186	434	30′
40′	398	853	412	5597	188	418	20′
50′	422	844	453	5497	190	403	10′
33° 0′	.5446	1.836	.6494	1.5399	1.192	.8387	57° 0′
10′	471	828	536	5301	195	371	50′
20′	495	820	577	5204	197	355	40′
30′	519	812	619	5108	199	339	30′
40′	544	804	661	5013	202	323	20′
50′	568	796	703	4919	204	307	10′
34° 0′	5592	1.788	.6745	1.4826	1.206	.8290	56° 0′
10′	616	781	787	4733	209	274	50′
20′	640	773	830	4641	211	258	40′
30′	664	766	873	550	213	241	30′
40′	688	758	916	4460	216	225	20′
50′	712	751	.6959	4370	218	208	10′
35° 0′	.5736	1.743	.7002	1.4281	1.221	.8192	55° 0′
10′	760	736	046	4193	223	175	50′
20′	783	729	089	4106	226	158	40′
30′	807	722	133	4019	228	141	30′
40′	831	715	177	3934	231	124	20′
50′	854	708	221	3848	233	107	10′
36° 0′	.5878	1.701	.7265	1.3764	1.236	.8090	54° 0′
	cos	sec	cot	tan	csc	sin	Degrees

Degrees	s n	csc	tan	cot	sec	cos	
36 0	5878	1 701	7265	1 3764	1 236	8090	54 0′
10	901	695	310	3680	239	073	50′
20	925	688	355	3597	241	056	40′
30′	948	681	400	3514	244	039	30
40	972	675	445	3432	247	021	20
50	5995	668	490	3351	249	8004	10′
37 0	6018	1 662	7536	1 3270	1 252	7986	53 0
10	041	655	581	3190	255	969	50
20	065	649	627	3111	258	951	40
30	088	643	673	3032	260	934	30
40′	111	636	720	2954	263	916	20
50	134	630	766	2876	266	898	10′
38 0	6157	1 624	7813	1 2799	1 269	7880	52 0
10	180	618	860	2723	272	862	50
20	202	612	907	2647	275	844	40
30	225	606	7954	2572	278	826	30
40	248	601	8002	2497	281	808	20′
50	271	595	050	2423	284	790	10
39 0	6293	1 589	8098	1 2349	1 287	7771	51 0
10	316	583	146	2276	290	753	50
20	338	578	195	2203	293	735	40
30	361	572	243	2131	296	716	30′
40	383	567	292	2059	299	698	20′
50	406	561	342	1988	302	679	10
40 0	6428	1 556	8391	1 1918	1 305	7660	50 0
10	450	550	441	1847	309	642	50
20	472	545	491	1778	312	623	40
30	494	540	541	1708	315	604	30
40	517	535	591	1640	318	585	20
50	539	529	642	1571	322	566	10
41 0	6561	1 524	8693	1 1504	1 325	7547	49 0
10	583	519	744	1436	328	528	50
20	604	514	796	1369	332	509	40
30	626	509	847	1303	335	490	30
40	648	504	899	1237	339	470	20
50	670	499	8952	1171	342	451	10
42 0	6691	1 494	9004	1 1106	1 346	7431	48 0
10	713	490	057	1041	349	412	50
20′	734	485	110	0977	353	392	40
30	756	480	163	0913	356	373	30
40	777	476	217	0850	360	353	20
50	799	471	271	0786	364	333	10
43 0	6820	1 466	9325	1 0724	1 367	7314	47 0
10	841	462	380	0661	371	294	50
20	862	457	435	0599	375	274	40
30	884	453	490	0538	379	254	30
40	905	448	545	0477	382	234	20
50	926	444	601	0416	386	214	10
44 0	6947	1 440	9657	1 0355	1 390	7193	46 0
10	967	435	713	0295	394	173	50
20	6988	431	770	0235	398	153	40
30′	7009	427	827	0176	402	133	30
40	030	423	884	0117	406	112	20′
50	050	418	9942	0058	410	092	10
45 0	7071	1 414	1 000	1 0000	1 414	7071	45 0
	cos	sec	cot	tan	csc	s n	Degrees

Table 2. TRIGONOMETRIC FUNCTIONS IN RADIAN MEASURE

Rad	sin	tan	cot	cos	Rad	sin	tan	cot	cos
.00	.0000	.0000	——	1.0000	.50	.4794	.5463	1.830	.8776
.01	.0100	.0100	99.997	1.0000	.51	.4882	.5594	1.788	.8727
.02	.0200	.0200	49.993	.9998	.52	.4969	.5726	1.747	.8678
.03	.0300	.0300	33.323	.9996	.53	.5055	.5859	1.707	.8628
.04	.0400	.0400	24.987	.9992	.54	.5141	.5994	1.668	.8577
.05	.0500	.0500	19.983	.9988	.55	.5227	.6131	1.631	.8525
.06	.0600	.0601	16.647	.9982	.56	.5312	.6269	1.595	.8473
.07	.0699	.0701	14.262	.9976	.57	.5396	.6410	1.560	.8419
.08	.0799	.0802	12.473	.9968	.58	.5480	.6552	1.526	.8365
.09	.0899	.0902	11.081	.9960	.59	.5564	.6696	1.494	.8309
.10	.0998	.1003	9.967	.9950	.60	.5646	.6841	1.462	.8253
.11	.1098	.1104	9.054	.9940	.61	.5729	.6989	1.431	.8196
.12	.1197	.1206	8.293	.9928	.62	.5810	.7139	1.401	.8139
.13	.1296	.1307	7.649	.9916	.63	.5891	.7291	1.372	.8080
.14	.1395	.1409	7.096	.9902	.64	.5972	.7445	1.343	.8021
.15	.1494	.1511	6.617	.9888	.65	.6052	.7602	1.315	.7961
.16	.1593	.1614	6.197	.9872	.66	.6131	.7761	1.288	.7900
.17	.1692	.1717	5.826	.9856	.67	.6210	.7923	1.262	.7838
.18	.1790	.1820	5.495	.9838	.68	.6288	.8087	1.237	.7776
.19	.1889	.1923	5.200	.9820	.69	.6365	.8253	1.212	.7712
.20	.1987	.2027	3.933	.9801	.70	.6442	.8423	1.187	.7648
.21	.2085	.2131	4.692	.9780	.71	6518	.8595	1.163	.7584
.22	.2182	.2236	4.472	.9759	.72	.6594	.8771	1.140	.7518
.23	.2280	.2341	4.271	.9737	.73	.6669	.8949	1.117	.7452
.24	.2377	.2447	4.086	.9713	.74	.6743	.9131	1.095	.7385
.25	.2474	.2553	3.916	.9689	.75	.6816	.9316	1.073	.7317
.26	.2571	.2660	3.759	.9664	.76	.6889	.9505	1.052	.7248
.27	.2667	.2768	3.613	.9638	.77	.6961	.9697	1.031	.7179
.28	.2764	.2876	3.478	.9611	.78	.7033	.9893	1.011	.7109
.29	.2860	.2984	3.351	.9582	.79	.7104	1.009	.9908	.7038
.30	.2955	.3093	3.233	.9553	.80	.7174	1.030	.9712	.6967
.31	.3051	.3203	3.122	.9523	.81	.7243	1.050	.9520	.6895
.32	.3146	.3314	3.018	.9492	.82	.7311	1.072	.9331	.6822
.33	.3240	.3425	2.920	.9460	.83	.7379	1.093	.9146	.6749
.34	.3335	.3536	2.827	.9428	.84	.7446	1.116	.8964	.6675
.35	.3429	.3650	2.740	.9394	.85	.7513	1.138	.8785	.6600
.36	.3523	.3764	2.657	.9359	.86	.7578	1.162	.8609	.6524
.37	.3616	.3879	2.578	.9323	.87	.7643	1.185	.8437	.6448
.38	.3709	.3994	2.504	.9287	.88	.7707	1.210	.8267	.6372
.39	.3802	.4111	2.433	.9249	.89	.7771	1.235	.8100	.6294
.40	.3894	.4228	2.365	.9211	.90	.7833	1.260	.7936	.6216
.41	.3986	.4346	2.301	.9171	.91	.7895	1.286	.7774	.6137
.42	.4078	.4466	2.239	.9131	.92	.7956	1.313	.7615	.6058
.43	.4169	.4586	2.180	.9090	.93	.8016	1.341	.7458	.5978
.44	.4259	.4708	2.124	.9048	.94	.8076	1.369	.7303	.5898
.45	.4350	.4831	2.070	.9004	.95	.8134	1.398	.7151	.5817
.46	.4439	.4954	2.018	.8961	.96	.8192	1.428	.7001	.5735
.47	.4529	.5080	1.969	.8916	.97	.8249	1.459	.6853	.5653
.48	.4618	.5206	1.921	.8870	.98	.8305	1.491	.6707	.5570
.49	.4706	.5334	1.875	.8823	.99	.8360	1.524	.6563	.5487
.50	.4794	.5463	1.830	.8776	1.00	.8415	1.557	.6421	.5403

Table 2 (Continued) TRIGONOMETRIC FUNCTIONS IN RADIAN MEASURE

Rad	sin	tan	cot	cos
1 00	8415	1 557	6421	5403
1 01	8468	1 592	6281	5319
1 02	8521	1 628	6142	5234
1 03	8573	1 665	6005	5148
1 04	8624	1 704	5870	5062
1 05	8674	1 743	5736	4976
1 06	8724	1 784	5604	4889
1 07	8772	1 827	5473	4801
1 08	8820	1 871	5344	4713
1 09	8866	1 917	5216	4625
1 10	8912	1 965	5090	4536
1 11	8957	2 014	4964	4447
1 12	9001	2 066	4840	4357
1 13	9044	2 120	4718	4267
1 14	9086	2 176	4596	4176
1 15	9128	2 234	4475	4085
1 16	9168	2 296	4356	3993
1 17	9208	2 360	4237	3902
1 18	9246	2 427	4120	3809
1 19	9284	2 498	4003	3717
1 20	9320	2 572	3888	3624
1 21	9356	2 650	3773	3530
1 22	9391	2 733	3659	3436
1 23	9425	2 820	3546	3342
1 24	9458	2 912	3434	3248
1 25	9490	3 010	3323	3153
1 26	9521	3 113	3212	3058
1 27	9551	2 224	3102	2963
1 28	9580	3 341	2993	2867
1 29	9608	3 467	2884	2771
1.30	9636	3 602	2776	2675

Rad	sin	tan	cot	cos
1 30	9636	3 602	2776	2675
1 31	9662	3 747	2669	2579
1 32	9687	3 903	2562	2482
1 33	9711	4 072	2456	2385
1 34	9735	4 256	2350	2288
1.35	9757	4 455	2245	2190
1 36	9779	4 673	2140	2092
1 37	9799	4 913	2035	1994
1 38	9819	5 177	1931	1896
1 39	9837	5 471	1828	1798
1 40	9854	5 798	1725	1700
1 41	9871	6 165	1622	1601
1 42	9887	6 581	1519	1502
1 43	9901	7 055	1417	1403
1 44	9915	7 602	1315	1304
1 45	9927	8 238	1214	1205
1 46	9939	8 989	1113	1106
1 47	9949	9 887	1011	1006
1 48	9959	10 983	0910	0907
1 49	9967	12 350	0810	0807
1.50	9975	14 101	0709	0707
1 51	9982	16 428	0609	0608
1 52	9987	19 670	0508	0508
1 53	9992	24 498	0408	0408
1 54	9995	32 461	0308	0308
1 55	9998	48 078	0208	0208
1 56	9999	92 620	0108	0108
1 57	1 0000	1255 8	0008	0008
1 58	1 0000	—108 65	— 0092	— 0092
1 59	9998	—52 067	— 0192	— 0192
1 60	9996	—34 233	— 0292	— 0292

Table 3 CONVERSION OF DEGREES, MINUTES, SECONDS TO RADIANS

Deg	Rad	Min	Rad	Sec	Rad
1	0 01745 33	1	0 00029 09	1	0 00000 48
2	0 03490 66	2	0 00058 18	2	0 00000 97
3	0 05235 99	3	0 00087 27	3	0 00001 45
4	0 06981 32	4	0 00116 36	4	0 00001 94
5	0 08726 65	5	0 00145 44	5	0 00002 42
6	0 10471 98	6	0 00174 53	6	0 00002 91
7	0 12217 30	7	0 00203 62	7	0 00003 39
8	0 13962 63	8	0 00232 71	8	0 00003 88
9	0 15707 96	9	0 00261 80	9	0 00004 36
10	0 17453 29	10	0 00290 89	10	0 00004 85
20	0 34906 59	20	0 00581 78	20	0 00009 70
30	0 52359 88	30	0 00872 66	30	0 00014 54
40	0 69813 17	40	0 01163 55	40	0 00019 39
50	0 87266 46	50	0 01454 44	50	0 00024 24
60	1 04719 76	60	0 01745 33	60	0 00029 09
70	1 22173 05				
80	1 39626 34				
90	1 57079 63				

Table 4. LOGARITHMS OF TRIGONOMETRIC FUNCTIONS

Degrees	log sin	log tan	log cot	log cos	
0° 0′	——	——	——	10.0000	90° 0′
10′	7.4637	7.4637	12.5363	.0000	50′
20′	.7648	.7648	.2352	.0000	40′
30′	7.9408	7.9409	12.0591	.0000	30′
40′	8.0658	8.0658	11.9342	.0000	20′
50′	.1627	.1627	.8373	10.0000	10′
1° 0′	8.2419	8.2419	11.7581	9.9999	89° 0′
10′	.3088	.3089	.6911	.9999	50′
20′	.3668	.3669	.6331	.9999	40′
30′	.4179	.4181	.5819	.9999	30′
40′	.4637	.4638	.5362	.9998	20′
50′	.5050	.5053	.4947	.9998	10′
2° 0′	8.5428	8.5431	11.4569	9.9997	88° 0′
10′	.5776	.5779	.4221	.9997	50′
20′	.6097	.6101	.3899	.9996	40′
30′	.6397	.6401	.3599	.9996	30′
40′	.6677	.6682	.3318	.9995	20′
50′	.6940	.6945	.3055	.9995	10′
3° 0′	8.7188	8.7194	11.2806	9.9994	87° 0′
10′	.7423	.7429	.2571	.9993	50′
20′	.7645	.7652	.2348	.9993	40′
30′	.7857	.7865	.2135	.9992	30′
40′	.8059	.8067	.1933	.9991	20′
50′	.8251	.8261	.1739	.9990	10′
4° 0′	8.8436	8.8446	11.1554	9.9989	86° 0′
10′	.8613	.8624	.1376	.9989	50′
20′	.8783	.8795	.1205	.9988	40′
30′	.8946	.8960	.1040	.9987	30′
40′	.9104	.9118	.0882	.9986	20′
50′	.9256	.9272	.0728	.9985	10′
5° 0′	8.9403	8.9420	11.0580	9.9983	85° 0′
10′	.9545	.9563	.0437	.9982	50′
20′	.9682	.9701	.0299	.9981	40′
30′	.9816	.9836	.0164	.9980	30′
40′	8.9945	8.9966	11.0034	.9979	20′
50′	9.0070	9.0093	10.9907	.9977	10′
6° 0′	9.0192	9.0216	10.9784	9.9976	84° 0′
10′	.0311	.0336	.9664	.9975	50′
20′	.0426	. 0453	.9547	.9973	40′
30′	.0539	.0567	.9433	.9972	30′
40′	.0648	.0678	.9322	.9971	20′
50′	.0755	.0786	.9214	.9969	10′
7° 0′	9.0859	9.0891	10.9109	9.9968	83° 0′
10′	.0961	.0995	.9005	.9966	50′
20′	.1060	.1096	.8904	.9964	40′
30′	.1157	.1194	.8806	.9963	30′
40′	.1252	.1291	.8709	.9961	20′
50′	.1345	.1385	.8615	.9959	10′
8° 0′	9.1436	9.1478	10.8522	9.9958	82° 0′
10′	.1525	.1569	.8431	.9956	50′
20′	.1612	.1658	.8342	.9954	40′
30′	.1697	.1745	.8255	.9952	30′
40′	.1781	.1831	.8169	.9950	20′
50′	.1863	.1915	.8085	.9948	10′
9° 0′	9.1943	9.1997	10.8003	9.9946	81° 0′
	log cos	log cot	log tan	log sin	Degrees

Subtract 10 from each entry in this table to obtain the proper logarithm of the indicated trigonometric function.

Degrees	log sin	log tan	log cot	log cos	
9 0	9 1943	9 1997	10 8003	9 9946	81° 0
10	2022	2078	7922	9944	50
20	2100	2158	7842	9942	40
30	2176	2236	7764	9940	30
40	2251	2313	7687	9938	20
50	2324	2389	7611	9936	10
10 0	9 2397	9 2463	10 7537	9 9934	80° 0
10	2468	2536	7464	9931	50
20	2538	2609	7391	9929	40
30	2606	2680	7320	9927	30
40	2674	2750	7250	9924	20
50	2740	2819	7181	9922	10
11° 0	9 2806	9 2887	10 7113	9 9919	79° 0
10	2870	2953	7047	9917	50
20	2934	3020	6980	9914	40
30	2997	3085	6915	9912	30
40	3058	3149	6851	9909	20
50	3119	3212	6788	9907	10
12° 0	9 3179	9 3275	10 6725	9 9904	78 0
10	3238	3336	6664	9901	50
20	3296	3397	6603	9899	40
30	3353	3458	6542	9896	30
40	3410	3517	6483	9893	20
50	3466	3576	6424	9890	10
13° 0	9 3521	9 3634	10 6366	9 9887	77° 0
10	3575	3691	6309	9884	50
20′	3629	3748	6252	9881	40′
30	3682	3804	6196	9878	30
40	3734	3859	6141	9875	20
50	3786	3914	6086	9872	10
14 0	9 3837	9 3968	10 6032	9 9869	76° 0
10	3887	4021	5979	9866	50
20	3937	4074	5926	9863	40
30	3986	4127	5873	9859	30
40	4035	4178	5822	9856	20
50	4083	4230	5770	9853	10
15° 0	9 4130	9 4281	10 5719	9 9849	75° 0
10	4177	4331	5669	9846	50
20	4223	4381	5619	9843	40′
30	4269	4430	5570	9839	30
40	4314	4479	5521	9836	20
50	4359	4527	5473	9832	10
16° 0	9 4403	9 4575	10 5425	9 9828	74° 0
10	4447	4622	5378	9825	50
20	4491	4669	5331	9821	40
30	4533	4716	5284	9817	30′
40	4576	4762	5238	9814	20
50	4618	4808	5192	9810	10
17° 0	9 4659	9 4853	10 5147	9 9806	73° 0
10	4700	4898	5102	9802	50
20	4741	4943	5057	9798	40
30	4781	4987	5013	9794	30
40	4821	5031	4969	9790	20′
50′	4861	5075	4925	9786	10
18° 0	9 4900	9 5118	10 4882	9 9782	72° 0
	log cos	log cot	log tan	log sin	Degrees

Subtract 10 from each entry in this table to obtain the proper logarithm of the indicated trigonometric function

Table 4 (Continued). **LOGARITHMS OF TRIGONOMETRIC FUNCTIONS**

Degrees	log sin	log tan	log cot	log cos	
18° 0′	9.4900	9.5118	10.4882	9.9782	72° 0′
10′	.4939	.5161	.4839	.9778	50′
20′	.4977	.5203	.4797	.9774	40′
30′	.5015	.5245	.4755	.9770	30′
40′	.5052	.5287	.4713	.9765	20′
50′	.5090	.5329	.4671	.9761	10′
19° 0′	9.5126	9.5370	10.4630	9.9757	71° 0′
10′	.5163	.5411	.4589	.9752	50′
20′	.5199	.5451	.4549	.9748	40′
30′	.5235	.5491	.4509	.9743	30′
40′	.5270	.5531	.4469	.9739	20′
50′	.5306	.5571	.4429	.9734	10′
20° 0′	9.5341	9.5611	10.4389	9.9730	70° 0′
10′	.5375	.5650	.4350	.9725	50′
20′	.5409	.5689	.4311	.9721	40′
30′	.5443	.5727	.4273	.9716	30′
40′	.5477	.5766	.4234	.9711	20′
50′	.5510	.5804	.4196	.9706	10′
21° 0′	9.5543	9.5842	10.4158	9.9702	69° 0′
10′	.5576	.5879	.4121	.9697	50′
20′	.5609	.5917	.4083	.9692	40′
30′	.5641	.5954	.4046	.9687	30′
40′	.5673	.5991	.4009	.9682	20′
50′	.5704	.6028	.3972	.9677	10′
22° 0′	9.5736	9.6064	10.3936	9.9672	68° 0′
10′	.5767	.6100	.3900	.9667	50′
20′	.5798	.6136	.3864	.9661	40′
30′	.5828	.6172	.3828	.9656	30′
40′	.5859	.6208	.3792	.9651	20′
50′	.5889	.6243	.3757	.9646	10′
23° 0′	9.5919	9.6279	10.3721	9.9640	67° 0′
10′	.5948	.6314	.3686	.9635	50′
20′	.5978	.6348	.3652	.9629	40′
30′	.6007	.6383	.3617	.9624	30′
40′	.6036	.6417	.3583	.9618	20′
50′	.6065	.6452	.3548	.9613	10′
24° 0′	9.6093	9.6486	10.3514	9.9607	66° 0′
10′	.6121	.6520	.3480	.9602	50′
20′	.6149	.6553	.3447	.9596	40′
30′	.6177	.6587	.3413	.9590	30′
40′	.6205	.6620	.3380	.9584	20′
50′	.6232	.6654	.3346	.9579	10′
25° 0′	9.6259	9.6687	10.3313	9.9573	65° 0′
10′	.6286	.6720	.3280	.9567	50′
20′	.6313	.6752	.3248	.9561	40′
30′	.6340	.6785	.3215	.9555	30′
40′	.6366	.6817	.3183	.9549	20′
50′	.6392	.6850	.3150	.9543	10′
26° 0′	9.6418	9.6882	10.3118	9.9537	64° 0′
10′	.6444	.6914	.3086	.9530	50′
20′	.6470	.6946	.3054	.9524	40′
30′	.6495	.6977	.3023	.9518	30′
40′	.6521	.7009	.2991	.9512	20′
50′	.6546	.7040	.2960	.9505	10′
27° 0′	9.6570	9.7072	10.2928	9.9499	63° 0′
	log cos	log cot	log tan	log sin	Degrees

Subtract 10 from each entry in this table to obtain the proper logarithm of the indicated trigonometric function.

Degrees	log sin	log tan	log cot	log cos	
27° 0	9 6570	9 7072	10 2928	9 9499	63 0
10	6595	7103	2897	9492	50
20	6620	7134	2866	9486	40
30	6644	7165	2835	9479	30′
40	6668	7196	2804	9473	20
50′	6692	7226	2774	9466	10
28 0	9 6716	9 7257	10 2743	9 9459	62° 0
10	6740	7287	2713	9453	50′
20	6763	7317	2683	9446	40′
30	6787	7348	2652	9439	30
40′	6810	7378	2622	9432	20′
50	6833	7408	2592	9425	10
29° 0	9 6856	9 7438	10 2562	9 9418	61° 0
10	6878	7467	2533	9411	50
20	6901	7497	2503	9404	40
30	6923	7526	2474	9397	30
40	6946	7556	24444	9390	20
50	6968	7585	2415	9383	10
30 0	9 6990	9 7614	10 2386	9 9375	60 0
10	7012	7644	2356	9368	50
20	7033	7673	2327	9361	40
30	7055	7701	2299	9353	30′
40	7076	7730	2270	9346	20
50	7097	7759	2241	9338	10′
31° 0	9 7118	9 7788	10 2212	9 9331	59° 0
10	7139	7816	2184	9323	50
20	7160	7845	2155	9315	40
30	7181	7873	2127	9308	30
40	7201	7902	2098	9300	20
50	7222	7930	2070	9292	10
32° 0	9 7242	9 7958	10 2042	9 9284	58° 0
10	7262	7986	2014	9276	50
20′	7282	8014	1986	9268	40
30	7302	8042	1958	9260	30′
40	7322	8070	1930	9252	20
50	7342	8097	1903	9244	10
33° 0	9 7361	9 8125	10 1875	9 9236	57° 0
10	7380	8153	1847	9228	50
20	7400	8180	1820	9219	40′
30	7419	8208	1792	9211	30
40	7438	8235	1765	9203	20′
50	7457	8263	1737	9194	10
34° 0	9 7476	9 8290	10 1710	9 9186	56° 0
10	7494	8317	1683	9177	50
20	7513	8344	1656	9169	40′
30	7531	8371	1629	9160	30
40	7550	8398	1602	9151	20
50	7568	8425	1575	9142	10
35° 0	9 7586	9 8452	10 1548	9 9134	55° 0
10′	7604	8479	1521	9125	50
20	7622	8506	1494	9116	40
30	7640	8533	1467	9107	30
40	7657	8559	1441	9098	20
50	7675	8586	1414	9089	10′
36° 0	9 7692	9 8613	10 1387	9 9080	54° 0
	log cos	log cot	log tan	log sin	Degrees

Subtract 10 from each entry in this table to obtain the proper logarithm of the indicated trigonometric function

Degrees	log sin	log tan	log cot	log cos	
36° 0′	9.7692	9.8613	10.1387	9.9080	54° 0′
10′	.7710	.8639	.1361	.9070	50′
20′	.7727	.8666	.1334	.9061	40′
30′	.7744	.8692	.1308	.9052	30′
40′	.7761	.8718	.1282	.9042	20′
50′	.7778	.8745	.1255	.9033	10′
37° 0′	9.7795	9.8771	10.1229	9.9023	53° 0′
10′	.7811	.8797	.1203	.9014	50′
20′	.7828	.8824	.1176	.9004	40′
30′	.7844	.8850	.1150	.8995	30′
40′	.7861	.8876	.1124	.8985	20′
50′	.7877	.8902	.1908	.8975	10′
38° 0′	9.7893	9.8928	10.1072	9.8965	52° 0′
10′	.7910	.8954	.1046	.8955	50′
20′	.7926	.8980	.1020	.8945	40′
30′	.7941	.9006	.0994	.8935	30′
40′	.7957	.9032	.0968	.8925	20′
50′	.7973	.9058	.0942	.8915	10′
39° 0′	9.7989	9.9084	10.0916	9.8905	51° 0′
10′	.8004	.9110	.0890	.8895	50′
20′	.8020	.9135	.0865	.8884	40′
30′	.8035	.9161	.0839	.8874	30′
40′	.8050	.9187	.0813	.8864	20′
50′	.8066	.9212	.0788	.8853	10′
40° 0′	9.8081	9.9238	10.0762	9.8843	50° 0′
10′	.8096	.9264	.0736	.8832	50′
20′	.8111	.9289	.0711	.8821	40′
30′	.8125	.9315	.0685	.8810	30′
40′	.8140	.9341	.0659	.8800	20′
50′	.8155	.9366	.0634	.8789	10′
41° 0′	9.8169	9.9392	10.0608	9.8778	49° 0′
10′	.8184	.9417	.0583	.8767	50′
20′	.8198	.9443	.0557	.8756	40′
30′	.8213	.9468	.0532	.8745	30′
40′	.8227	.9494	.0506	.8733	20′
50′	.8241	.9519	.0481	.8722	10′
42° 0′	9.8255	9.9544	10.0456	9.8711	48° 0′
10′	.8269	.9570	.0430	.8699	50′
20′	.8283	.9595	.0405	.8688	40′
30′	.8297	.9621	.0379	.8676	30′
40′	.8311	.9646	.0354	.8665	20′
50′	.8324	.9671	.0329	.8653	10′
43° 0′	9.8338	9.9697	10.0303	9.8641	47° 0′
10′	.8351	.9722	.0278	.8629	50′
20′	.8365	.9747	.0253	.8618	40′
30′	.8378	.9772	.0228	.8606	30′
40′	.8391	.9798	.0202	.8594	20′
50′	.8405	.9823	.0177	.8582	10′
44° 0′	9.8418	9.9848	10.0152	9.8569	46° 0′
10′	.8431	.9874	.0126	.8557	50′
20′	.8444	.9899	.0101	.8545	40′
30′	.8457	.9924	.0076	.8532	30′
40′	.8469	.9949	.0051	.8520	20′
50′	.8482	9.9975	.0025	.8507	10′
45° 0′	9.8495	10.0000	10.0000	9.8495	45° 0′
	log cos	log cot	log tan	log sin	Degrees

Subtract 10 from each entry in this table to obtain the proper logarithm of the indicated trigonometric function.

Table 5. COMMON LOGARITHMS

n	0	1	2	3	4	5	6	7	8	9
10	0000	0043	0086	0128	0170	0212	0253	0294	0334	0374
11	0414	0453	0492	0531	0569	0607	0645	0682	0719	0755
12	0792	0828	0864	0899	0934	0969	1004	1038	1072	1106
13	1139	1173	1206	1239	1271	1303	1335	1367	1399	1430
14	1461	1492	1523	1553	1584	1614	1644	1673	1703	1732
15	1761	1790	1818	1847	1875	1903	1931	1959	1987	2014
16	2041	2068	2095	2122	2148	2175	2201	2227	2253	2279
17	2304	2330	2355	2380	2405	2430	2455	2480	2504	2529
18	2553	2577	2601	2625	2648	2672	2695	2718	2742	2765
19	2788	2810	2833	2856	2878	2900	2923	2945	2967	2989
20	3010	3032	3054	3075	3096	3118	3139	3160	3181	3201
21	3222	3243	3263	3284	3304	3324	3345	3365	3385	3404
22	3424	3444	3464	3483	3502	3522	3541	3560	3579	3598
23	3617	3636	3655	3674	3692	3711	3729	3747	3766	3784
24	3802	3820	3838	3856	3874	3892	3909	3927	3945	3962
25	3979	3997	4014	4031	4048	4065	4082	4099	4116	4133
26	4150	4166	4183	4200	4216	4232	4249	4265	4281	4298
27	4314	4330	4346	4362	4378	4393	4409	4425	4440	4456
28	4472	4487	4502	4518	4533	4548	4564	4579	4594	4609
29	4624	4639	4654	4669	4683	4698	4713	4728	4742	4757
30	4771	4786	4800	4814	4829	4843	4857	4871	4886	4900
31	4914	4928	4942	4955	4969	4983	4997	5011	5024	5038
32	5051	5065	5079	5092	5105	5119	5132	5145	5159	5172
33	5185	5198	5211	5224	5237	5250	5263	5276	5289	5302
34	5315	5328	5340	5353	5366	5378	5391	5403	5416	5428
35	5441	5453	5465	5478	5490	5502	5514	5527	5539	5551
36	5563	5575	5587	5599	5611	5623	5635	5647	5658	5670
37	5682	5694	5705	5717	5729	5740	5752	5763	5775	5786
38	5798	5809	5821	5832	5843	5855	5866	5877	5888	5899
39	5911	5922	5933	5944	5955	5966	5977	5988	5999	6010
40	6021	6031	6042	6053	6064	6075	6085	6096	6107	6117
41	6128	6138	6149	6160	6170	6180	6191	6201	6212	6222
42	6232	6243	6253	6263	6274	6284	6294	6304	6314	6325
43	6335	6345	6355	6365	6375	6385	6395	6405	6415	6425
44	6435	6444	6454	6464	6474	6484	6493	6503	6513	6522
45	6532	6542	6551	6561	6571	6580	6590	6599	6609	6618
46	6628	6637	6646	6656	6665	6675	6684	6693	6702	6712
47	6721	6730	6739	6749	6758	6767	6776	6785	6794	6803
48	6812	6821	6830	6839	6848	6857	6866	6875	6884	6893
49	6902	6911	6920	6928	6937	6946	6955	6964	6972	6981
50	6990	6998	7007	7016	7024	7033	7042	7050	7059	7067
51	7076	7084	7093	7101	7110	7118	7126	7135	7143	7152
52	7160	7168	7177	7185	7193	7202	7210	7218	7226	7235
53	7243	7251	7259	7267	7275	7284	7292	7300	7308	7316
54	7324	7332	7340	7348	7356	7364	7372	7380	7388	7396

Table 5 (Continued). COMMON LOGARITHMS

n	0	1	2	3	4	5	6	7	8	9
55	7404	7412	7419	7427	7435	7443	7451	7459	7466	7474
56	7482	7490	7497	7505	7513	7520	7528	7536	7543	7551
57	7559	7566	7574	7582	7589	7597	7604	7612	7619	7627
58	7634	7642	7649	7657	7664	7672	7679	7686	7694	7701
59	7709	7716	7723	7731	7738	7745	7752	7760	7767	7774
60	7782	7789	7796	7803	7810	7818	7825	7832	7839	7846
61	7853	7860	7868	7875	7882	7889	7896	7903	7910	7917
62	7924	7931	7938	7945	7952	7959	7966	7973	7980	7987
63	7993	8000	8007	8014	8021	8028	8035	8041	8048	8055
64	8062	8069	8075	8082	8089	8096	8102	8109	8116	8122
65	8129	8136	8142	8149	8156	8162	9169	8176	8182	8189
66	8195	8202	8209	8215	8222	8228	8235	8241	8248	8254
67	8261	8267	8274	8280	8287	8293	8299	8306	8312	8319
68	8325	8331	8338	8344	8351	8357	8363	8370	8376	8382
69	8388	8395	8401	8407	8414	8420	8426	8432	8439	8445
70	8451	8457	8463	8470	8476	8482	8488	8494	8500	8506
71	8513	8519	8525	8531	8537	8543	8549	8555	8561	8567
72	8573	8579	8585	8591	8597	8603	8609	8615	8621	8627
73	8633	8639	8645	8651	8657	8663	8669	8675	8681	8686
74	8692	8698	8704	8710	8716	8722	8727	8733	8739	8745
75	8751	8756	8762	8768	8774	8779	8785	8791	8797	8802
76	8808	8814	8820	8825	8831	8837	8842	8848	8854	8859
77	8865	8871	8876	8882	8887	8893	8899	8904	8910	8915
78	8921	8927	8932	8938	8943	8949	8954	8960	8965	8971
79	8976	8982	8987	8993	8998	9004	9009	9015	9020	9025
80	9031	9036	9042	9047	9053	9058	9063	9069	9074	9079
81	9085	9090	9096	9101	9106	9112	9117	9122	9128	9133
82	9138	9143	9149	9154	9159	9165	9170	9175	9180	9186
83	9191	9196	9201	9206	9212	9217	9222	9227	9232	9238
84	9243	9248	9253	9258	9263	9269	9274	9279	9284	9289
85	9294	9299	9304	9309	9315	9320	9325	9330	9335	9340
86	9345	9350	9355	9360	9365	9370	9375	9380	9385	9390
87	9395	9400	9405	9410	9415	9420	9425	9430	9435	9440
88	9445	9450	9455	9460	9465	9469	9474	9479	9484	9489
89	9494	9499	9504	9509	9513	9518	9523	9528	9533	9538
90	9542	9547	9552	9557	9562	9566	9571	9576	9581	9586
91	9590	9595	9600	9605	9609	9614	9619	9624	9628	9633
92	9638	9643	9647	9652	9657	9661	9666	9671	9675	9680
93	9685	9689	9694	9699	9703	9708	9713	9717	9722	9727
94	9731	9736	9741	9745	9750	9754	9759	9763	9768	9773
95	9777	9782	9786	9791	9795	9800	9805	9809	9814	9818
96	9823	9827	9832	9836	9841	9845	9850	9854	9859	9863
97	9868	9872	9877	9881	9886	9890	9894	9899	9903	9908
98	9912	9917	9921	9926	9930	9934	9939	9943	9948	9952
99	9956	9961	9965	9969	9974	9978	9983	9987	9991	9996

Table 6 HYPERBOLIC FUNCTIONS

x	sinh x	cosh x	tanh x	x	sinh x	cosh x	tanh x
00	0000	1 000	0000	50	5211	1 128	4621
01	0100	1 000	0100	51	5324	1 133	4700
02	0200	1 000	0200	52	5438	1 138	4777
03	0300	1 000	0300	53	5552	1 144	4854
04	0400	1 001	0400	54	5666	1 149	4930
05	0500	1 001	0500	55	5782	1 155	5005
06	0600	1 002	0599	56	5897	1 161	5080
07	0701	1 002	0699	57	6014	1 167	5154
08	0801	1 003	0798	58	6131	1 173	5227
09	0901	1 004	0898	59	6248	1 179	5299
10	1002	1 005	0997	60	6367	1 185	5370
11	1102	1 006	1096	61	6485	1 192	5441
12	1203	1 007	1194	62	6605	1 198	5511
13	1304	1 008	1293	63	6725	1 205	5581
14	1405	1 010	1391	64	6846	1 212	5649
15	1506	1 011	1489	65	6967	1 219	5717
16	1607	1 013	1587	66	7090	1 226	5784
17	1708	1 014	1684	67	7213	1 233	5850
18	1810	1 016	1781	68	7336	1 240	5915
19	1911	1 018	1878	69	7461	1.248	5980
.20	2013	1 020	1974	70	7586	1 255	6044
21	2115	1 022	2070	71	7712	1 263	6107
22	2218	1 024	2165	72	7838	1 271	6169
23	2320	1 027	2260	73	7966	1 278	6231
24	2423	1 029	2355	74	8094	1 287	6291
25	2526	1 031	2449	75	8223	1 295	6352
26	2629	1 034	2543	76	8353	1 303	6411
27	2733	1 037	2636	77	8484	1 311	6469
28	2837	1 039	2729	78	8615	1 320	6527
29	2941	1 042	2821	79	8748	1 329	6584
.30	3045	1 045	2913	80	8881	1 337	6640
31	3150	1 048	3004	81	9015	1 346	6696
32	3255	1 052	3095	82	9150	1 355	6751
33	3360	1 055	3185	83	9286	1 365	6805
34	3466	1 058	3275	84	9423	1 374	6858
35	3572	1 062	3364	85	9561	1 384	6911
36	3678	1 066	3452	86	9700	1 393	6963
37	3785	1 069	3540	87	9840	1 403	7014
38	3892	1 073	3627	88	9981	1 413	7064
39	4000	1 077	3714	89	1 012	1 423	7114
40	4108	1 081	3800	90	1 027	1 433	7163
41	4216	1 085	3885	91	1 041	1 443	7211
42	4325	1 090	3969	92	1 055	1 454	7259
43	4434	1 094	4053	93	1 070	1 465	7306
44	4543	1 098	4136	94	1 085	1 475	7352
45	4653	1 103	4219	95	1 099	1 486	7398
46	4764	1 108	4301	96	1 114	1 497	7443
47	4875	1 112	4382	97	1 129	1 509	7487
48	4986	1 117	4462	98	1 145	1 520	7531
49	5098	1 122	4542	99	1 160	1 531	7574
50	5211	1 128	4621	1 00	1 175	1 543	7616

Table 6 (Continued). HYPERBOLIC FUNCTIONS

x	sinh x	cosh x	tanh x
1.0	1.175	1.543	.7616
1.1	1.336	1.669	.8005
1.2	1.509	1.811	.8337
1.3	1.698	1.971	.8617
1.4	1.904	2.151	.8854
1.5	2.129	2.352	.9052
1.6	2.376	2.577	.9217
1.7	2.646	2.828	.9354
1.8	2.942	3.107	.9468
1.9	3.268	3.418	.9562
2.0	3.627	3.762	.9640
2.1	4.022	4.144	.9705
2.2	4.457	4.568	.9757
2.3	4.937	5.037	.9801
2.4	5.466	5.557	.9837
2.5	6.050	6.132	.9866
2.6	6.695	6.769	.9890
2.7	7.406	7.473	.9910
2.8	8.192	8.253	.9926
2.9	9.060	9.115	.9940
3.0	10.02	10.07	.9951
3.1	11.08	11.12	.9960
3.2	12.25	12.29	.9967
3.3	13.54	13.57	.9973
3.4	14.97	15.00	.9978
3.5	16.54	16.57	.9982
3.6	18.29	18.31	.9985
3.7	20.21	20.24	.9988
3.8	22.34	22.36	.9990
3.9	24.69	24.71	.9992
4.0	27.29	27.31	.9993
4.1	30.16	30.18	.9995
4.2	33.34	33.35	.9996
4.3	36.84	36.86	.9996
4.4	40.72	40.73	.9997
4.5	45.00	45.01	.9998
4.6	49.74	49.75	.9998
4.7	54.97	54.98	.9998
4.8	60.75	60.76	.9999
4.9	67.14	67.15	.9999
5.0	74.20	74.21	.9999
5.1	82.01	82.01	.9999
5.2	90.63	90.64	.9999
5.3	100.17	100.17	1.0000
5.4	110.70	110.71	1.0000
5.5	122.34	122.35	1.0000

x	sinh x	cosh x	tanh x
5.5	122.34	122.35	1.0000
5.6	135.21	135.22	1.0000
5.7	149.43	149.44	1.0000
5.8	165.15	165.15	1.0000
5.9	182.52	182.52	1.0000
6.0	201.71	201.72	1.0000
6.1	222.93	222.93	1.0000
6.2	246.37	246.38	1.0000
6.3	272.29	272.29	1.0000
6.4	300.92	300.92	1.0000
6.5	332.57	332.57	1.0000
6.6	367.55	367.55	1.0000
6.7	406.20	406.20	1.0000
6.8	448.92	448.92	1.0000
6.9	496.14	496.14	1.0000
7.0	548.32	548.32	1.0000
7.1	605.98	605.98	1.0000
7.2	669.72	669.72	1.0000
7.3	740.15	740.15	1.0000
7.4	817.99	817.99	1.0000
7.5	904.02	904.02	1.0000
7.6	999.10	999.10	1.0000
7.7	1104.2	1104.2	1.0000
7.8	1220.3	1220.3	1.0000
7.9	1348.6	1348.6	1.0000
8.0	1490.5	1490.5	1.0000
8.1	1647.2	1647.2	1.0000
8.2	1820.5	1820.5	1.0000
8.3	2011.9	2011.9	1.0000
8.4	2223.5	2223.5	1.0000
8.5	2457.4	2457.4	1.0000
8.6	2715.8	2715.8	1.0000
8.7	3001.5	3001.5	1.0000
8.8	3317.1	3317.1	1.0000
8.9	3666.0	3666.0	1.0000
9.0	4051.5	4051.5	1.0000
9.1	4477.6	4477.6	1.0000
9.2	4948.6	4948.6	1.0000
9.3	5469.0	5469.0	1.0000
9.4	6044.2	6044.2	1.0000
9.5	6679.9	6679.9	1.0000
9.6	7382.4	7382.4	1.0000
9.7	8158.8	8158.8	1.0000
9.8	9016.9	9016.9	1.0000
9.9	9965.2	9965.2	1.0000
10.0	11013.2	11013.2	1.0000

Table 7. POWERS, ROOTS, RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1000/n$
0	0	0	000 000	000 000	000 000	000 000	000 000	—
1	1	1	1 000 000	3 162 278	1 000 000	2 154 435	4 641 589	1000 000
2	4	8	1 414 214	4 472 136	1 259 921	2 714 418	5 848 035	500 000 0
3	9	27	1 732 051	5 477 226	1 442 250	3 107 233	6 694 330	333 333 3
4	16	64	2 000 000	6 324 555	1 587 401	3 419 952	7 368 063	250 000 0
5	25	125	2 236 068	7 071 068	1 709 976	3 684 031	7 937 005	200 000 0
6	36	216	2 449 490	7 745 967	1 817 121	3 914 868	8 434 327	166 666 7
7	49	343	2 645 751	8 366 600	1 912 931	4 121 285	8 879 040	142 857 1
8	64	512	2 828 427	8 944 272	2 000 000	4 308 869	9 283 178	125 000 0
9	81	729	3 000 000	9 486 833	2 080 084	4 481 405	9 654 894	111 111 1
10	100	1 000	3 162 278	10 000 00	2 154 435	4 641 589	10 000 00	100 000 0
11	121	1 331	3 316 625	10 488 09	2 223 980	4 791 420	10 322 80	90 909 09
12	144	1 728	3 464 102	10 954 45	2.289 428	4 932 424	10 626 59	83 333 33
13	169	2 197	3 605 551	11 401 75	2 351 335	5 065 797	10 913 93	76 923 08
14	196	2 744	3 741 657	11 832 16	2 410 142	5 192 494	11 186 89	71 428 57
15	225	3 375	3 872 983	12 247 45	2 466 212	5 313 293	11 447 14	66 666 67
16	256	4 096	4 000 000	12 649 11	2 519 842	5 428 835	11 696 07	62 500 00
17	289	4 913	4 123 106	13 038 40	2 571 282	5 539 658	11 934 83	58 823 53
18	324	5 832	4 242 641	13 416 41	2 620 741	5 646 216	12.164 40	55 555 56
19	361	6 859	4 358 899	13 784 05	7 668 402	5 748 897	12 385 62	52 631 58
20	400	8 000	4 472 136	14 142 14	2 714 418	5 848 035	12 599 21	50 000 00
21	441	9 261	4 582 576	14 491 38	2 758 924	4 943 922	12 805 79	47 619 05
22	484	10 648	4 690 416	14 832 40	2 802 039	6 036 811	13 005 91	45 454 55
23	529	12 167	4 795 832	15 165 75	2 843 867	6 126 926	13 200 06	43 478 26
24	576	13 824	4 898 979	15 491 93	2 884 499	6 214 465	13 388 66	41 666 67
25	625	15 625	5 000 000	15 811 39	2 924 018	6 299 605	13.572 09	40 000 00
26	676	17 576	4 099 020	16 124 52	2 962 496	6 382 504	13 750 69	38 461 54
27	729	19 683	4 196 152	16 431 68	3 000 000	6 463 304	13 924 77	37 037 04
28	784	21 952	5 291 503	16 733 70	3 036 589	6 542 133	14 094 60	35 714 29
29	841	24 389	5 385 165	17 029 39	3 072 317	6 619 106	14 260 43	34 482 76
30	900	27 000	5 477 226	17 320 51	3 107 233	6 694 330	14 422 50	33 333 33
31	961	29 791	5 567 764	17 606 82	3 141 381	6 767 899	14 581 00	32 258 06
32	1 024	32 768	5 656 854	17 888 54	3 174 802	6 839 904	14 736 13	31 250 00
33	1 089	35 937	5 744 563	18 165 90	3 207 534	6 910 423	14 888 06	30 303 03
34	1 156	39 304	5 830 952	18 439 09	3 239 612	6 979 532	15 036 95	29 411 76
35	1 225	42 875	5 916 080	18 708 29	3 271 066	7 047 299	15 182 94	28 571 43
36	1 296	46 656	6 000 000	18 973 67	3 301 927	7 113 787	15 326 19	27 777 78
37	1 369	50 653	6 082 763	19 235 38	3 332 222	7 179 054	15 466 80	27 027 03
38	1 444	54 872	6 164 414	19 493 59	3 361 975	7 243 156	15 604 91	26 315 79
39	1 521	59 319	6 244 998	19 748 42	3 391 211	7 306 144	15 740 61	25 641 03
40	1 600	64 000	6 324 555	20 000 00	3 419 952	7 368 063	15 874 01	25 000 00
41	1 681	68 921	6 403 124	20 248 46	3 448 217	7 428 959	16 005 21	24 390 24
42	1 764	74 088	6 480 741	20 493 90	3 476 027	7 488 872	16 134 29	23 809 52
43	1 849	79 507	6 557 439	20 736 44	3 503 398	7 547 842	16 261 33	23 255 81
44	1 936	85 184	6 633 250	20 976 18	3 530 348	7 605 905	16 386 43	22 727 27
45	2 025	91 125	6 708 204	21 213 20	3 556 893	7 663 094	16 509 64	22.222 22
46	2 116	97 336	6 782 330	21 447 61	3 583 048	7 719 443	16 631 03	21 739 13
47	2 209	103 823	6 855 655	21 679 48	3 608 826	7 774 980	16 750 69	21 276 60
48	2 304	110 592	6 928 203	21 908 90	3 634 241	7 829 735	16 868 65	20 833 33
49	2 401	117 649	7 000 000	22 135 94	3 659 306	7 883 735	16 984 99	20 408 16
50	2 500	125 000	7 071 068	22 360 68	3 684 031	7 937 005	17 099 76	20 000 00

Table 7 (Continued). POWERS, ROOTS, RECIPROCALS

n	n^2	n^3	$\sqrt{n}$	$\sqrt{10n}$	$\sqrt[3]{n}$	$\sqrt[3]{10n}$	$\sqrt[3]{100n}$	$1000/n$
50	2 500	125 000	7.071 068	22.360 68	3.684 031	7.937 005	17.099 76	20.000 00
51	2 601	132 651	7.141 428	22.583 18	3.708 430	7.989 570	17.213 01	19.607 84
52	2 704	140 608	7.211 103	22.803 51	3.732 511	8.041 452	17.324 78	19.230 77
53	2 809	148 877	7.280 110	23.021 73	3.756 286	8.092 672	17.435 13	18.867 92
54	2 916	157 464	7.348 469	23.237 90	3.779 763	8.143 253	17.544 11	18.518 52
55	3 025	166 375	7.416 198	23.452 08	3.802 952	8.193 213	17.651 74	18.181 82
56	3 136	175 616	7.483 315	23.664 32	3.825 862	8.242 571	17.758 08	17.857 14
57	3 249	185 193	7.549 834	23.874 67	3.848 501	8.291 344	17.863 16	17.543 86
58	3 364	195 112	7.615 773	24.083 19	3.870 877	8.339 551	17.967 02	17.241 38
59	3 481	205 379	7.681 146	24.289 92	3.892 996	8 387 207	18 069 69	16.949 15
60	3 600	216 000	7.745 967	24.494 90	3.914 868	8.434 327	18.171 21	16.666 67
61	3 721	226 981	7.810 250	24.698 18	3.936 497	8.480 926	18.271 60	16.393 44
62	3 844	238 328	7.874 008	24.899 80	3.957 982	8.527 019	18.370 91	16.129 03
63	3 969	250 047	7.937 254	25.099 80	3.979 057	8.572 619	18.469 15	15.873 02
64	4 096	262 144	8.000 000	25.298 22	4.000 000	8.617 739	18.566 36	15.625 00
65	4 225	274 625	8.062 258	25.495 10	4.020 726	8.662 391	18.662 56	15.384 62
66	4 356	287 496	8.124 038	25.690 47	4.041 240	8.706 588	18.757 77	15.151 52
67	4 489	300 763	8.185 353	25.884 36	4.061 548	8.750 340	18.852 04	14.925 37
68	4 624	314 432	8.246 211	26.076 81	4.081 655	8.793 659	18.945 36	14.705 88
69	4 761	328 509	8.306 624	26.267 85	4.101 566	8.836 556	19.037 78	14.492 75
70	4 900	343 000	8.366 600	26.457 51	4.121 285	8.897 040	19.129 31	14.285 71
71	5 041	357 911	8.426 150	26.645 83	4.140 818	8.921 121	19.219 97	14.084 51
72	5 184	373 248	8.485 281	26.832 82	4.160 168	8.962 809	19.309 79	13.888 89
73	5 329	389 017	8.544 004	27.018 51	4.179 339	9.004 113	19.398 77	13.698 63
74	5 476	405 224	8.602 325	27.202 94	4.198 336	9.045 042	19.486 95	13.513 51
75	5 625	421 875	8.660 254	27.386 13	4.217 163	9.085 603	19.574 34	13.333 33
76	5 776	438 976	8.717 798	27.568 10	4.235 824	9.125 805	19.660 95	13.157 89
77	5 929	456 533	8.774 964	27.748 87	4.254 321	9.165 656	19.746 81	12.987 01
78	6 084	474 552	8.831 761	27.928 48	4.272 659	9.205 164	19.831 92	12.820 51
79	6 241	493 039	8.888 194	28.106 94	4.290 840	9.244 335	19.916 32	12.658 23
80	6 400	512 000	8.944 272	28.284 27	4.308 869	9.283 178	20.000 00	12.500 00
81	6 561	531 441	9.000 000	28.460 50	4.326 749	9.321 698	20.082 99	12.345 68
82	6 724	551 368	9.055 385	28.635 64	4.344 481	9.359 902	20.165 30	12.195 12
83	6 889	571 787	9.110 434	28.809 72	4.362 071	9.397 796	20.246 94	12.048 19
84	7 056	592 704	9.165 151	28.982 75	4.379 519	9.435 388	20.327 93	11.904 76
85	7 225	614 125	9.219 544	29.154 76	4.395 830	9.472 682	20.408 28	11.764 71
86	7 396	636 056	9.273 618	29.325 76	4.414 005	9.509 685	20.488 00	11.627 91
87	7 569	658 503	9.327 379	29.495 76	4.431 048	9.546 403	20.567 10	11.494 25
88	7 744	681 472	9.380 832	29.664 79	4.447 960	9.582 840	20.645 60	11.363 64
89	7 921	704 969	9.433 981	29.832 87	4.464 745	9.619 002	20.723 51	11.235 96
90	8 100	729 000	9.486 833	30.000 00	4.481 405	9.654 894	20.800 84	11.111 11
91	8 281	753 571	9.539 392	30.166 21	4.497 941	9.690 521	20.877 59	10.989 01
92	8 464	778 688	9.591 663	30.331 50	4.514 357	9.725 888	20.953 79	10.869 57
93	8 649	804 357	9.643 651	30.495 90	4.530 655	9.761 000	21.029 44	10.752 69
94	8 836	830 584	9.695 360	30.659 42	4.546 836	9.795 861	21.104 54	10.638 30
95	9 025	857 375	9.746 794	30.822 07	4.562 903	9.830 476	21.179 12	10.526 32
96	9 216	884 736	9.797 959	30.983 87	4.578 857	9.864 848	21.253 17	10.416 67
97	9 409	912 673	9.848 858	31.144 82	4.594 701	9.898 983	21.326 71	10.309 28
98	9 604	941 192	9.899 495	31.304 95	4.610 436	9.932 884	21.399 75	10.204 08
99	9 801	970 299	9.949 874	31.464 27	4.626 065	9.966 555	21.472 29	10.101 01
100	10 000	1 000 000	10.000 000	31.622 78	4.641 589	10.000 000	21.544 35	10.000 00

ANSWERS TO EXERCISES AND PROBLEMS

PART I
ODD-NUMBERED EXERCISES*

EXERCISE 1-0

1. ± 0.714
3. ± 0.600
5. ± 0.995
7. ± 0.866
9. ± 0.707
11. 5
13. 10.95
15. 12.08
17. 0.5
19. 4.12
21. 8.60
23. 12.05
25. 2.24
27. 3.00
29. 0.22

EXERCISE 1-1

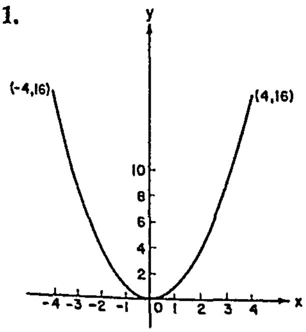
1. 14.2600°
3. 61.2378°
5. 315.0539°
7. 255.8633°
9. 77.9992°
11. 62° 30′ 54″
13. 309° 12′ 54″
15. 341° 05′ 35″
17. 209° 26′ 24″
19. 94° 11′ 24″

EXERCISE 1-2

1. 45°
3. 45°
5. 62°
7. 68°
9. 10°

EXERCISE 1-3

1.



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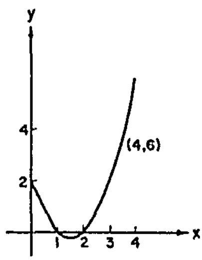

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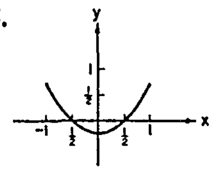


7. $4y = -x + 15$
9. $3y = -8x + 39$

* Answers to the even-numbered exercises in Part I are published in a separate pamphlet.

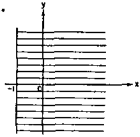
EXERCISE 1-4

1. 5 **3.** 11.31 **5.** 7.62

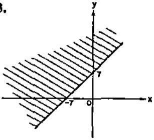
7. 12 04 **9.** 10.00

EXERCISE 1-5

1.



3.



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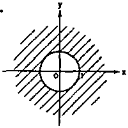

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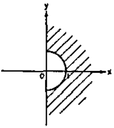

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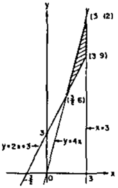


11. $y > 0$

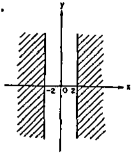
13. $y > 7x$ and $2y > -3x$ and $3y < 4x + 17$

15. $x^2 + y^2 > 25$

17. $[|x| < 1$ and $|y| < 1]$ and $[x^2 + y^2 > 1]$

19. $|x| > 0$ and $|y| > 0$

21.



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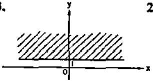

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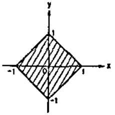
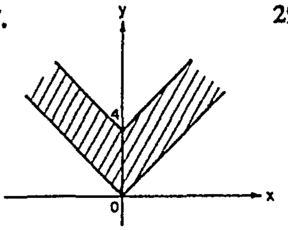
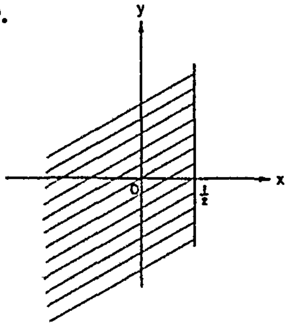


27.



29.



EXERCISE 2-1

	sin	cos	tan	cot	sec	csc
1.	3/5	4/5	3/4	4/3	5/4	5/3
3.	4/5	3/5	4/3	3/4	5/3	5/4
5.	$1/\sqrt{5}$	$2/\sqrt{5}$	1/2	2	$\sqrt{5}/2$	$\sqrt{5}$
7.	$\sqrt{15}/8$	7/8	$\sqrt{15}/7$	$7/\sqrt{15}$	8/7	$8/\sqrt{15}$
9.	$2/\sqrt{5}$	$1/\sqrt{5}$	2	1/2	$\sqrt{5}$	$\sqrt{5}/2$

EXERCISE 2-2

1. $\dfrac{2+\sqrt{2}}{2}$
3. $\dfrac{5}{4}$
5. $\dfrac{3+2\sqrt{2}}{4}$
7. $\dfrac{4+\sqrt{2}}{4}$
9. $\dfrac{4}{3}$
11. 0.94
13. 5.7
15. 0.82
17. 0.64
19. 0.64
21. 0.1765
23. 0.3939
25. 0.9999
27. 0.9555
29. 13.23
31. 0.0901
33. 0.3961
35. 1.499
37. 0.9239
39. 0.9985
41. 67° 0′
43. 30° 40′
45. 55° 50′
47. 45° 50′
49. 6° 10′
51. 14° 10′
53. 14° 40′
55. 25° 10′
57. 60° 10′
59. 18° 30′

EXERCISE 2-3

1. $\sqrt{3}/2$
3. −1
5. $-\sqrt{3}/2$
7. $\sqrt{3}/2$
9. 1/2
11. cos 37°
13. −cot 34°
15. −tan 2°
17. −sin 7°
19. −tan 11°
21. −cot 9° 11′
23. sin 23° 36′
25. −cos 32° 30′
27. cos 27° 35′
29. sin 44° 4′

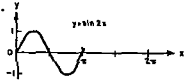
31. cos 43° 48′ 1″	**33.** −cos 18° 21′ 48″	**35.** cos 24′ 34″
37. cos 29° 45′ 24″	**39.** −sin 33° 46′ 48″	**41.** 183°, 357°
43. 14°, 166°	**45.** 19° 30′, 340° 30′	**47.** 54° 50′, 125° 10′
49. 29° 30′, 330° 30′	**51.** 37° 10′, 217° 10′	**53.** 58° 30′, 301° 30′
55. 35° 30′, 144° 30′	**57.** 71° 50′, 288° 10′	**59.** 68° 50′, 291° 10′
61. 0.5760	**63.** −0.3311	**65.** −0.1305
67. −0.9929	**69.** 0.3311	**71.** sin 45°
73. −tan 29°	**75.** cos 12°	**77.** sin 40°
79. cos 9°		

EXERCISE 2-4

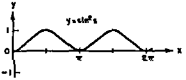
	sin	cos	tan	cot	sec	csc
1.	3/4	$\sqrt{7}/4$	$3/\sqrt{7}$	$\sqrt{7}/3$	$4/\sqrt{7}$	4/3 (I Quad)
	3/4	$-\sqrt{7}/4$	$-3/\sqrt{7}$	$-\sqrt{7}/3$	$-4/\sqrt{7}$	4/3 (II Quad)
3.	4/5	3/5	4/3	3/4	5/3	5/4
5.	$\sqrt{7}/4$	−3/4	$-\sqrt{7}/3$	$-3/\sqrt{7}$	−4/3	$4/\sqrt{7}$
7.	$2\sqrt{2}/3$	−1/3	$-2\sqrt{2}$	$-\sqrt{2}/4$	−3	$3\sqrt{2}/4$
9.	$4/\sqrt{17}$	$-1/\sqrt{17}$	−4	−1/4	$-\sqrt{17}$	$\sqrt{17}/4$
11.	$\sqrt{2}/10$	$-7\sqrt{2}/10$	−1/7	−7	$-5\sqrt{2}/7$	$5\sqrt{2}$
13.	$2\sqrt{14}/9$	5/9	$2\sqrt{14}/5$	$5\sqrt{14}/28$	9/5	$9\sqrt{14}/28$
15.	$1/\sqrt{17}$	$4/\sqrt{17}$	1/4	4	$\sqrt{17}/4$	$\sqrt{17}$
17.	$2\sqrt{2}/3$	−1/3	$-2\sqrt{2}$	$-\sqrt{2}/4$	−3	$3\sqrt{2}/4$
19.	$4\sqrt{2}/9$	−7/9	$-4\sqrt{2}/7$	$-7\sqrt{2}/8$	−9/7	$9\sqrt{2}/8$

EXERCISE 2-5

1.



3.



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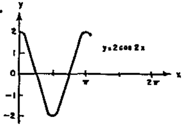

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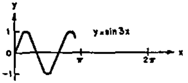

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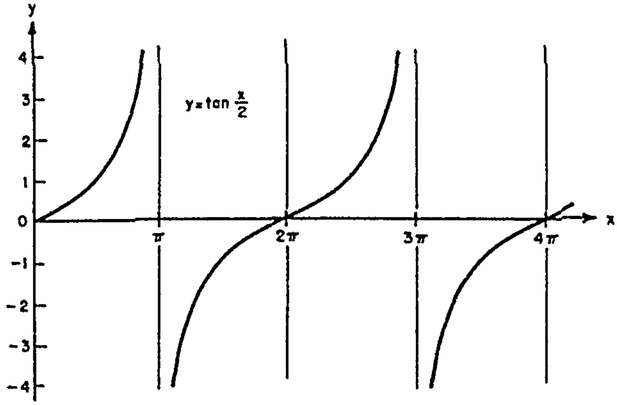


EXERCISE 3-1

1. 9.5493°
3. 22.9183°
5. 71.0296°
7. 112.4473°
9. 177.5428°
11. 42° 35′ 7″
13. 103° 16′ 8″
15. 132° 54′ 20″
17. 86° 13′ 58″
19. 268° 4′ 50″
21. 1.047198
23. 0.837758
25. 1.697042
27. 0.487756
29. 1.617043
31. 7.8540
33. 11.8333
35. 125.1052
37. 0.1699
39. 161.4754
41. 0.707
43. 0.8415
45. 0.0699
47. 1.491
49. 1.0000

EXERCISE 3-2

1. 0.00002
3. 0.9976
5. 16.52
7. 0.0398
9. 0.0376

EXERCISE 3-3

1. $\cos(\theta + 10°)$
3. $\sin(\theta + 40°)$
5. $\cos(\theta + 44°)$
7. $10\cos(\omega t - 53°)$
9. $4\sin(\alpha - 80°)$
11. $49\sin(x - 70°)$
13. $2\sin(t - 20°)$
15. $5\sin(y + 70°)$
17. $\pi/7$
19. 2π

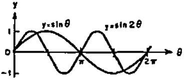
EXERCISE 4-1

1. $\frac{1}{2}\sqrt{2 + \sqrt{2}}$
3. $2 + \sqrt{3}$
5. $-\frac{1}{2}\sqrt{2 - \sqrt{3}}$
7. $\frac{1}{2}\sqrt{2 - \sqrt{2 + \sqrt{3}}}$
9. 0.96
11. 0.2492

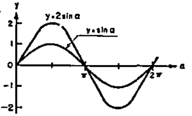
13. −0 1912 **15.** 0 90 **17.** −0 5630
19. −0 9949 **21.** cos 65° **23.** cos 39°
25. 1

EXERCISE 4-2

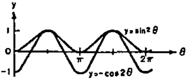
1.



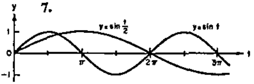
3.



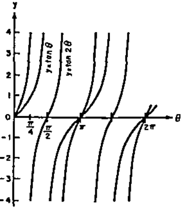
5.



7.



9.



	$\sin 2A$	$\sin \frac{A}{2}$	$\cos 2A$	$\cos \frac{A}{2}$
11.	$\frac{120}{169}$	$1/\sqrt{26}$	$\frac{119}{169}$	$5/\sqrt{26}$
13.	$\frac{12}{13}$	$\sqrt{\frac{13-2\sqrt{13}}{26}}$	$-5/13$	$\sqrt{\frac{13+2\sqrt{13}}{26}}$
15.	$-4\sqrt{21}/25$	$-\sqrt{30}/10$	$-17/25$	$-\sqrt{70}/10$

EXERCISE 5-1

1. 45°, 135° **3.** 90° **5.** 120°, 240°
7. 32° 30′, 147° 30′ **9.** 15° 10′, 344° 50′ **11.** 221° 30′, 138° 30′
13. 44° 22′, 135° 38′ **15.** 22°, 158° **17.** 105° 11′, 254° 49′

19. 75°, 105°
21. x
23. x
25. $x_0 = x - 2n\pi$ or $x_0 = \pi(2n - 1) - x$
27. $\sqrt{2}/2$
29. 0
31. $-4/3$
33. $1/\sqrt{15}$
35. -1

EXERCISE 5-2

1. 56/65
3. $\dfrac{\sqrt{3} + 2\sqrt{2}}{6}$
5. 33/65
11. 7/25
13. 0.9293
15. 0.2241

EXERCISE 5-3

1. 13°, 193°
3. 123°, 303°
5. 46° 40′, 226° 40′
7. 23°, 203°
9. 75°, 255°
11. $y = x$
13. $y = x + n\pi$
15. $6/\sqrt{61}$
17. $-\pi/4$
19. $\dfrac{1 - \theta^2}{1 + \theta^2}$

EXERCISE 5-4

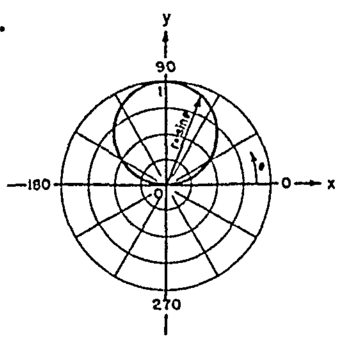
1. 0
3. 30°
5. 71° 20′

EXERCISE 5-5

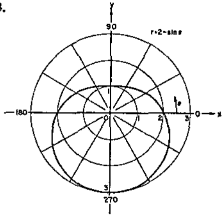
1. $x = y - 3$
3. $x = \frac{1}{5}(y - 11)$
5. $x = -1 \pm \sqrt{y - 7}$

EXERCISE 5-6

1. $r = 3\sqrt{2}$, $\theta = 45°$
3. $x = -5$, $r = 5\sqrt{2}$
5. $x = 5$, $\theta = 315°$
7. $x = \dfrac{17\sqrt{2}}{2}$, $y = \dfrac{17\sqrt{2}}{2}$
9. $x = 0$, $r = 17$
11. $x = 0$, $\theta = 270°$
13. $y = -8$, $r = 8\sqrt{2}$
15. $y = \sqrt{2}$, $\theta = 45°$
17. $x = 7.22$, $y = 14.3$
19. $x = -7.5$, $y = 11.7$
21.

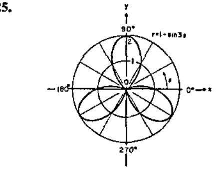


23.

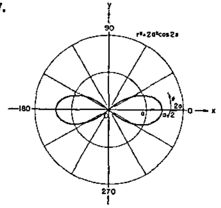


Limaçon

25.

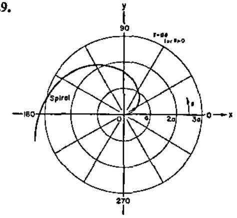


27.



Lemniscate

29.



Spiral of Archimedes

EXERCISE 6-3

1. No
3. Yes
5. Yes
7. Yes
9. Yes

EXERCISE 7-1

1. 30°, 150°, 210°, 330°
3. 0°, 120°, 240°
5. 0°, 90°, 120°, 180°, 240°, 270°
7. 0, $\pi/4$, π, $7\pi/4$, 2π
9. $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$
11. $\pi/4$, $3\pi/4$, $5\pi/4$, $7\pi/4$
13. $\pi/4$, $3\pi/4$, $5\pi/4$, $3\pi/2$
15. 45°, 60°, 120°, 135°, 225°, 240°, 300°, 315°
17. 60°
19. 120°, 240°

EXERCISE 7-2

1. 0 2496
3. −0 0575
5. −1.738

7. 1.406	9. −0.8812	11. −0.6182
13. −0.4353	15. 2.796	17. 0.9388
19. −0.9646	21. 0.1847	23. 3.166
25. 1.415	27. 1.133	29. 0.7589
31. −0.8105	33. −0.8011	35. 0.4496
37. −2.295	39. 0.7696	41. 45° 49′
43. 20° 3′	45. 76° 10′	47. 5° 32′
49. 150° 5′	51. 58° 26′	53. 149° 56′
55. 116° 54′	57. 72° 39′	59. 137° 47′
61. 45° 27′	63. 187° 2′	65. 41° 34′
67. 16° 1′	69. 107° 15′	71. 63° 15′
73. 15° 3′	75. 252° 33′	77. 8° 51′
79. 125° 56′	81. 77° 20′	83. $6° 29' + m120°$, $53° 31' + m120°$, $m = 1, 2$

85. 18° 26′, 198° 26′	87. 204° 28′, 335° 32′
89. arcsin $\frac{3}{4}$, arcsin 1	91. 67° 30′, 90°, 157° 30′, 247° 30′, 270°
93. 105° 3′, 254° 57′	95. arccos −1, arccos $\frac{3}{4}$
97. 5° 43′, 185° 43′	99. 46° 55′, 100° 33′, 259° 27′, 313° 5′

EXERCISE 7-3

1. $C = \sqrt{41}$, $\phi = 38° 40'$
3. $C = 13$, $\phi = 157° 23'$
5. $A = \dfrac{5\sqrt{3}}{2}$, $B = -2.5$
7. $C = 14.05$, $\phi = 85° 6'$
9. $B = 0.6411$, $C = 1.188$
11. 0°, 73° 44′
13. $3\pi/2 + 2n\pi$, $\pi - 0.643 + 2n\pi$ $\quad n$ an integer
15. 15°

EXERCISE 7-4

1. $\pm\sqrt{10}$
3. 0.540
5. $\frac{4}{3}$

EXERCISE 7-5

1. 0
3. 50° 15′, 0
5. 0, ±1.32
7. 0, 39° 50′
9. 0

EXERCISE 8-1

1. $A = 42° 50'$, $b = 5.40$, $c = 7.36$
3. $B = 75° 41'$, $a = 0.075$, $c = 3.02$
5. $B = 52° 29'$, $b = 7579$, $c = 9557$
7. $A = 64° 36'$, $a = 235.9$, $c = 261.1$
9. $A = 67° 30'$, $a = 30.7$, $c = 33.2$
11. $c = 2.24$, $A = 26° 34'$, $B = 63° 26'$
13. $b = 6.32$, $A = 54° 55'$, $B = 35° 5'$
15. $b = 3.71$, $A = 40° 46'$, $B = 49° 14'$
17. $c = 0.944$, $A = 12° 51'$, $B = 77° 9'$
19. $c = 66.5$, $A = 59° 21'$, $B = 30° 39'$
21. $c = 15.6$, $A = 39° 48'$, $B = 50° 12'$
23. $c = 5.88$, $A = 23° 4'$, $B = 66° 56'$
25. $a = 0.581$, $A = 49° 51'$, $B = 40° 9'$
27. $b = 18.4$, $A = 39°$, $B = 51°$
29. $b = 1450$, $A = 40° 10'$, $B = 49° 50'$

EXERCISE 8-3

1. $A = 43°2', C = 72°58', b = 6\ 58$
3. $A = 33°19', B = 47°41', c = 8\ 99$
5. $A = 12°41', B = 150°9', c = 3\ 5$
7. $A = 31°21', B = 17°59', c = 0\ 86$
9. $A = 73°47', B = 101°7', c = 0\ 483$
11. $B = 16°48', C = 152°45', a = 7.53$
13. $B = 82°37', C = 55°22', a = 1\ 62$
15. $A = 36°6', C = 91°45', b = 13\ 4$
17. $B = 34°1', C = 61°37', a = 90\ 5$
19. $A = 57°39', B = 93°49', c = 3.11$
21. $A = 19°12', B = 80°24', C = 80°24'$
23. $A = 40°32', B = 27°40', C = 111°48'$
25. $A = 114°58', B = 42°50', C = 22°12'$
27. $A = 41°51', B = 110°55', C = 27°14'$
29. $A = 29°46', B = 83°3', C = 67°11'$

EXERCISE 8-4

1. $C = 76°10', b = 9\ 15, c = 11\ 8$
3. $B = 59°24', a = 129, b = 120$
5. $C = 23°51', a = 5391, c = 2417$
7. $A = 54°11', a = 1296, c = 1039$
9. $A = 63°29', a = 6896, c = 7417$
11. $B = 90°, a = 10\ 1, c = 8\ 33$
13. $A = 74°16', b = 1\ 49, c = 1\ 07$
15. $B = 22°43', a = 38\ 9, c = 61.6$
17. $B = 28°23', a = 62\ 1, c = 65\ 5$
19. $B = 85°5', a = 3\ 534, c = 2\ 742$
21. $b = 7\ 19$
23. $a = 8\ 84$
25. $c = 239\ 3$
27. $b = 1307$
29. $b = 11\ 75$
31. $B = 70°3', C = 62°27', c = 69\ 8$
    $B = 109°57', C = 22°33', c = 30\ 2$
33. $A = 69°17', B = 40°33', a = 41\ 8$
35. $A = 57°2', B = 80°46', b = 0\ 815$
    $A = 122°58', B = 14°50', b = 0\ 211$
37. $A = 37°16', C = 33°22', c = 0\ 113$
39. $B = 83°49', C = 55°31', b = 5\ 165$
    $B = 14°51', C = 124°29', b = 1.332$
41. $A = 40°23', C = 15°56', c = 85\ 0$
43. $A = 46°5', B = 111°40', b = 82\ 5$
    $A = 134°55', B = 22°50', b = 36\ 6$
45. $B = 47°39', C = 93°52', c = 6\ 94$
    $B = 132°21', C = 9°10, c = 1\ 11$
47. $A = 40°10', B = 87°26', b = 3240$
49. No solution

EXERCISE 8-5

1. $A = 94°54', B = 41°54', c = 0\ 82$
3. $A = 157°56', C = 7°44', b = 0\ 596$
5. $A = 71°59, C = 49°13', b = 15\ 49$
7. $A = 63°27', C = 56°16', b = 52\ 3$
9. $B = 8°5', C = 155°44', a = 22\ 03$

EXERCISE 8-6

1. 76 86
3. 505 6
5. 102 0
7. 185 0
9. 1063
11. 14 2
13. 14 85
15. 855 4
17. 35 77
19. $19\ 116 \times 10^6$
21. 9.9
23. 78 2
25. 733 9
27. $22\ 368 \times 10^6$
29. 10 81

EXERCISE 8-7

1. $A = 99°$, $B = 44° 40'$, $C = 36° 20'$
3. $A = 146° 48'$, $B = 17° 12'$, $C = 16°$
5. $A = 88° 46'$, $B = 31° 44'$, $C = 59° 30'$
7. $A = 32° 34'$, $B = 91° 40'$, $C = 55° 46'$
9. $A = 98° 50'$, $B = 47° 50'$, $C = 33° 20'$
11. $A = 123° 30'$, $B = 29° 16'$, $C = 27° 14'$
13. $A = 114° 4'$, $B = 18° 56'$, $C = 47°$
15. $A = 63° 14'$, $B = 53° 40'$, $C = 63° 6'$

EXERCISE 9-1

1. x^{10}
3. $a^9b^3x^{12}$
5. 4
7. $\sqrt{a + b}$
9. $\dfrac{a + b}{ab}$

EXERCISE 9-2

1. 32,768
3. 50.12
5. 4.266×10^{-6}

EXERCISE 9-3

1. 3.0000
3. 0
5. −2.000
7. 2.000
9. 2.000
11. 0.0128
13. 0.3598
15. 0.8500
17. 0.9872
19. 0.6702
21. 0.9719
23. 0.5000
25. 0.8880
27. 0.6599
29. 0.9098
31. 1.2455
33. 0.6444 − 2
35. 8.7210
37. 6.8376
39. 0.0719
41. 1.7612
43. 0.8082 − 1
45. 2.9360
47. 4.7332
49. 0.3820 − 3
51. 3.4972
53. 0.6346 − 3
55. 0.7732 − 7
57. 3.4993
59. 1.4465
61. 23.52
63. 386.5
65. 293.8
67. 648.4
69. 38850
71. 1.178
73. 919.5
75. 0.9683
77. 101.0
79. 0.104
81. 3381000
83. 23300
85. 0.002717
87. 17.22
89. 0.1971
91. 1.414
93. 1.259
95. 1.059
97. 1.109×10^{27}
99. 6.474×10^{32}
101. 2.8076
103. 2.2696
105. −0.2488

EXERCISE 9-4

1. 9.9866 − 10
3. 9.0818 − 10
5. (−)9.5816 − 10
7. (−)8.8363 − 10
9. 9.9045 − 10
11. (−)9.8941 − 10
13. 9.6450 − 10
15. 9.9927 − 10
17. 0.3016
19. (−)9.4896 − 10
21. 8° 49′, 188° 49′, 171° 11′, 351° 11′

23. 29° 52′, 209° 52′, 150° 8′, 330° 8′
25. 7° 11′, 187° 11′, 172° 49′, 352° 49′
27. 1° 17′, 181° 17′, 178° 43′, 358° 43′
29. 71° 7′, 251° 7′, 108° 53′, 288° 53′
31. 65° 11′, 245° 11′, 114° 49 , 294° 49′
33. 70° 6 , 250° 6′, 109° 54′, 289° 54′
35. 16° 27′, 196° 27′, 163° 33′, 343° 33′
37. 26° 50 , 206° 50′, 153° 10′, 333° 10′
39. 36° 59′, 216° 59′, 143° 1′, 323° 1′
41. $A = 120° 18′$, $C = 25° 25′$, $b = 3\ 130$
43. $A = 137° 11′$, $B = 19° 15′$, $c = 1\ 386$
45. $C = 15° 2′$, $a = 288\ 4$, $b = 255\ 8$
47. $A = 39° 24′$, $B = 62° 14′$, $C = 78° 22′$
49. $B = 27° 55′$, $C = 84° 13′$, $c = 50\ 81$

EXERCISE 9-6

1. $A = 75\ 0$, 12° 42′ **3.** $E = 1\ 769$, 115° 31′ **5.** $Z = 120$, −56° 43′
7. $V = 38\ 2$, −27° 38′ **9.** $Q = 9546$, −53° 52′ **11.** $A_x = 767\ 3$, $A_y = 603\ 2$
13. $J_x = -12\ 2$, $J_y = -16\ 7$ **15.** $W_x = 1031\ 6$, $W_y = -990\ 4$
17. $S_x = 0\ 377$, $S_y = -4\ 304$ **19.** $Z_x = -8\ 34$, $Z_y = -3\ 01$

EXERCISE 9-7

1. 189 5 knots, 20° 55′ **3.** 86 65 knots, 270° 49′
5. 308 knots, 213° 5′ **7.** 343 knots, 52° 25′
9. 178 4 knots, 141° 36
11. $DA = 609\ 8$, $\measuredangle DAB = 129° 49′$, $\measuredangle CDA = 39° 22′$
13. $CD = 1003\ 5$, $\measuredangle CDA = 71° 9′$, $\measuredangle BCD = 54° 54′$
15. $\measuredangle ABC = 204° 54′$, $\measuredangle BCD = 19° 41′$, $CD = 979\ 2$
17. $EA = 556\ 4$, $\measuredangle EAB = 174° 43′$, $\measuredangle DEA = 39° 58′$
19. $EA = 651\ 4$, $\measuredangle EAB = 136° 18′$, $\measuredangle DEA = 55° 43′$

EXERCISE 9-8

1. 114 7 at 54° **3.** 857 at 275° 23′ **5.** 676 at 204° 10′
7. 899 at 145° 13′ **9.** 1857 at 148°
11. Boom = 1192 4, cable = 649 4 **13.** Boom = 2466, cable = 1137
15. Boom = 3277, cable = 2728, mast = 2717, guide wire = 4469

PART I
ODD-NUMBERED PROBLEMS*

CHAPTER 1

3. (a) $(-8,6)$ (b) $(-5,7)$ (c) $(-4,8)$
(d) $(-7,2)$ (e) $(-4,2)$ (f) $(0,3)$
(g) $(1,0)$ (h) $(3,7)$ (i) $(6,8)$
(j) $(4,2)$ (k) $(7,3)$ (m) $(8,6)$
(n) $(-8,-2)$ (o) $(-7,-4)$ (p) $(-2,-2)$
(q) $(-9,-9)$ (r) $(-5,-10)$ (s) $(-3,-7)$
(t) $(3,-3)$ (u) $(8,-2)$ (v) $(7,-4)$
(w) $(4,-6)$ (x) $(2,-8)$ (y) $(7,-7)$
(z) $(10,-10)$

CHAPTER 2

3. 38.9 feet

5. HINT: $\sin x \cos x = \frac{1}{2}\sin 2x$

CHAPTER 3

5. 6.70 feet

7. 81.3 feet

CHAPTER 4

1. $\cos\frac{\theta}{4} = \pm\sqrt{\dfrac{1 \pm \sqrt{\dfrac{1+\cos\theta}{2}}}{2}}$

9. $\sin^{2n+1}\theta = \dfrac{(-1)^n}{2^{2n}}\sum_{j=0}^{n}(-1)^j\binom{2n+1}{j}\sin(2n+1-2j)\theta$

CHAPTER 5

3. $\operatorname{Arcsin} x + \operatorname{Arcsin}(-y) + \frac{\pi}{2}$

* Answers to the even-numbered problems in Part I are published in a separate pamphlet.

CHAPTER 6

1. a $\dfrac{\cos^2 x}{1 + \sin x + \sin^2 x}$ b 1 c. 1 d $5 \cot \phi$ e $\sin 4\alpha$

CHAPTER 7

1. a 20° 58′ b 6° 25′ c 24° 3. $x = 35° 31'$ 5. 47° 35′

CHAPTER 8

1. No angle is given.
7. (a) $B = 25° 36'$, $C = 106° 49'$, $a = 1.08$
   (b) $C = 71° 20'$, $a = 12\ 2$, $b = 9\ 22$
   (c) $A = 37° 23'$, $B = 126° 21'$, $C = 16° 16'$
   (d) $B = 51° 24'$, $a = 27\ 8$, $c = 54\ 5$
   (e) $B = 65° 40$, $C = 56° 57'$, $c = 10\ 9$
   $B = 114° 20'$, $C = 8° 17'$, $c = 1\ 88$
   (f) $A = 17° 6'$, $B = 131° 35'$, $b = 11\ 0$
   (g) $A = 69° 17'$, $C = 37° 26'$, $b = 0\ 0646$
   (h) $A = 27° 37'$, $b = 10\ 0$, $c = 7.10$
   (i) $A = 55° 39'$, $B = 72° 31'$, $C = 51° 50'$
   (j) $A = 162° 27'$, $B = 2° 23'$, $c = 95\ 5$

CHAPTER 9

5. 1 70 ft
7. a = 12,000 lb b = 16,971 lb c = 12,000 lb
   d = 16,000 lb e = 5657 lb f = 12,000 lb
   g = 8000 lb h = 16,000 lb j = 5657 lb
   k = 12,000 lb m = 12,000 lb n = 12,000 lb
   p = 16,971 lb
9.

Member	Force (+ is tension) (thousands of pounds)	Member	Force (thousands of pounds)
A–B	−14 9	A′–B′	−8 23
A–F	+15 5	A′–F′	+4 43
B–F	+3 30	B′–F′	0
B–C	−14 9	B′–C′	−8 23
C–F	+3 69	C′–F′	0
F–G	+11 8	F′–G′	+4 43
C–G	−6 62	C′–G′	0
C–D	−14 9	C′–D′	−8 23
C–H	+3 69	C′–H′	0
G–H	+7 4	G′–H′	0
D–H	−3 30	D′–H′	0
D–E	−14 9	D′–E′	−8 23
E–H	+11 0	E′–H′	0
G–G′	+4 43		

PART II

ODD- AND EVEN-NUMBERED PROBLEMS

CHAPTER 10

1. (a) $\Delta f(x) = \dfrac{\tan a \sec^2 (ax + b)}{1 - \tan a \tan (ax + b)}$ (b) $\Delta f(x) = \dfrac{-\csc^2 (ax + b)}{\cot a + \csc(ax + b)}$
   (c) $\Delta f(x) = 20x^3 + 30x^2 + 20x + 5$ (d) $\Delta f(x) = 4x(x - 1)(x - 2)$
   (e) $\Delta f(x) = 2^x$
3. (a) $f(x) = -\dfrac{\cos 3(x - \frac{1}{2})}{2 \sin \frac{3}{2}}$ (b) $f(x) = \dfrac{x}{2} + \dfrac{\sin 5(2x - 1)}{4 \sin 5}$
   (c) $f(x) = \frac{3}{2}x^{(4)} + 6x^{(3)} + 3x^{(2)}$ (d) $f(x) = -\dfrac{1}{x}$
   (e) $f(x) = \frac{1}{2}3^x$
4. $\dfrac{\sin (N + 1)}{(2 \sin \frac{1}{2})^2} - \dfrac{(N + 1) \cos (N + \frac{1}{2})}{2 \sin \frac{1}{2}}$
6. $\displaystyle\sum_{x=1}^{N} x^3 = \frac{N^2(N + 1)^2}{4}$
7. $\cos \alpha - \cos \beta$
8. (a) $\dfrac{1}{m} [1 - (-1)^m],\ m \neq 0$
   (b) 0
   (b) 0
9. (a) 0 if $m \neq n$ and π if $m = n$
   (c) 0 if $m \neq n$ and π if $m = n$
10. 156
11. $\sec^2 x_0$
12. $-\csc^2 x_0$
13. $3x_0^2$
14. $\dfrac{1}{2\sqrt{x_0}}$
15. $y = 8x - 4$

CHAPTER 11

3. (a) 3.606(cos 56° 19′ + i sin 56° 19′)
   (b) 1.732(cos 324° 44′ + i sin 324° 44′)
   (c) 2.646(cos 139° 6′ + i sin 139° 6′)
   (d) cos (Arccos a) + i sin (Arccos a)
   (e) 1.732(cos 45° + i sin 45°)
4. (a) $3.759 + 1.368i$ (b) $0.7660 + 0.5000i$
   (c) $2.819 + 0.3420i$ (d) $2 \cos \theta + i2 \sin \phi$
   (e) $\sin \theta + i \cos \theta$

5 (a) -4 (b) $-0.5128 - 0.5166i$
(c) $-2.061 - 3.836i$ (d) $0.6839 + 0.2964i$
(e) $(-0.5912 + 1.913i)e^{2n\pi}$, $n = 0, \pm 1, \pm 2,$

7. (a) 1.260, $-0.6300 + 1.091i$, $-0.6300 - 1.091i$
(b) $2, 2i, -2, -2i$
(c) $2.236i, -2.236i$
(d) $1.150 + 1.150i$, $1.150 - 1.150i$, $-1.150 + 1.150i$, $-1.150 - 1.150i$
(e) $e^{2\pi ik/n} = \cos\dfrac{2\pi k}{n} + i\sin\dfrac{2\pi k}{n}$, $k = 0, 1, \quad , n - 1$

8 (a) $1.099 + 0.4550i$, $-1.099 - 0.4550i$
(b) $\dfrac{\sqrt{3}}{2} + \dfrac{1}{2}i$, $\dfrac{\sqrt{3}}{2} - \dfrac{1}{2}i$ $-i$
(c) $2.207 - 0.3583i$, $-2.207 + 0.3583i$, $0.3583 + 2.207i$, $-0.3583 - 2.207i$
(d) Same as (a)
(e) $a + ib$, $-\frac{1}{2}(a + b\sqrt{3}) - \frac{1}{2}(b - a\sqrt{3})i$, $-\frac{1}{2}(a - b\sqrt{3}) - \frac{1}{2}(b + a\sqrt{3})i$

CHAPTER 12

1 (a) Converges to 0 (b) Converges to 1 (c) Converges to 0
(d) Converges to 0 (e) Converges to $\frac{1}{2}$

5 (a) Converges (b) Converges (c) Diverges
(d) Converges (e) Diverges

CHAPTER 13

4 $\log_e(1 + x) = x - \dfrac{x^2}{2} + \dfrac{x^3}{3} - \quad + (-1)^{n+1}\dfrac{x^n}{n} +$

CHAPTER 14

5 $y = \pm\dfrac{b}{a}x$

7 $\sinh x = \sum_{n=0}^{\infty} \dfrac{x^{2n+1}}{(2n+1)!}$

$\cosh x = \sum_{n=0}^{\infty} \dfrac{x^{2n}}{(2n)!}$

9 (a) $\cosh x = 2$, $\tanh x = \dfrac{\sqrt{3}}{2}$

(b) $\sinh x = \dfrac{4}{3}$, $\cosh x = \dfrac{5}{3}$

10 (a) $0.8415i$ (b) $3.591 + 0.5309i$ (c) -0.9900
(d) $-1.640 + 1.742i$ (e) $0.2407 - 1.161i$ (f) $-0.2178 - 1.163i$

11. (a) $\dfrac{\pi}{2} + 1.317i$ (b) $\pi + 1.763i$ (c) $\dfrac{\pi}{3}i$
(d) 1.763 (e) -1.684 (f) $\pm(0.9046 - 1.061i)$
(g) $\pm(0.5073 + 1.469i)$ (h) $1.444i$
(i) $0.4271 + 1.529i$ and $2.714 - 1.529i$ (j) $1.099i$

CHAPTER 15

1. You cannot form their derivatives.

2. $E(x) = \frac{3}{2} + \frac{\sqrt{3}}{3} \sin x$

3. $f(x) = \frac{1}{2} + \frac{1}{2} \cos 2x + \frac{9}{2} \sin x - \frac{3}{2} \sin 3x$

4. $f(t) = 4.167 - 1.250 \cos \frac{\pi t}{3} - 0.4167 \cos \frac{2\pi t}{3} - 0.7217 \sin \frac{\pi t}{3} - 0.7217 \sin \frac{2\pi t}{3}$

5. $f(x) = \frac{35}{54} + \frac{4}{9} \cos \pi x - \frac{4}{27} \cos 2\pi x + \frac{1}{18} \cos 3\pi x$

6. $f(t) = 10 - \frac{26}{3} \cos \frac{\pi t}{6} - 2 \cos \frac{\pi t}{3} + \frac{2}{3} \cos \frac{\pi t}{2} - \frac{2\sqrt{3}}{3} \sin \frac{\pi t}{6} - \frac{2\sqrt{3}}{3} \sin \frac{\pi t}{3}$

CHAPTER 16

1. $T_9(x) = 256x^9 - 576x^7 + 432x^5 - 120x^3 + 9x$
$T_{10}(x) = 512x^{10} - 1280x^8 + 1120x^6 - 400x^4 + 50x^2 - 1$
$T_{11}(x) = 1024x^{11} - 2816x^9 + 2816x^7 - 1232x^5 + 220x^3 - 11x$
$T_{12}(x) = 2048x^{12} - 6144x^{10} + 6912x^8 - 3584x^6 + 840x^4 - 72x^2 + 1$

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